Research Article

Energy Conservation for 3D Tropical Climate Model in Periodic Domains

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In this paper, we prove the energy conservation for the weak solutions of the 3D tropical climate model under some sufficient conditions. Our results are similar to Onsager's conjecture which is on energy conservation for weak solutions of Euler equations.

1. Introduction

The incompressible Euler equations in $\mathbb{R}^3$ are read as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1)$$

where u is the velocity and p is the pressure.

Given that the solution to system (1) is sufficiently smooth, it is easy to see that the total kinetic energy of the flow is conserved. However, turbulent flows are highly irregular; one major character of turbulent fluids is the energy dissipation. In the context of hydrodynamics, this leads to the famous Onsager's conjecture [1]. Specifically, Onsager's conjecture that every weak solution to the Euler equations with Hölder continuity exponent $\alpha > (1/3)$ conserves energy; and anomalous dissipation of energy occurs when $\alpha < (1/3)$. The first part of the conjecture was proved by Eyink [2] and extended to Besov space in subsequent work of Constantin et al. [3], while significant progress has recently been made on the second part, for example, see [4, 5], and some prior work on the compressible Onsager's conjecture can be found in the literature, for example, see [6, 7].

The object of this study is to address the relation between the energy conservation and the degree of regularity of the

solution for the tropical climate model in $\Omega = [0, 1]^3$ with periodic boundary conditions

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\ \partial_t v + v \cdot \nabla u + u \cdot \nabla v + \nabla \theta = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \nabla \cdot v = 0, \\ \nabla \cdot u = 0, \end{cases} \quad (2)$$

with initial data

$$\begin{aligned} u(x, 0) &= u_0(x), \\ v(x, 0) &= v_0(x), \\ \theta(x, 0) &= \theta_0(x), \end{aligned} \quad (3)$$

where u and v denote the vector fields and θ and p denote the temperature and the pressure, respectively.

The system (2) is derived from the inviscid primitive equations by Frierson et al. [8]. For more details of the related background about the tropical climate model, for instance, see [9, 10] and the references therein. Here, it is worth particularly mentioning the solution of (2) with enough regularity to justify integration by parts, and it is clear that the following total energy is conserved:

$$\begin{aligned} \text{total energy: } E(t) &= \|u(x, t)\|_{L^2(\Omega)}^2 + \|v(x, t)\|_{L^2(\Omega)}^2 \\ &\quad + \|\theta(x, t)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4)$$

So, it seems very reasonable to investigate the analogue of Onsager's conjecture about system (2). To the best of our knowledge, there is no available result for the weak solutions of the tropical climate model. Our study is motivated by [3, 11]. However, we are unable to obtain the desired result using the same processes as [3, 12]. The obstacle is due to $\nabla \cdot v \neq 0$ and the nonlinear term $u \cdot \nabla \theta$. In fact, we require a stronger regularity about v due to the presence of the term $u \nabla \cdot v$. Now, we are ready to state the main results.

Theorem 1. *Let $u \in L^3([0, T]; B_3^{\alpha, \infty}(\Omega)) \cap C([0, T], L^2(\Omega))$, $\nabla \cdot v \in L^3([0, T]; L^3(\Omega))$, $v \in L^3([0, T]; B_3^{\beta, \infty}(\Omega)) \cap C([0, T], L^2(\Omega))$, and $\theta \in L^3([0, T]; B_3^{\gamma, \infty}(\Omega)) \cap C([0, T], L^2(\Omega))$ be a weak solution of systems (2) and (3), i.e.,*

$$\begin{aligned} (u, \partial_t \varphi) + (u \otimes u, \nabla \varphi) + (p, \nabla \varphi) + (v \otimes v, \nabla \varphi) &= 0, \\ (v, \partial_t \varphi) + (u \otimes v, \nabla \varphi) + (v \otimes u, \nabla \varphi) + (u \cdot \nabla v, \varphi) + (\theta, \nabla \cdot \varphi) &= 0, \\ (\theta, \partial_t \varphi) + (u \theta, \nabla \varphi) + (v, \nabla \varphi) &= 0, \end{aligned} \quad (5)$$

where $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^+)$ is a vector field and $\phi \in C_c^\infty(\Omega \times \mathbb{R}^+)$ is a scalar field. If

$$\begin{aligned} \alpha &> \frac{1}{3}, \\ \alpha + 2\beta &> 1, \\ \alpha + 2\gamma &> 1, \end{aligned} \quad (6)$$

then we have

$$E(t) = E(0), \quad \text{for } t \in [0, T]. \quad (7)$$

2. Preliminary Remarks

In this section, we briefly discuss some properties of the Besov space $B_3^{\alpha, \infty}(\Omega)$.

$$\|u\|_{B_3^{\alpha, \infty}(\Omega)} := \|u\|_{L^3(\Omega)} + \sup_{x \in \Omega} \frac{\|u(x+y) - u(x)\|_{L^3(\Omega)}}{|y|^\alpha} < +\infty. \quad (8)$$

From the above definition, we have the following inequalities [3]:

$$\|u(x+y) - u(x)\|_{L^3(\Omega)} \leq C|y|^\alpha \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \quad (9)$$

Let $\eta \in C_c^\infty(\mathbb{R}^3)$ be a standard mollifying kernel and set

$$\eta^\epsilon = \frac{1}{\epsilon^3} \eta\left(\frac{x}{\epsilon}\right). \quad (10)$$

With the notation

$$\begin{aligned} u^\epsilon &= \eta^\epsilon * u, \\ x \in \Omega^\epsilon &= \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}, \end{aligned} \quad (11)$$

we have the following inequalities [3]:

$$\|u - u^\epsilon\|_{L^3} \leq C\epsilon^\alpha \|u\|_{B_3^{\alpha, \infty}(\Omega)}, \quad (12)$$

$$\|\nabla u^\epsilon\|_{L^3} \leq C\epsilon^{\alpha-1} \|u\|_{B_3^{\alpha, \infty}(\Omega)}, \quad (13)$$

$$\|u^\epsilon\|_{L^3} \leq \|u - u^\epsilon\|_{L^3} + \|u\|_{L^3} \leq (C\epsilon^\alpha + 1) \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \quad (14)$$

Now, we define

$$\begin{aligned} r_\epsilon(u, v)(x) &= \int_\Omega \eta^\epsilon(y) [(u(x-y) - u(x)) \otimes (v(x-y) - v(x))] dy, \\ q_\epsilon(u, v)(x) &= \int_\Omega \eta^\epsilon(y) [(u(x-y) - u(x)) \nabla \cdot (v(x-y) - v(x))] dy, \\ h_\epsilon(u, \theta)(x) &= \int_\Omega \eta^\epsilon(y) [(u(x-y) - u(x)) (\theta(x-y) - \theta(x))] dy. \end{aligned} \quad (15)$$

Then, it is easy to check the following equalities, it is pointed out that the first equality was discovered in [3]:

$$\begin{aligned} (u \otimes v)^\epsilon &= u^\epsilon \otimes v^\epsilon + r_\epsilon(u, v) - (u - u^\epsilon) \otimes (v - v^\epsilon), \\ (u \nabla \cdot v)^\epsilon &= u^\epsilon \nabla \cdot v^\epsilon + q_\epsilon(u, v) - (u - u^\epsilon) \nabla \cdot (v - v^\epsilon), \\ (u \theta)^\epsilon &= u^\epsilon \theta^\epsilon + h_\epsilon(u, \theta) - (u - u^\epsilon) (\theta - \theta^\epsilon). \end{aligned} \quad (16)$$

3. Proof of the Main Theorem

For the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments need to be modified in time are straightforward.

Proof. First, we apply the test function $\varphi = (u^\epsilon)^\epsilon$ on the first equation and then we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |u^\epsilon(x, t)|^2 dx &= \int_\Omega (u \otimes u)^\epsilon \nabla u^\epsilon dx + \int_\Omega (v \otimes v)^\epsilon \nabla u^\epsilon dx \\ &= I_1 + I_2. \end{aligned} \quad (17)$$

We point out that the I_1 term can be estimated as same as that in [3], and we omit the details:

$$|I_1| \leq C\epsilon^{3\alpha-1} \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \quad (18)$$

Now, we estimate I_2 :

$$\begin{aligned}
I_2 &= \int_{\Omega} (\nu \otimes \nu)^\epsilon \nabla u^\epsilon dx \\
&= \int_{\Omega} [\nu^\epsilon \otimes \nu^\epsilon + r_\epsilon(\nu, \nu) - (\nu - \nu^\epsilon) \otimes (\nu - \nu^\epsilon)] \nabla u^\epsilon dx \\
&= I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{19}$$

Using the following inequality and (12) and (13), we obtain

$$\|r_\epsilon(\nu, \nu)\|_{L^{(3/2)}} \leq C\epsilon^{2\beta} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}^2. \tag{20}$$

We can also obtain the following estimate through the above inequalities:

$$|I_{22}|; |I_{23}| \leq C\epsilon^{\alpha+2\beta-1} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}^2 \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \tag{21}$$

Because $\nabla \cdot \nu \neq 0$, I_{21} cannot be zero, and we will estimate this term later.

We rewrite the second equation as follows:

$$\partial_t \nu + \nabla \cdot (u \otimes \nu) + \nabla \cdot (\nu \otimes u) - u \nabla \cdot \nu + \nabla \theta = 0. \tag{22}$$

Second, we apply the test function $\varphi = (\nu^\epsilon)^\epsilon$ on the above equation and then we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nu^\epsilon\|_2^2 &= \int_{\Omega} (u \otimes \nu)^\epsilon \nabla \nu^\epsilon dx + \int_{\Omega} (\nu \otimes u)^\epsilon \nabla \nu^\epsilon dx \\
&\quad + \int_{\Omega} (u \nabla \cdot \nu)^\epsilon \nu^\epsilon dx + \int_{\Omega} \theta^\epsilon \nabla \cdot \nu^\epsilon dx \\
&= H_1 + H_2 + H_3 + H_4.
\end{aligned} \tag{23}$$

Using (12), one can easily obtain

$$\|r_\epsilon(u, \nu)\|_{L^{(3/2)}} \leq C\epsilon^{\alpha+\beta} \|u\|_{B_3^{\alpha, \infty}(\Omega)} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}. \tag{24}$$

Then,

$$\begin{aligned}
H_1 &= \int_{\Omega} [u^\epsilon \otimes \nu^\epsilon + r_\epsilon(u, \nu) - (u - u^\epsilon) \otimes (\nu - \nu^\epsilon)] \nabla \nu^\epsilon dx \\
&= H_{11} + H_{12} + H_{13}.
\end{aligned} \tag{25}$$

Similar to (21), we have

$$|H_{12}|; |H_{13}| \leq C\epsilon^{\alpha+2\beta-1} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}^2 \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \tag{26}$$

We can rewrite H_{11} as follows:

$$\begin{aligned}
H_{11} &= - \int_{\Omega} \nabla \cdot (u^\epsilon \otimes \nu^\epsilon) \nu^\epsilon dx \\
&= - \int_{\Omega} \nu^\epsilon \cdot \nabla u^\epsilon \nu^\epsilon dx - \int_{\Omega} u^\epsilon \nu^\epsilon \nabla \cdot \nu^\epsilon dx \\
&= - \int_{\Omega} (\nu^\epsilon \otimes \nu^\epsilon) \nabla u^\epsilon dx - \int_{\Omega} u^\epsilon \nu^\epsilon \nabla \cdot \nu^\epsilon dx.
\end{aligned} \tag{27}$$

We can estimate H_2 using a similar process as follows:

$$\begin{aligned}
H_2 &= \int_{\Omega} [\nu^\epsilon \otimes u^\epsilon + r_\epsilon(\nu, u) - (\nu - \nu^\epsilon) \otimes (u - u^\epsilon)] \nabla \nu^\epsilon dx \\
&= H_{21} + H_{22} + H_{23}.
\end{aligned} \tag{28}$$

Then, we have the following estimate:

$$|H_{22}|; |H_{23}| \leq C\epsilon^{\alpha+2\beta-1} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}^2 \|u\|_{B_3^{\alpha, \infty}(\Omega)}. \tag{29}$$

Here, we have used the following equality due to $\nabla \cdot u = 0$:

$$\begin{aligned}
H_{21} &= - \int_{\Omega} \nabla \cdot (\nu^\epsilon \otimes u^\epsilon) \nu^\epsilon dx \\
&= - \int_{\Omega} u^\epsilon \cdot \nabla \nu^\epsilon \nu^\epsilon dx - \int_{\Omega} u^\epsilon \nu^\epsilon \nabla \cdot u^\epsilon dx \\
&= 0.
\end{aligned} \tag{30}$$

Now, we estimate H_3 as follows:

$$\begin{aligned}
H_3 &= \int_{\Omega} [u^\epsilon \nabla \cdot \nu^\epsilon + q_\epsilon(u, \nu) - (u - u^\epsilon) \nabla \cdot (\nu - \nu^\epsilon)] \nu^\epsilon dx \\
&= H_{31} + H_{32} + H_{33}.
\end{aligned} \tag{31}$$

Using (12), one can obtain the following inequality:

$$\|q_\epsilon(u, \nu)\|_{L^{(3/2)}} \leq C\epsilon^\alpha \|u\|_{B_3^{\alpha, \infty}(\Omega)} \|\nabla \cdot \nu\|_{L^3(\Omega)}. \tag{32}$$

Then, we have

$$|H_{32}|; |H_{33}| \leq C\epsilon^\alpha (C\epsilon^\beta + 1) \|\nabla \cdot \nu\|_{L^3(\Omega)} \|u\|_{B_3^{\alpha, \infty}(\Omega)} \|\nu\|_{B_3^{\beta, \infty}(\Omega)}. \tag{33}$$

From the above, we can easily find that

$$I_{21} + H_{11} + H_{31} = 0. \tag{34}$$

We will estimate H_4 later. At last, we apply the test function $\phi = (\theta^\epsilon)^\epsilon$ on the first equation and then we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\theta^\epsilon\|_{L^2(\Omega)}^2 &= \int_{\Omega} (u\theta)^\epsilon \nabla \theta^\epsilon dx - \int_{\Omega} \nabla \cdot \nu^\epsilon \theta^\epsilon dx = N_1 + N_2.
\end{aligned} \tag{35}$$

Now, we have

$$\begin{aligned}
N_1 &= \int_{\Omega} [u^\epsilon \theta^\epsilon + h_\epsilon(u, \theta) - (u - u^\epsilon)(\theta - \theta^\epsilon)] \nabla \theta^\epsilon dx \\
&= N_{11} + N_{12} + N_{13}.
\end{aligned} \tag{36}$$

Similarly, we also have the following inequality:

$$\|h_\epsilon(u, \theta)\|_{L^{(3/2)}} \leq C\epsilon^{\alpha+\gamma} \|u\|_{B_3^{\alpha, \infty}(\Omega)} \|\theta\|_{B_3^{\gamma, \infty}(\Omega)}. \tag{37}$$

Then, we have

$$|N_2|; |N_3| \leq C\epsilon^{\alpha+2\gamma-1} \|u\|_{B_3^{\alpha, \infty}(\Omega)} \|\theta\|_{B_3^{\gamma, \infty}(\Omega)}^2. \tag{38}$$

Because $\nabla \cdot u^\epsilon = 0$, we can easily find that

$$N_{11} = 0. \quad (39)$$

We can also see that

$$H_4 + N_2 = 0. \quad (40)$$

Finally, we can establish the following inequality through the above estimates and condition (6):

$$|E^\epsilon(t) - E^\epsilon(0)| \leq C\epsilon^{3\alpha-1} + C\epsilon^{\alpha+2\beta-1} + C\epsilon^{\alpha+2\gamma-1} \longrightarrow 0, \quad (41)$$

as $\epsilon \longrightarrow 0$, which completes the proof. $\square$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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