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Research Article

Strong Convergence on the Split Feasibility Problem by Mixing W-Mapping

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In this paper, we concern with the split feasibility problem (SFP) in real Hilbert space whenever the sets involved are nonempty, closed, and convex. By mixing W-mapping with the viscosity, we introduce a new iterative algorithm for solving the split feasibility problem, and we prove that our proposed algorithm is convergent strongly to a solution of the split feasibility problem.

1. Introduction

Throughout this article, we assume that H_1 and H_2 are two real Hilbert spaces. The split feasibility problem (SFP) was introduced by Censor and Elfving [1], and it is formulated as finding a point x in C such that Ax is in Q, namely,

$$\begin{aligned} x &\in C, \\ Ax &\in Q, \end{aligned} \tag{1}$$

where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator from H_1 to H_2.

Many inverse problems arising from various fields of science and technology, such as intensity-modulated radiation therapy [2], signal processing, and image reconstruction, can be summarized as SFP. Due to its applications, many algorithms have been invented to solve SFP (see, for instance, [3–11]).

To solve problem (1), in 2002, Byren [3] introduced a popular algorithm which is called the CQ-algorithm as follows:

$$x_{n+1} = P_C\left[I - \mu_n A^*\left(I - P_Q\right)A\right]x_n, \tag{2}$$

where I is the identity operator on H, P_C and P_Q denote the metric projection onto the closed convex subsets C and Q, respectively, and A^* is the adjoint operator of A and $0 < \mu_n < (2/\|A\|^2)$. In 2018, Wang [10] proposed his algorithm as follows:

$$x_{n+1} = x_n - \tau_n\left[\left(I - P_{C_n}\right)x_n + A^*\left(I - P_{Q_n}\right)Ax_n\right], \tag{3}$$

where $\{C_n\}$, $\{Q_n\}$, and $\{\tau_n\}$ were given by

$$\begin{aligned} C_n &= \{x \in H_1 \colon c(x_n) \le \langle \xi_n, x_n - x\rangle\}, \quad \xi_n \in \partial c(x_n), \\ Q_n &= \{y \in H_2 \colon q(Ax_n) \le \langle \zeta_n, Ax_n - y\rangle\}, \quad \zeta_n \in \partial q(Ax_n), \end{aligned} \tag{4}$$

in which $c\colon H_1 \longrightarrow R$ and $q\colon H_2 \longrightarrow R$ are two given convex functions, and

$$\tau_n = \lambda_n \frac{\left\|x_n - P_{C_n}x_n\right\|^2 + \left\|\left(I - P_{Q_n}\right)Ax_n\right\|^2}{2\left\|\left(I - P_{C_n}\right)x_n + A^*\left(I - P_{Q_n}\right)Ax_n\right\|^2}, \tag{5}$$

in which $\lambda_n \in (0, 4)$.

To obtain strong convergence theorem, Wang [10] modified his algorithm as follows:

$$\begin{aligned} y_n &= x_n - \tau_n\left[\left(I - P_{C_n}\right)x_n + A^*\left(I - P_{Q_n}\right)Ax_n\right], \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n, \end{aligned} \tag{6}$$

where $\{C_n\}$, $\{Q_n\}$, and $\{\tau_n\}$ were given as the same to the weak convergence theorem; $\{\alpha_n\}$ is a sequence in $[0, 1]$ which is chosen so that

$$\lim_{n\longrightarrow\infty}\alpha_n = 0,$$
$$\sum_{n=0}^{\infty}\alpha_n = \infty,$$
$$\text{either } \sum_{n=0}^{\infty}\left|\alpha_{n+1}-\alpha_n\right| < \infty \tag{7}$$
$$\text{or } \lim_{n\longrightarrow\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) = 1.$$

On the other hand, another problem which is similar to the split feasibility problem is the proximal split feasibility problem (PSFP), and the proximal split feasibility problem is to find a point x^* satisfying the property:

$$\begin{aligned} x^* &\in \arg\min f, \\ Ax^* &\in \arg\min g, \end{aligned} \tag{8}$$

where $f\colon H_1 \longrightarrow R\cup\{\infty\}$ and $g\colon H_2 \longrightarrow R\cup\{\infty\}$ are two proper and lower semicontinuous convex functions, $A\colon H_1 \longrightarrow H_2$ is a linear bounded operator, and $\arg\min f\colon = \{\overline{x}\in H_1\colon f(\overline{x})\le f(x), \forall x\in H_1\}$, $\arg\min g\colon = \{\overline{y}\in H_2\colon g(\overline{y})\le g(y), \forall y\in H_2\}$.

To solve problem (8), in 2014, Moudafi and Thakur [12] introduced the split proximal algorithm by the following iterative scheme:

$$x_{n+1} = \operatorname{prox}_{\lambda\mu_n f}\left(x_n - \mu_n A^*\left(I-\operatorname{prox}_{\lambda g}\right)Ax_n\right), \tag{9}$$

where the stepsize $\{\mu_n\}$ was the suitable positive real number sequence, and they also proved the weak convergence of the sequence generated by the above equation to a solution of problem (8). In 2015, Shehu et al. [13] introduced a viscosity-type algorithm for solving proximal split feasibility problems as follows:

$$\begin{aligned} y_n &= x_n - \mu_n A^*\left(I-\operatorname{prox}_{\lambda g}A\right)x_n, \\ x_{n+1} &= \alpha_n\psi(x_n) + (1-\alpha_n)\operatorname{prox}_{\lambda\mu_n f}y_n, \end{aligned} \tag{10}$$

where $\psi\colon H_1 \longrightarrow H_1$ is a contraction mapping. They also proved a strong convergence of the sequence generated by iterative scheme (10) in Hilbert spaces. Recently, Shehu and Iyiola [14] introduced the following algorithm for solving split proximal problems and fixed point problems in Hilbert spaces:

$$\begin{aligned} u_n &= (1-\alpha_n)x_n, \\ y_n &= \operatorname{prox}_{\lambda\gamma_n f}\left(u_n - \gamma_n A^*\left(I-\operatorname{prox}_{\lambda g}\right)Au_n\right), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n T y_n, \end{aligned} \tag{11}$$

where T is a k-strictly pseudocontractive mapping. They also showed that, under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ was generated by the algorithm that they introduced converges strongly to $x^*\in F(T)\cap\arg\min f\cap A^{-1}(\arg\min g)$.

If we defined $f = i_C$ and $g = i_Q$ as indicated functions of sets C and Q, where C and Q are nonempty, closed, and convex sets of H_1 and H_2, respectively, then the proximal split feasibility problem (8) becomes the split feasibility problem (1). In this paper, inspired and motivated by these works that have been done, we focus on the split feasibility problem in Hilbert spaces.

The rest of this paper is organized as follows. In Section 2, we review some definitions and lemmas that we need. In Section 3, we introduce a new iterative algorithm based on the viscosity method and W-mapping which is defined in Section 2 for finding a solution of the split feasibility problem and prove a strong convergence theorem under some mild conditions.

2. Preliminaries

Throughout this paper, let $H_i\,(i=1,2)$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by I the identity operator on $H_i\,(i=1,2)$ and by $w_w(x_n)$ the set of all weak cluster points of $\{x_n\}$. The notation $\longrightarrow$ stands for strong convergence and $\rightharpoonup$ stands for weak convergence.

Definition 1 (see [15]). Let $T\colon H\longrightarrow H$ be a nonlinear mapping. Then, T is

(1) Nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|, \quad \text{for } \forall x,y\in H_1. \tag{12}$$

(2) Firmly nonexpansive if

$$\|Tx-Ty\|^2+\|(I-T)x-(I-T)y\|^2\le\|x-y\|^2, \quad \text{for } \forall x,y\in H_1. \tag{13}$$

Definition 2. Let C be a nonempty closed convex subset of H_1. Then, an orthogonal projection $P_C\colon H_1\longrightarrow C$ is defined by

$$P_C x = \arg\min_{y\in C}\|x-y\|^2, \quad x\in H_1. \tag{14}$$

Lemma 1. *Let C be a nonempty closed convex subset of H_1, then*

(1) $\langle x-P_Cx, z-P_Cx\rangle\le 0, \forall x\in H_1, z\in C$.
(2) P_C and $I-P_C$ both are (firmly) nonexpansive.
(3) $\langle P_Cx-P_Cy, x-y\rangle\ge\|P_Cx-P_Cy\|^2, \forall x,y\in H_1$.
(4) $\langle x-z, x-P_Cx\rangle\ge\|x-P_Cx\|^2, \forall x\in H_1, z\in C$.

Definition 3. Let $T\colon H_1\longrightarrow H_1$ be an operator with $\operatorname{Fix}(T)\ne\varnothing$. If for any $\{x_n\}$ in H_1, $x_n\rightharpoonup x$ and $(I-T)x_n\longrightarrow 0$, we can obtain $x\in\operatorname{Fix}(T)$, then we claim that $I-T$ is demiclosed at zero.

Lemma 2. *If $T\colon H_1\longrightarrow H_1$ is a nonexpansive operator, then $I-T$ is demiclosed at zero.*

Lemma 3. *For all $x,y\in H_1$, and $\eta>0$, we have*

(1) $\|x+y\|^2 \le \|y\|^2 + 2\langle x, x+y\rangle$.

(2) $\|x+y\|^2 \le (1+\eta)\|x\|^2 + (1+(1/\eta))\|y\|^2$.

Definition 4. If for any $n \ge 1$ and $z \in C$, let $S_i\,(1 \le i \le n)$ be a nonexpansive mapping on C and η_i be real numbers with $0 < \eta_i \le \eta < 1$. We define a mapping W_n on C for each $n \ge 1$ by

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= (1-\eta_n)I + \eta_n S_n U_{n,n+1} \\ U_{n,n-1} &= (1-\eta_{n-1})I + \eta_{n-1} S_{n-1} U_{n,n} \\ &\vdots \\ U_{n,k} &= (1-\eta_k)I + \eta_k S_k U_{n,k+1} \\ U_{n,k-1} &= (1-\eta_{k-1})I + \eta_{k-1} S_{k-1} U_{n,k} \\ &\vdots \\ U_{n,2} &= (1-\eta_2)I + \eta_2 S_2 U_{n,3} \\ U_{n,1} &= (1-\eta_1)I + \eta_1 S_1 U_{n,2} \\ W_n &= U_{n,1}, \end{aligned} \tag{15}$$

then we say that W_n is the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\eta_n, \eta_{n-1}, \ldots, \eta_1$.

Lemma 4 (see [16]). *Let W_n: $H_1 \longrightarrow H_1$ be a W-mapping, then W_n is a nonexpansive mapping.*

Lemma 5 (see [17]). *Assume that $\{s_k\}$ is a sequence of nonnegative real number such that*

$$s_{k+1} \le (1-\lambda_k)s_k + \lambda_k b_k + c_k, \tag{16}$$

where $\{\lambda_k\}$, $\{b_k\}$, and $\{c_k\}$ satisfy the following conditions:

(1) $\sum_{k=0}^{\infty} \lambda_k = \infty$;

(2) *either* $\limsup_{k\longrightarrow\infty} b_k \le 0$ *or* $\sum_{k=0}^{\infty} |\lambda_k b_k| < \infty$;

(3) $c_k \ge 0$ *for all* k *and* $\sum_{k=0}^{\infty} c_k < \infty$.

Then, $\lim_{k\longrightarrow\infty} s_k = 0$.

3. Main Results

To introduce our iterative algorithm for solving the split feasibility problem in real Hilbert spaces, firstly, we shall assume that problem (1) is consistent, namely, its solution set, denoted by S, is nonempty. Secondly, we need to define a special W-mapping $W_n\,(n \ge 1)$ as follows:

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= (1-\eta_n)I + \eta_n P_C U_{n,n+1} \\ U_{n,n-1} &= (1-\eta_{n-1})I + \eta_{n-1} P_C U_{n,n} \\ &\vdots \\ U_{n,k} &= (1-\eta_k)I + \eta_k P_C U_{n,k+1} \\ U_{n,k-1} &= (1-\eta_{k-1})I + \eta_{k-1} P_C U_{n,k} \\ &\vdots \\ U_{n,2} &= (1-\eta_2)I + \eta_2 P_C U_{n,3} \\ U_{n,1} &= (1-\eta_1)I + \eta_1 P_C U_{n,2} \\ W_n &= U_{n,1}, \end{aligned} \tag{17}$$

where $\eta_i = (1/n-i+1)\,(1 \le i \le n)$. From Lemma 4, we know that W_n is nonexpansive.

Now, we will introduce our iterative algorithms for the split feasibility problem.

Algorithm 1. Given an initial point $x_0 \in H_1$, let ψ: $H_1 \longrightarrow H_1$ be a contraction mapping with $\delta \in [0,1)$. Assume that x_n has been constructed and compute x_{n+1} by the following iterative scheme:

$$\begin{aligned} y_n &= x_n - \tau_n\left[x_n - P_C x_n + A^*\left(I - P_Q\right)Ax_n\right], \\ x_{n+1} &= \alpha_n \psi(x_n) + (1-\alpha_n)W_n y_n, \end{aligned} \tag{18}$$

where $\{\tau_n\}$ and $\{\alpha_n\}$ satisfy

(1) $0 < \varepsilon \le \tau_n \le (2/1 + \|A\|^2) - \varepsilon$;

(2) $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_0^{\infty} \alpha_n = \infty$.

Theorem 1. *Let $\{x_n\}$ be the sequence generated by Algorithm 1, then $\{x_n\}$ converges strongly to a solution x^* of the SFP (1), where $x^* = P_S\psi(x^*)$.*

Proof. Since ψ: $H_1 \longrightarrow H_1$ is a contraction mapping with $\delta \in [0,1)$ and the fact that P_S is nonexpansive, it is clear that $P_S\psi$: $H_1 \longrightarrow S$ is also a contraction mapping. By Banach fixed point theorem, there exists $x^* \in S$, such that $x^* = P_S\psi(x^*)$.

Since $x^* \in S$, that is, $x^* \in C$ and $Ax^* \in Q$. By the definition of W_n, we have $x^* = W_n x^*$. In what follows, we will divide the proof into four steps.

Firstly, we prove that the sequence $\{x_n\}$ is bounded.

From (18) and Lemmas 1 and 3, we have

$$\begin{aligned} \left\|y_n - x^*\right\|^2 &= \left\|x_n - x^* - \tau_n\left[x_n - P_C x_n + A^*\left(I - P_Q\right)Ax_n\right]\right\|^2 \\ &= \left\|x_n - x^*\right\|^2 - 2\tau_n\left[\langle x_n - x^*, x_n - P_C x_n\rangle + \langle Ax_n - P_Q Ax_n, Ax_n - Ax^*\rangle\right] \\ &\quad + \tau_n^2\left\|x_n - P_C x_n + A^*\left(I - P_Q\right)Ax_n\right\|^2 \end{aligned}$$

$$\begin{aligned}
&\le \|x_n - x^*\|^2 - 2\tau_n\left(\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q Ax_n\|^2\right) \\
&\quad + \tau_n^2\left[\left(1 + \|A\|^2\right)\|x_n - P_C x_n\|^2 + \left(1 + \frac{1}{\|A\|^2}\right)\|Ax_n - P_Q Ax_n\|^2\right] \\
&\le \|x_n - x^*\|^2 - \tau_n\left[2 - \tau_n\left(1 + \|A\|^2\right)\right]\left(\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q Ax_n\|^2\right).
\end{aligned} \tag{19}$$

By condition (1), we have $\tau_n[2 - \tau_n(1 + \|A\|^2)] > 0$, so $\|y_n - x^*\| \le \|x_n - x^*\|$. Therefore, from (18), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(\psi(x_n) - x^*) + (1 - \alpha_n)(W_n y_n - x^*)\| \\
&\le \alpha_n\|\psi(x_n) - x^*\| + (1 - \alpha_n)\|W_n y_n - x^*\| \\
&\le \alpha_n\|\psi(x_n) - \psi(x^*)\| + \alpha_n\|\psi(x^*) - x^*\| + (1 - \alpha_n)\|W_n y_n - x^*\| \\
&\le \alpha_n\delta\|x_n - x^*\| + \alpha_n\|\psi(x^*) - x^*\| + (1 - \alpha_n)\|y_n - x^*\| \\
&\le \alpha_n\delta\|x_n - x^*\| + \alpha_n\|\psi(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&= [1 - \alpha_n(1 - \delta)]\|x_n - x^*\| + \alpha_n(1 - \delta)\frac{\|\psi(x^*) - x^*\|}{1 - \delta} \\
&\le \max\left\{\|x_n - x^*\|, \frac{\|\psi(x^*) - x^*\|}{1 - \delta}\right\}.
\end{aligned} \tag{20}$$

By introduction, we obtain

$$\|x_{n+1} - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|\psi(x^*) - x^*\|}{1 - \delta}\right\}, \tag{21}$$

for all $n \ge 0$. The above inequality implies that the sequence $\{x_n\}$ is bounded. Combining with (18), we know that $\{y_n\}$, $\{W_n y_n\}$, and $\{\psi(x_n)\}$ are also bounded.

Secondly, we show that the following inequality holds:

$$\|x_{n+1} - x^*\|^2 \le (1 - \tilde{\alpha}_n)\|x_n - x^*\|^2 + \tilde{\alpha}_n\delta_n, \tag{22}$$

where $\tilde{\alpha}_n = \alpha_n(1 - \delta^2)$ and

$$\delta_n = \frac{2\alpha_n\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle - t_n\left(\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q Ax_n\|^2\right)}{\alpha_n\left(1 - \delta^2\right)}, \tag{23}$$

with $t_n = \tau_n(1 - \alpha_n)[2 - \tau_n(1 + \|A\|^2)]$.

From equations (18) and (19) and Lemma 3, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(\psi(x_n) - \psi(x^*)) + (1 - \alpha_n)(W_n y_n - x^*) + \alpha_n(\psi(x^*) - x^*)\|^2 \\
&\le \|\alpha_n(\psi(x_n) - \psi(x^*)) + (1 - \alpha_n)(W_n y_n - x^*)\|^2 + 2\alpha_n\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n\|\psi(x_n) - \psi(x^*)\|^2 + (1 - \alpha_n)\|W_n y_n - x^*\|^2 + 2\alpha_n\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n\delta^2\|x_n - x^*\|^2 + (1 - \alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n\delta^2\|x_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \tau_n(1 - \alpha_n)\left[2 - \tau_n\left(1 + \|A\|^2\right)\right] \\
&\quad \times\left(\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q Ax_n\|^2\right) + 2\alpha_n\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \left[1 - \alpha_n\left(1 - \delta^2\right)\right]\|x_n - x^*\|^2 + \alpha_n\left(1 - \delta^2\right) \\
&\quad \left[\frac{2\langle\psi(x^*) - x^*, x_{n+1} - x^*\rangle}{1 - \delta^2} - \frac{\tau_n(1 - \alpha_n)\left[2 - \tau_n\left(1 + \|A\|^2\right)\right]\left(\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q Ax_n\|^2\right)}{\alpha_n\left(1 - \delta^2\right)}\right].
\end{aligned} \tag{24}$$

So, inequality (22) holds.

Thirdly, we show that $\limsup_{n\longrightarrow\infty}\delta_n$ is finite.

Since $\{x_n\}$ is bounded, we have

$$\begin{aligned}\delta_n &\leq \frac{2\langle \psi(x^*)-x^*, x_{n+1}-x^*\rangle}{1-\delta^2}\\ &\leq \frac{2\|\psi(x^*)-x^*\|\cdot\|x_{n+1}-x^*\|}{1-\delta^2}<\infty.\end{aligned} \tag{25}$$

This implies that $\limsup_{n\longrightarrow\infty}\delta_n<\infty$. Next, we will show that $\limsup_{n\longrightarrow\infty}\delta_n<-1$ by contraction. If we assume that $\limsup_{n\longrightarrow\infty}\delta_n<-1$, then there exists $n_0\in N$, such that $\delta_n\leq -1$ for all $n\geq n_0$. From (22), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq (1-\tilde{\alpha}_n)\|x_n-x^*\|^2+\tilde{\alpha}_n\delta_n\\ &\leq (1-\tilde{\alpha}_n)\|x_n-x^*\|^2-\tilde{\alpha}_n\\ &\leq \|x_n-x^*\|^2-\tilde{\alpha}_n.\end{aligned} \tag{26}$$

By introduction, we have

$$\|x_{n+1}-x^*\|^2\leq\|x_{n_0}-x^*\|^2-\sum_{i=n_0}^{n}\tilde{\alpha}_i. \tag{27}$$

Since $\sum_{i=0}^{\infty}\alpha_i=\infty$, $\sum_{i=0}^{\infty}\tilde{\alpha}_i=(1-\delta^2)\sum_{i=0}^{\infty}\alpha_i=\infty$, then there exists $N>n_0$, such that $\sum_{i=n_0}^{N}\tilde{\alpha}_i>\|x_{n_0}-x^*\|^2$. Combining with the last inequality, we have

$$\|x_{N+1}-x^*\|^2\leq\|x_{n_0}-x^*\|^2-\sum_{i=n_0}^{N}\tilde{\alpha}_i<0, \tag{28}$$

which is contradicted with the fact that $\|x_{N+1}-x^*\|^2$ is nonnegative. Thus, $\limsup_{n\longrightarrow\infty}\delta_n\geq -1$. So, $\limsup_{n\longrightarrow\infty}\delta_n$ is finite.

Lastly, we show that $\limsup_{n\longrightarrow\infty}\delta_n\leq 0$.

Since $\limsup_{n\longrightarrow\infty}\delta_n$ is finite, there exists a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\delta_n &= \lim_{k\longrightarrow\infty}\delta_{n_k}=\lim_{k\longrightarrow\infty}\frac{2\alpha_{n_k}\langle\psi(x^*)-x^*, x_{n_k+1}-x^*\rangle}{\alpha_{n_k}(1-\delta^2)}\\ &-\frac{t_{n_k}\left(\|x_{n_k}-P_Cx_{n_k}\|^2+\|Ax_{n_k}-P_QAx_{n_k}\|^2\right)}{\alpha_{n_k}}(1-\delta^2).\end{aligned} \tag{29}$$

Since $\langle\psi(x^*)-x^*, x_{n_k+1}-x^*\rangle$ is bounded, without loss of generality, we may assume the limit of $\langle\psi(x^*)-x^*, x_{n_k+1}-x^*\rangle$ exists. From (29), we may also assume the following limit exists:

$$\lim_{k\longrightarrow\infty}\frac{t_{n_k}\left(\|x_{n_k}-P_Cx_{n_k}\|^2+\|Ax_{n_k}-P_QAx_{n_k}\|^2\right)}{\alpha_{n_k}(1-\delta^2)}. \tag{30}$$

These conditions $\lim_{n\longrightarrow\infty}\alpha_n=0, 0<\varepsilon\leq\tau_{n_k}\leq(2/1+\|A\|^2)-\varepsilon$ and $t_n=\tau_n(1-\alpha_n)[2-\tau_n(1+\|A\|^2)]$ imply $(t_{n_k}/\alpha_{n_k}(1-\delta^2))\longrightarrow\infty\,(k\longrightarrow\infty)$. So, we obtain

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-P_Cx_{n_k}\|^2+\|Ax_{n_k}-P_QAx_{n_k}\|^2=0, \tag{31}$$

that is,

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-P_Cx_{n_k}\|=\lim_{k\longrightarrow\infty}\|Ax_{n_k}-P_QAx_{n_k}\|^2=0. \tag{32}$$

Next, we prove that any weak cluster point of the sequence $\{x_{n_k}\}$ is a solution of the SFP (1).

Since $\{x_{n_k}\}$ is bounded, let $\overline{x}$ be a weak cluster point of the sequence $\{x_{n_k}\}$; without loss of generality, we assume that $x_{n_k}\rightharpoonup\overline{x}$; then, we obtain $Ax_{n_k}\rightharpoonup A\overline{x}$. From the fact that P_C and P_Q are nonexpansive, Lemma 2 implies $I-P_C$ and $I-P_Q$ are demiclosed at zero; from (32), we obtain $\overline{x}=P_C\overline{x}$ and $A\overline{x}=P_QA\overline{x}$, i.e., $\overline{x}\in C, A\overline{x}\in Q$, hence $\overline{x}\in S$.

Finally, we show that $\|x_{n_{k+1}}-x_{n_k}\|\longrightarrow 0\,(k\longrightarrow\infty)$.

From (18) and the definition of W_n, we know

$$\begin{aligned}\|W_{n_k}-y_{n_k}\| &= \|(1-\eta_1)y_{n_k}+\eta_1P_CU_{n_k,2}y_{n_k}-y_{n_k}\|\\ &\leq \eta_1\|y_{n_k}-P_CU_{n_k,2}y_{n_k}\|\\ &= \frac{\|y_{n_k}-P_CU_{n_k,2}y_{n_k}\|}{n_k}\longrightarrow 0,\quad (k\longrightarrow\infty),\end{aligned} \tag{33}$$

and, from (32), we have

$$\begin{aligned}\|y_{n_k}-x_{n_k}\| &= \tau_{n_k}\|x_{n_k}-P_Cx_{n_k}+A^*(I-P_Q)Ax_{n_k}\|\\ &\leq \tau_{n_k}\|x_{n_k}-P_Cx_{n_k}\|+\|A\|\cdot\|Ax_{n_k}-P_QAx_{n_k}\|\\ &\longrightarrow 0,\quad (k\longrightarrow\infty).\end{aligned} \tag{34}$$

So,

$$\begin{aligned}\|x_{n_{k+1}}-x_{n_k}\| &= \|\alpha_{n_k}\psi(x_{n_k})+(1-\alpha_{n_k})W_{n_k}y_{n_k}-x_{n_k}\|\\ &\leq \alpha_{n_k}\|\psi(x_{n_k})-x_{n_k}\|+(1-\alpha_{n_k})\|W_{n_k}y_{n_k}-x_{n_k}\|\\ &+(1-\alpha_{n_k})\|y_{n_k}-x_{n_k}\|\longrightarrow 0,\quad (k\longrightarrow\infty).\end{aligned} \tag{35}$$

This implies that any weak cluster point of $\{x_{n_{k+1}}\}$ also belongs to S. Without loss of generality, we assume that $\{x_{n_{k+1}}\}$ converges weakly to $\hat{x}\in S$. Now, combing (29), Lemma 1, and the fact that $x^*=P_S\psi(x^*)$, we can obtain

$$\limsup_{n\longrightarrow\infty}\delta_n\leq\lim_{k\longrightarrow\infty}\frac{2\langle\psi(x^*)-x^*, x_{n_{k+1}}-x^*\rangle}{1-\delta^2}=\frac{2\langle\psi(x^*)-x^*,\hat{x}-x^*\rangle}{1-\delta^2}\leq 0. \tag{36}$$

From Lemma 5, we get $\lim_{n\longrightarrow\infty}\|x_n - x^*\| = 0$, which ends the proof.

From Theorem 1, we obtain the following subresult on the split feasibility problem (1). □

Algorithm 2. Given an initial point $x_0 \in H_1$, let $u \in H_1$ be fixed. Assume that x_n has been constructed and compute x_{n+1} by the following iterative scheme:

$$\begin{aligned} y_n &= x_n - \tau_n\left[x_n - P_C x_n + A^*\left(I - P_Q\right)Ax_n\right], \\ x_{n+1} &= \alpha_n u + \left(1 - \alpha_n\right)W_n y_n, \end{aligned} \tag{37}$$

where $\{\tau_n\}$ and $\{\alpha_n\}$ satisfy

(1) $0 < \varepsilon \le \tau_n \le (2/1 + \|A\|^2) - \varepsilon$;

(2) $\lim_{n\longrightarrow\infty}\alpha_n = 0, \sum_0^\infty \alpha_n = \infty$.

Corollary 1. *Let $\{x_n\}$ be the sequence generated by Algorithm 2, then $\{x_n\}$ converges strongly to a solution x^* of the SFP (1), where $x^* = P_S u$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

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