Research Article

Some Smooth and Nonsmooth Traveling Wave Solutions for KP-MEW(2, 2) Equation

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In this paper, we consider the KP-MEW(2, 2) equation by the theory of bifurcations of planar dynamical systems when integral constant is considered. The periodic peakon solution and peakon and smooth periodic solutions are given.

1. Introduction

The KdV equation [1]

$$q_t + aqq_x + q_{xxx} = 0, \quad (1)$$

is a model that governs the one-dimensional propagation of small-amplitude and weakly dispersive waves. In (1), the nonlinear term qq_x causes steepening of wave form, and the linear dispersion term q_{xxx} gives rise to the spread of the wave.

There are many researches on higher dimensional models recently. One of the well-known 2-dimensional generalizations of the KdV equation is KP equation [2]:

$$(q_t + aqq_x + q_{xxx})_x + qyy = 0, \quad (2)$$

which was proposed by Kadomtsev and Petviashvili, described as a nonlinear model for shallow water waves with weakly nonlinear restoring forces and waves in the media of ferromagnet. MEW equation

$$q_t + a(q^3)_x - bq_{xxt} = 0, \quad (3)$$

plays an important role in many physical applications [3]. Wazwaz [4] proposed a KP-MEW equation

$$(q_t + (q^n)_x + (q)_{xxt})_x + q_{yy} = 0, \quad (4)$$

and investigated the exact solutions with different physical structure. Moreover, KP-MEW equation was investigated on some methods [4–6]. Particularly, Saha [7] considered the generalized KP-MEW equation

$$(q_t \pm (q^m)_x \pm (q^n)_{xxt})_x \pm q_{yy} = 0, \quad (5)$$

by using the theory of bifurcations of planar dynamical systems [7–11]. In [7], integral constant is neglected; it is said that, on the condition integral constant $g = 0$, the author obtained traveling wave solutions when m, n, c varied. More precisely, for $m = n = 2$, what called KP-MEW (2, 2) equation in the form

$$(q_t + (q^2)_x + (q^2)_{xxt})_x + q_{yy} = 0, \quad (6)$$

is investigated by Li and Song [12]. They used the theory of bifurcations of planar dynamical systems to find compacton-like wave and a kink-like wave for (6) when integral constant g was not neglected. After that, (6) was investigated to find the peakon soliton, cuspon soliton, and smooth soliton solutions on the boundary condition by using the phase portrait analytical technique [13, 14]. Particularly, the generalized KP-MEW equation is the nonlinear PDEs which described the propagation of long wave with dissipation and dispersion in nonlinear media. In [15], the qualitative change of the traveling wave solutions of the KP-MEW-Burgers equation is investigated by using numerical simulations.

More recently, the solitary wave solutions for KP-MEW equation are constructed with the help of a new technique which is a modification form of the extended auxiliary equation mapping method.

In present paper, we consider the KP-MEW (2, 2) equation in the form

$$(q_t - (q^2)_x - (q^2)_{xxt})_x - q_{yy} = 0, \quad (7)$$

by the theory of bifurcations of planar dynamical systems when integral constant $g \neq 0$ and $g = 0$, which is not investigated by the same method before. In (7), the first term is the evolution term, while the second term is the dissipative term and the third term is the dispersion term.

It is well known that nonlinear complex wave phenomena appear in many fields, such as plasma physics, biology, fluid mechanics, solid state physics, and optical fibers. They are related to nonlinear partial differential equations. As mathematical models of the phenomena, investigations of exact solutions for nonlinear partial differential equations will help understand these phenomena better. With the development of nonlinear partial differential equations theory, there exist many different approaches to search for their exact solutions, such as Hirota bilinear method [16], inverse scattering method [17], Darboux transformation method [18], and so on. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

This paper is organized as follows. In Section 2, depending on the changes of parameters c and g , bifurcations of the phase portraits of systems (2) are shown. In Section 3, the parametric expressions of periodic peakon solution and peakon and smooth periodic solutions are given. In Section 4, for given periodic wave solutions, we prove that the KP-MEW (2, 2) equation has two isochronous centers under certain parameter conditions and there exist two families of periodic solutions with equal period.

2. Phase Portrait for (7)

Making the transformations $q(x, t) = q(x - ct) = u(\xi)$ to (7) and integrating it twice, we get

$$-(1+c)u - u^2 + c(u^2)'' = g, \quad (8)$$

where c is the wave speed, g is the integral constant, and $'$ is the derivative with respect to ξ .

Equation (8) is equivalent to the planar dynamical system

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{g + (1+c)u + u^2 - 2cy^2}{2cu}. \end{cases} \quad (9)$$

Using the "timescale" transformation $d\xi = 2cud\tau$, (9) reduces to the regular system

$$\begin{cases} \frac{du}{d\tau} = 2cuy, \\ \frac{dy}{d\tau} = g + (1+c)u + u^2 - 2cy^2, \end{cases} \quad (10)$$

with the first integral

$$y^2 = \frac{(g/2)u^2 + (1+c/3)u^3 + (1/4)u^4 + h}{cu^2}, \quad (11)$$

and Hamiltonian function

$$H(u, y) = cu^2 y^2 - \left(\frac{g}{2}u^2 + \frac{1+c}{3}u^3 + \frac{1}{4}u^4 \right) = h, \quad (12)$$

where h is an integral constant. Thus, systems (9) and (10) have the same topological phase portraits except for the straight line $u = 0$. Under some parametric conditions, the variable τ is a fast variable while the variable ξ is a slow variable in the sense of the geometric singular perturbation theory [19, 20].

Suppose that $u(\xi)$ is a continuous solution of (9) for $\xi \in (-\infty, +\infty)$ and $\lim_{\xi \rightarrow -\infty} u(\xi) = \alpha$ and $\lim_{\xi \rightarrow +\infty} u(\xi) = \beta$. If $\alpha = \beta$, $u(x, t)$ is called a solitary wave solution. If $\alpha \neq \beta$, $u(x, t)$ is called a kink (anti-kink) wave solution. A solitary wave solution corresponds to a homoclinic orbit, a kink (anti-kink) wave solution corresponds to a heteroclinic orbit, and a periodic orbit corresponds to a periodic traveling wave solution. According to the bifurcation theory of dynamical systems, let $\Delta = (1+c)^2 - 4g$, and the following holds:

- (i) For $\Delta > 0$, system (10) has two equilibrium points $E_1(- (1+c) + \sqrt{\Delta}/2, 0)$ and $E_2(- (1+c) - \sqrt{\Delta}/2, 0)$ (see Figures 1(a), 2(a)–2(c), 3(b)–3(d), and 4(a)–4(c)).
- (ii) For $\Delta = 0$, system (10) has unique equilibrium point $E_1(- (1+c)^2/2, 0)$ (see Figures 2(d), 3(a), and 4(d)).
- (iii) For $\Delta < 0$, system (10) has no equilibrium point.
- (iv) On the straight line $u = c$, system (10) has two equilibrium points $S_1(0, \sqrt{g/2c})$ and $S_2(0, -\sqrt{g/2c})$ for $cg > 0$ (see Figures 1(a), 2(a), 3(a)–3(d), and 4(a)); system (10) has unique equilibrium point $S_1(0, 0)$ for $g = 0, c \neq -1$ on the straight line $u = c$ (see Figures 2(b) and 4(b)).

Let $M(u_i, y_i)$ be the coefficient matrix of the linearized system of (10) at an equilibrium point (u_i, y_i) and $J(u_i, y_i) = \det M(u_i, y_i)$. Hence, the following holds:

$$J(u_i, y_i) = -8c^2 y_i^2 - 2cu_i(1+c+2u_i). \quad (13)$$

It has the following proposition: for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$, then it is a center point; if $J = 0$ and the Poincaré index of the equilibrium point is zero, then it is cusped.

For convenience, in order to discuss the given constants c and g , set $g_1 = (1+c)^2/4, g_2 = 2(1+c)^2/9$. In the following, we will discuss four cases for c : $c = -1, -1 < c < 0, c > 0$, and $c > -1$. The phase portraits are shown in Figures 1–4.

FIGURE 1: Phase portraits of system (9) when $c = -1$, $g < 0$. (a) $g < 0$. (b) $g = 0$.

3. Some Smooth and Nonsmooth Traveling Wave Solutions of System (7)

In this section, some traveling wave solutions of system (7) are given, with the exact parametric representations of its solutions presented. With the aid of [21], we will obtain the exact traveling wave solutions of KP-MEW (2, 2) equation.

3.1. Periodic Peakon Solution. In this subsection, we shall to find the exact expression of periodic peak as solution for system (7) under the parametric condition $c > 0, 0 < g < g_2$. The phase portrait is seen in Figure 3(d). For $h_2 < h < 0$, on the left-hand side of the straight line $u = 0$, there exists an arch orbit connecting S_1 and S_2 . At the same time,

$$y^2 = \frac{1}{4c} \left(u^2 + \frac{4(1+c)}{3}u + 2g \right) = \frac{1}{4c} \left[\left(u + \frac{2(1+c)}{3} \right)^2 - \left(\frac{4(1+c)^2}{9} - 2g \right) \right]. \quad (14)$$

Consequently, from the first equation of (11), the parametric representations are as follows:

$$u(\xi) = -\frac{2(1+c)}{3} + \sqrt{\frac{4(1+c)^2}{9} - 2g} \cosh\left(\frac{1}{2\sqrt{c}}\xi\right), \quad (15)$$

$$\xi \in \left(0, 2\sqrt{c} \cosh^{-1} \sqrt{\frac{4(1+c)^2}{4(1+c)^2 - 18g}} \right),$$

and equation (15) gives rise to a periodic peakon solution; the profile is shown in Figure 5(a).

3.2. Peakon Solutions. In this section, we will consider the curve triangle S_2SS_1 in Figure 3(c). There exists an equilibrium point S of (10) at the vertex of the triangle S_2SS_1 , which is far from the singular straight line $u = 0$. If $\xi \rightarrow \pm\infty$, the phase point $(u(\xi), y(\xi))$ of (10) tends to the equilibrium point S along two curves S_2S and SS_1 as ξ varies. In Figure 3(c), the curve triangle S_2SS_1 defined by $H(u, y) = 0$ is the limit curve of the periodic orbits which is given by

$H(u, y) = h, h \in (H(P), 0)$. As h varies from $H(P)$ to 0, the period of the periodic orbits approach to infinity. Thus, we know that, as the limit curve of the family of periodic cusp wave, this curve triangle S_2SS_1 gives rise to a solitary cusp wave of $u(\xi)$ (called peakon).

On the above analysis, it easy to know that, for $h = 0$, there are two arch orbits, which connect S_1 to S and connect S_2 to S , respectively (see Figure 3(c)). At the same time,

$$y^2 = \frac{1}{4c}(u + t_0)^2, \quad (16)$$

where $t_0 > 0$ from the first equation of (11); parametric representations of the peakon solution (see Figure 5(b)) are obtained as follows:

$$u(\xi) = -t_0 + (u_1 + t_0) \exp\left(-\frac{1}{2\sqrt{c}}|\xi|\right). \quad (17)$$

Remark 1. Note that the periodic peakon solution (15) and peakon solution (17) are not obtained in [6, 10].

FIGURE 2: Phase portraits of system (9) when $-1 < c < 0$. (a) $g < 0$. (b) $g = 0$. (c) $0 < g < g_1$. (d) $g = g_1$.

3.3. *Smooth Periodic Solutions.* For $c = -1, g < 0$ (see Figure 1(a)), more precisely, we color the orbits for different values of h to better understand (see Figure 6).

- (I) Corresponding to the level curves defined by $H(u, y) = h = 0$, there exist three heteroclinic orbits (the green orbits in Figure 6) connecting the equilibrium points $S_{1,2}(0, \pm \sqrt{g/2c})$ of system (7), enclosing the centers $E_1(-(1+c) + \sqrt{\Delta}/2, 0)$ and $E_2(-(1+c) - \sqrt{\Delta}/2, 0)$, respectively. For the oval orbit loop, it has

$$y^2 = \frac{u^2(t_1^2 - u^2)}{-4u^2}, \quad (18)$$

where $t_1 > u$; then, it can arrive to the following smooth periodic wave solutions (see Figure 7(a)) of system (7):

$$u(\xi) = \pm t_1 \sin\left(\frac{\xi}{2}\right). \quad (19)$$

- (II) Corresponding to the level curves defined by $H(u, y) = h, h \in (0, h_1)$, there exist two families of periodic orbits (the red and blue orbits in Figure 6) enclosing the centers $E_1(-(1+c) + \sqrt{\Delta}/2, 0)$ and $E_2(-(1+c) - \sqrt{\Delta}/2, 0)$, respectively:

$$y^2 = \frac{(t_2^2 - u^2)(u^2 - t_3^2)}{-4u^2}, \quad (20)$$

where $t_2 > u > t_3 > 0$. Consequently, it can obtain the parametric representations for the two families periodic wave solutions by quoting [21], see Figures 7(b) and 7(c):

FIGURE 3: Phase portraits of system (9) when $c > 0$. (a) $g = g_1$. (b) $g_2 < g < g_1$. (c) $g = g_2$. (d) $0 < g < g_2$.

FIGURE 4: Continued.

FIGURE 4: Phase portraits of system (9) when $c < -1$. (a) $g < 0$. (b) $g = 0$. (c) $g_2 < g < g_1$. (d) $g = g_1$.

FIGURE 5: Profiles of periodic peakon solution and peakon solution. (a) Periodic peakon solution for $c = 1$, $g = 2/3$, and $h = 0$. (b) Peakon solution for $c = 1$, $g = 8/9$, and $h = 0$.

$$u(\xi) = \pm \frac{\sqrt{2}}{2} \sqrt{t_2^2 + t_3^2 + (t_2^2 - t_3^2) \sin(\xi)}. \quad (21)$$

Therefore, as $h \rightarrow 0$, (21) gives rise to periodic peakon solutions (see Figure 5(a)). Furthermore, the presented two families of periodic solutions are of equal period from

Figures 7(b) and 7(c), and their period is half of the smooth periodic solution in Figure 7(a).

Remark 2. Note that smooth periodic solutions (18) and (21) for KP-MEW (2, 2) equation are not obtained by the same method in the reference.

FIGURE 6: The phase portrait of system (2) for $c = -1, g < 0$.

FIGURE 7: Continued.

FIGURE 7: When $c = 1, g = -1$, profiles of smooth periodic solutions for $h = 0$ (a) and $h = 0.1$ (b, c), respectively.

4. Conclusion

In present paper, the method of bifurcation theory of dynamical systems is used to investigate KP-MEW equation. We obtain the parametric representations of peakon, periodic peakon, and smooth periodic wave solutions. The phase portrait bifurcations of the traveling wave system corresponding to the equation are shown.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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