New Perspectives on Classical Meanness of Some Ladder Graphs

A. M. Alanazi, G. Muhiuddin, A. R. Kannan, and V. Govindan

1Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
2Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi 626 005, Tamil Nadu, India
3Department of Mathematics, Sri Vidya Mandir Arts & Science College, Katteri, Uthangarai 636902, Tamilnadu, India

Correspondence should be addressed to G. Muhiuddin; chishtyg@ymail.com

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In this study, we investigate a new kind of mean labeling of graph. The ladder graph plays an important role in the area of communication networks, coding theory, and transportation engineering. Also, we found interesting new results corresponding to classical mean labeling for some ladder-related graphs and corona of ladder graphs with suitable examples.

1. Introduction and Preliminaries

All through this paper, by a graph, we mean an undirected, simple, and finite graph. For documentations and phrasing, we follow [1–6]. For a point-by-point review on graph labeling, refer [7].

Let $P_n$ be a path on $n$ nodes denoted by $u_1, u_2, \ldots, u_n$, and with $n-1$ lines denoted by $e_1, e_2, \ldots, e_{n-1}$, where $1 \leq e \leq n - 1$, and where $e_y$ is the line joining the vertices $u_y$ and $u_{y+1}$. On each edge $e_y$, erect a ladder with $n - (y - 1)$ steps including the edge $e_y$, for $y = 1, 2, 3, \ldots, n - 1$. The resulting graph is called the one-sided step graph, and it is denoted by ST$_n$.

Let $P_{2n}$ be a path on $2n$ vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,n}$, and with $2n-1$ edges $e_{1,1}, e_{1,2}, \ldots, e_{2n-1}$, where $e_y$ is the line joining the vertices $u_{1,y}$ and $u_{1,y+1}$. On each edge $e_y$, erect a ladder with $'y + 1'$ steps including the edge $e_y$, for $y = 1, 2, 3, \ldots, n$, and on each edge $e_0$, erect a ladder with $2n+1-y$ steps including $e_0$, for $y = n+1, n+2, \ldots, 2n-1$. The graph thus obtained is called the double-sided step graph, and it is denoted by DST$_{2n}$. Let $G_1$ and $G_2$ be any two graphs with $p_1$ and $p_2$ vertices, respectively. Then, $G_1 \times G_2$ is the cartesian product of two graphs. A ladder graph $L_n$ is the graph $P_2 \times P_n$. The graph $G \ast S_m$ is obtained from $G$ by attaching $m$ pendant vertices to each vertex of $G$. The triangular ladder $T_{mn}$ for $n \geq 2$, is a graph obtained from two paths by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by adding the edges $u_yv_y, 1 \leq y \leq n$ and $u_yv_{y+1}, 1 \leq y \leq n - 1$. The slanting ladder $SL_n$ is a graph obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining each $v_y$ with $u_{y+1}, 1 \leq y \leq n - 1$. The graph $D_n^*$ having the vertices \{$a_{y,\delta}$: $1 \leq y \leq n, \delta = 1, 2, 3, 4$\}, and its edge set is \{$a_{y,1}a_{y+1,1}, a_{y,2}a_{y+1,2}, a_{y,3}a_{y+1,3}$: $1 \leq y \leq n - 1$\} U \{$a_{y,1}a_{y,2}, a_{y,2}a_{y,3}, a_{y,3}a_{y,4}, a_{y,4}a_{y,1}$: $1 \leq y \leq 3$\}.

2. Literature Survey

The origin of graph labeling called graceful labeling was characterized by Rosa in [8] and the mean labeling of graphs was introduced by Somasundram et al. in [9]. In [10], Arockiaraj et al. presented the idea of $F$-root square mean labeling of the graphs and examined its meanness [11]. Durai Baskar and Arockiaraj talked about the $C$-geometric meanness of some ladders in [12]. Dafik et al. researched the antimagicness of the graphs including the graph $D_n^*$. Durai Baskar considered the logarithmic meanness in [13] and Rajesh Kannan et al. characterized idea of exponential mean graphs in [15]. In addition, more concepts of ladder graphs and related concepts have been studied in [16–24]. Recently, Muhiuddin et al. have applied various related concepts on graphs in different aspects (see, e.g., [25–31]).

3. Methodology

A labeling $\chi$ on a graph $G(V, E)$ with $p$ vertices and $q$ edges is called a Smarandache $(m, k)$ mean labeling, for an integer $m \geq 1$ and $k \geq 2$, if $\chi$: $V(G) \longrightarrow \{1, 2, 3, \ldots, q + 1\}$ is injective.
and the induced function $\chi^*: E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ defined by

$$
\chi^*(uv) = \frac{1}{4} \left( \frac{\sqrt{\chi(u)^m + \chi(v)^m}}{2} + \sqrt{\chi(u)^m \chi(v)^m} + \frac{2\chi(u)^m \chi(v)^m}{\chi(u)^m + \chi(v)^m} + \frac{\chi(u)^k + \chi(v)^k}{2} \right),
$$

for all $uv \in E(G)$, is bijective.

Particularly, if $m = 1$ and $k = 2$, such a Smarandache $(m, k)$ mean labeling is the classical mean labeling on the graph. A function $\chi$ is known as a classical mean labeling of a graph $G(V, E)$ with $p$ nodes and $q$ edges if $\chi: V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ is injective and the incited edge assignment function $\chi^*: E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ characterized as

$$
\chi^*(uv) = \frac{1}{4} \left( \frac{\sqrt{\chi(u) + \chi(v)}}{2} + \sqrt{\chi(u) \chi(v)} + \frac{2\chi(u) \chi(v)}{\chi(u) + \chi(v)} + \frac{\chi(u)^2 + \chi(v)^2}{2} \right),
$$

for all $uv \in E(G)$, is bijective. A graph that concedes a classical mean labeling is said to be classical mean graph.

A classical mean labeling of $C_3$ is shown in Figure 1.

Here, we found interesting new results corresponding to classical mean labeling for some ladder-related graphs and corona of ladder graphs.

4. Main Results

Theorem 1. The one-sided step graph $ST_n$ is a classical mean graph, for $n \geq 2$.

Proof. Let $P_n$ be a path on $n$ nodes denoted by $u_{1,y}$, where $1 \leq y \leq n$ and with $n - 1$ lines denoted by $e_{1,y}$, where $1 \leq \delta \leq n - 1$, where $e_{1,y}$ is the line joining the vertices $u_{1,y}$ and $u_{2,y}$ on $P_n$. On each edge $e_{1,y}$, erect a ladder with $n - (y - 1)$ steps including the edge $e_{1,y}$ for $y = 1, 2, 3, \ldots, n - 1$. The resulting graph is called the one-sided step graph, and it is denoted by $ST_n$. Let $u_{1,a}, u_{2,a}, u_{3,a}, \ldots, u_{n,a}$, for $1 \leq a \leq n$, $1 \leq b \leq n - 1$, and $1 \leq c \leq n - 2$, be the nodes of $ST_n$.

Construct a mapping $\chi$ from $V(G)$ to $\{1, 2, 3, \ldots, 1 + n + n^2\}$:

$$
\chi(u_{1,y}) = (1 - y + n)^2 + \delta - 1, \quad \text{for} 2 \leq y \leq n, 1 \leq \delta \leq n + 2 - y,
$$

$$
\chi(u_{1,\delta}) = n^2 + \delta - 1, \quad \text{for} 2 \leq \delta \leq n.
$$

Hence, one-sided step graph $ST_n$ is a classical mean graph, for $n \geq 2$.

Figure 1: A classical mean labeling of $C_3$.

A classical mean labeling of $ST_7$ is shown in Figure 2.
**Theorem 2.** The double-sided step graph $2ST_{2n}$ is a classical mean graph, for $n \geq 1$.

**Proof.** Let $P_{2n}$ be a path on $2n$ vertices $u_{i,y}$, where $1 \leq y \leq 2n$ and with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$, where $e_i$ is the line joining the vertices $u_{i,y}$ and $u_{i+1,y}$. On each edge $e_y$, we erect a ladder with $\gamma + 1$ steps including the edge $e_y$, for $\gamma = 1, 2, 3, \ldots, n$, and on each $e_y$, we erect a layer with $2n + 1 - \gamma$ steps including $e_y$, for $\gamma = n + 1, n + 2, \ldots, 2n - 1$. The graph thus obtained is called the double-sided step graph, and it is denoted by $2ST_{2n}$.

Let $\{u_{1n}, u_{2n}, u_{3n}, u_{4n}, \ldots, u_{n+1n}, u_{2n+1}\}$, for $1 \leq a \leq 2n$, $1 \leq b \leq 2n - 2$, and $1 \leq c \leq 2n - 4$ be the nodes of $2ST_{2n}$. Construct a mapping $\chi$ from $V(G)$ to $\{1, 2, 3, \ldots, 3n + 2n^2\}$:

\[
\chi(u_{i,\delta}) = \begin{cases} 
1 + n + 2n^2 + 2(-1 + \delta), & 1 \leq \delta \leq n, \\
3n + 2n^2 - 2\delta + 2n + 2, & 1 + n \leq \delta \leq 2n,
\end{cases}
\]

for $2 \leq \delta \leq 2 + n - \gamma$ and $2 \leq \gamma \leq n$,

\[
\chi(u_{\gamma,\delta}) = 2(1 - \gamma + n)^2 + (n + 2 - \gamma) + (2\delta - 2),
\]

for $2 \leq \gamma \leq n$ and $n + 3 - \gamma \leq \delta \leq 2n + 3 - 2\gamma$,

\[
\chi(u_{\gamma,\delta}) = 2(1 + n - \gamma)^2 + 3(n + 1 - \gamma) - 2(\gamma + \delta - n - 3),
\]

\[
\chi(u_{\gamma,1}) = 2(\gamma + n)^2 - \gamma + n, \quad 3 \leq \gamma \leq n + 1,
\]

\[
\chi(u_{\gamma,2n+4-2\gamma}) = 2(n + 2 - \gamma)^2 + 1 + n - \gamma, \quad 2 \leq \gamma \leq n + 1.
\]

Therefore,

\[
\chi^*(u_{\gamma,\delta}) = \begin{cases} 
1 + n + 2n^2 + 2(-1 + \delta), & 1 \leq \delta \leq n, \\
3n + 2n^2 - 2\delta + 2n, & 1 + n \leq \delta \leq 2n,
\end{cases}
\]

for $2 \leq \delta \leq 2 + n - \gamma$ and $2 \leq \gamma \leq n - 1$,

\[
\chi^*(u_{\gamma,\delta}) = 2(-\gamma + 1 + n)^2 + (\gamma + n + 2) + 2(-2 + \delta),
\]

for $-\gamma + 3 + n \leq \delta \leq 2 + 2n - 2\gamma$ and $2 \leq \gamma \leq n - 1$.

Hence, the double-sided step graph $2ST_{2n}$ is a classical mean graph, for $n \geq 1$.

A classical mean labeling of $2ST_{10}$ is shown in Figure 3. \qed
Theorem 3. For \( n \geq 2 \) and \( m \leq 4 \), the graph \( P_m \times P_n \) is a classical mean graph.

Proof. Take \( V(P_m \times P_n) = \{v_{\delta} : 1 \leq \delta \leq n, 1 \leq y \leq m\} \) and \( E(P_m \times P_n) = \{v_{\delta}v_{(\delta+1)\delta} : 1 \leq \delta \leq n, 1 \leq y \leq m-1\} \cup \{v_{\delta}v_{\delta(\delta+1)} : 1 \leq \delta \leq n-1, 1 \leq y \leq m\} \).

Case (i). \( m = 2 \).

Construct a mapping \( \chi \) from \( V(P_2 \times P_n) \) to \( \{1, 2, 3, \ldots, 3n-1\} \):

\[
\chi(v_{\delta y}) = y + 3(\delta - 1), \quad \text{for } 1 \leq y \leq 2 \text{ and } 1 \leq \delta \leq n.
\]  

Therefore,

\[
\chi^*(v_{\delta 0}v_{\delta 2\delta}) = 3\delta - 2, \quad \text{for } 1 \leq \delta \leq n,
\]

\[
\chi^*(v_{\delta 0}v_{(\delta+1)\delta}) = y + 3\delta - 2, \quad \text{for } 1 \leq y \leq 2 \text{ and } 1 \leq \delta \leq n - 1.
\]  

(14)

Case (ii). \( m = 3 \).

Construct a mapping \( \chi \) from \( V(P_3 \times P_n) \) to \( \{1, 2, 3, \ldots, 5n-2\} \):

\[
\chi(v_{\delta y}) = \begin{cases} y, & 1 \leq y \leq 2, \\ 4, & y = 3. \end{cases}
\]

Therefore,

\[
\chi^*(v_{\delta 0}v_{\delta 2\delta}) = 3\delta - 2, \quad \text{for } 1 \leq \delta \leq n,
\]

\[
\chi^*(v_{\delta 0}v_{(\delta+1)\delta}) = y + 3\delta - 2, \quad \text{for } 1 \leq y \leq 2 \text{ and } 1 \leq \delta \leq n - 1.
\]  

(15)
\[
\chi^*(v_1v_1(v_{(1+1)}) = \gamma, \quad \text{for } 1 \leq \gamma \leq 2,
\]
\[
\chi^*(v_1v_2) = \gamma + 2, \quad \text{for } 1 \leq \gamma \leq 3,
\]
\[
\chi^*(v_2v_3v_3(v_{(1+2)}) = \gamma + 5\delta - 5, \quad \text{for } 1 \leq \gamma \leq 2 \text{ and } 2 \leq \delta \leq n,
\]
\[
\chi^*(v_2v_3v_3(v_{(1+3)}) = \gamma + 5\delta - 3, \quad \text{for } 1 \leq \gamma \leq 3 \text{ and } 2 \leq \delta \leq n - 1.
\]
(16)

Case (iii). \( m = 4. \)

Consider the graph \( P_4 \times P_n, \) for \( n \geq 3. \)

Construct a mapping \( \chi \) from \( \chi\{P_4 \times P_n\} \) to \( \{1, 2, 3, \ldots, 7n - 3\} : \)
\[
\chi(v_1) = \begin{cases} 
\gamma, & 1 \leq \gamma \leq 2, \\
\gamma + 1, & 3 \leq \gamma \leq 4,
\end{cases}
\]
\[
\chi(v_2) = \gamma + 7, \quad \text{for } 1 \leq \gamma \leq 4,
\]
\[
\chi(v_3) = \gamma + 7(\delta - 1), \quad \text{for } 1 \leq \gamma \leq 4 \text{ and } 3 \leq \delta \leq n.
\]

Therefore,
\[
\chi^*(v_{(1)}v_{(1+1)}) = \begin{cases} 
8\delta - 5, & 1 \leq \delta \leq 2, \\
7\delta - 3, & 3 \leq \delta \leq n - 1,
\end{cases}
\]
\[
\chi^*(v_{(1)}v_{(1+2)}) = \gamma + 7\delta - 4, \quad \text{for } 2 \leq \gamma \leq 4 \text{ and } 1 \leq \delta \leq n - 1,
\]
\[
\chi^*(v_{(1)}v_{(1+3)}) = \begin{cases} 
\gamma, & 1 \leq \gamma \leq 2, \\
4, & \gamma = 3,
\end{cases}
\]
\[
\chi^*(v_{(1)}v_{(1+4)}) = \gamma + 7, \quad \text{for } 1 \leq \gamma \leq 3,
\]
\[
\chi^*(v_{(1)}v_{(1+5)}) = \gamma + 7\delta - 7, \quad \text{for } 1 \leq \gamma \leq 3 \text{ and } 3 \leq \delta \leq n.
\]
(18)

Hence, for \( n \geq 2 \) and \( m \leq 4, \) the graph \( P_n \times P_n \) is a classical mean graph.

A classical mean labeling of \( P_2 \times P_2, P_3 \times P_2, \) and \( P_4 \times P_2 \) are shown in Figure 4.

**Corollary 1.** Every Ladder graph \( L_n = P_2 \times P_n \) is a classical mean graph.

A classical mean labeling of \( P_2 \times P_3 \) is shown in Figure 5.

**Theorem 4.** For \( m \leq 2 \) and \( n \geq 2, \) the graph \( L_n \ast S_m \) is a classical mean graph.

**Proof.** Case (i). \( m = 1. \)

Construct a mapping \( \chi \) from \( \chi\{L_n \ast S_1\} \) to \( \{1, 2, 3, \ldots, 5n - 1\} : \)
\[
\chi(u_r) = \begin{cases} 
3, & y = 1, \\
5y - 3, & 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(v_r) = \begin{cases} 
4, & y = 1, \\
5y - 2, & 2 \leq y \leq n
\end{cases}
\]
\[
\chi(x^{(1)}_r) = \begin{cases} 
5y - 4, & 1 \leq y \leq n,
\end{cases}
\]
(19)

Therefore,
\[
\chi^*(u_ru_{r+1}) = 5y - 1, \quad \text{for } 1 \leq y \leq n - 1,
\]
\[
\chi^*(v_rv_{r+1}) = 5y, \quad \text{for } 1 \leq y \leq n - 1,
\]
\[
\chi^*(u_rw_r) = \begin{cases} 
3, & y = 1, \\
5y - 3, & 2 \leq y \leq n,
\end{cases}
\]
(20)
\[
\chi^*(u_rw^{(1)}_r) = \begin{cases} 
5y - 4, & 1 \leq y \leq n,
\end{cases}
\]
\[
\chi^*(v_rw^{(1)}_r) = \begin{cases} 
2, & y = 1, \\
5y - 2, & 2 \leq y \leq n
\end{cases}
\]
Case (ii). \( m = 2. \)

Construct a mapping \( \chi \) from \( \chi\{L_n \ast S_1\} \) to \( \{1, 2, 3, \ldots, 7n - 1\} : \)
\[
\chi(u_r) = \begin{cases} 
3, & y = 1, \\
-2 + 7y, & y \text{ is even and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(v_r) = \begin{cases} 
7y - 5, & y \text{ is odd and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(w^{(1)}_r) = \begin{cases} 
1, & y = 1, \\
-3 + 7y, & y \text{ is even and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(w^{(2)}_r) = \begin{cases} 
-1 + 7y, & y \text{ is even and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(x^{(1)}_r) = \begin{cases} 
-4 + 7y, & y \text{ is odd and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(x^{(2)}_r) = \begin{cases} 
2y + 3, & 1 \leq y \leq 2,
\end{cases}
\]
\[
\chi(x^{(2)}_r) = \begin{cases} 
-6 + 7y, & y \text{ is even and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(x^{(2)}_r) = \begin{cases} 
-3 + 7y, & y \text{ is odd and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(x^{(2)}_r) = \begin{cases} 
8, & y = 1, \\
-5 + 7y, & y \text{ is even and } 2 \leq y \leq n,
\end{cases}
\]
\[
\chi(x^{(2)}_r) = \begin{cases} 
-2 + 7y, & y \text{ is odd and } 2 \leq y \leq n.
\end{cases}
\]
(21)

Therefore,
\[
\chi^*(u_{i+1}) = \begin{cases} 
5, & \text{if } i = 1, \\
7\gamma - 1, & 2 \leq \gamma \leq n - 1,
\end{cases}
\]
\[
\chi^*(v_{\gamma+1}) = 7\gamma, \quad \text{for } 1 \leq \gamma \leq n - 1,
\]
\[
\chi^*(u_{i}v_{\gamma}) = 7\gamma - 4, \quad \text{for } 1 \leq \gamma \leq n,
\]
\[
\chi^*(u_{i}w_{1}) = \begin{cases} 
1, & \gamma = 1, \\
7\gamma - 3, & 2 \leq \gamma \leq n \text{ and } \gamma \text{ is even}, \\
7\gamma - 6, & 1 \leq \gamma \leq n \text{ and } \gamma \text{ is odd},
\end{cases}
\]
\[
\chi^*(u_{i}w_{2}) = \begin{cases} 
7\gamma - 2, & 1 \leq \gamma \leq n \text{ and } \gamma \text{ is even}, \\
7\gamma - 5, & 1 \leq \gamma \leq n \text{ and } \gamma \text{ is odd},
\end{cases}
\]
\[
\chi^*(v_{\gamma}x_{1}) = \begin{cases} 
7\gamma - 6, & 1 \leq \gamma \leq n \text{ and } \gamma \text{ is even}, \\
7\gamma - 3, & 4 \leq \gamma \leq n \text{ and } \gamma \text{ is odd},
\end{cases}
\]
\[
\chi^*(v_{\gamma}x_{2}) = \begin{cases} 
7\gamma - 5, & 2 \leq \gamma \leq n \text{ and } \gamma \text{ is even}, \\
7\gamma - 2, & 1 \leq \gamma \leq n \text{ and } \gamma \text{ is odd}.
\end{cases}
\]

Hence, for \( m \leq 2 \) and \( n \geq 2 \), the graph \( L_n \circ S_m \) is a classical mean graph. 
A classical mean labeling of \( L_3 \circ S_1 \) is shown in Figure 6. 
A classical mean labeling of \( L_5 \circ S_2 \) is shown in Figure 7. \( \square \)

**Theorem 5.** The triangular ladder graph \( TL_n \) is a classical mean graph, for \( n \geq 2 \).

**Proof.** Construct a mapping \( \chi \) from \( V(TL_n) \) to \( \{1, 2, 3, \ldots, 4n - 2\} \):
\( \chi(u) = 4y - 3, \quad \text{for } 1 \leq y \leq n, \)

\( \chi(v) = \begin{cases} 
4y - 1, & 1 \leq y \leq n - 1, \\
4n - 2, & y = n.
\end{cases} \tag{23} \)

Therefore,

\( \chi^*(u, u_{y+1}) = 4y - 2, \quad \text{for } 1 \leq y \leq n - 1, \)

\( \chi^*(v, v_{y}) = 4y - 3, \quad \text{for } 1 \leq y \leq n, \)

\( \chi^*(v_{y}, v_{y+1}) = 4y, \quad \text{for } 1 \leq y \leq n - 1, \)

\( \chi^*(u_{y}, u_{y+1}) = -1 + 4y, \quad \text{for } 1 \leq y \leq n - 1. \) \( \tag{24} \)

Hence, the triangular ladder graph TL\(_n\) is a classical mean graph, for \( n \geq 2. \)

A classical mean labeling of TL\(_n\) is shown in Figure 8.

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**Theorem 6.** For \( m \leq 2 \) and \( n \geq 2, \) the graph TL\(_n\) \( \circ \) \( S_m \) is a classical mean graph.

**Proof.** Case (i). \( m = 1. \)

Construct a mapping \( \chi \) from \( V(L_n \circ S_1) \) to \( \{1, 2, 3, \ldots, 6n - 2\}: \)

\( \chi(u) = \begin{cases} 
5y - 3, & 1 \leq y \leq 2, \\
6y - 4, & 3 \leq y \leq n,
\end{cases} \)

\( \chi(v) = 6y - 2, \quad \text{for } 1 \leq y \leq n, \)

\( \chi(w^{(y)}) = \begin{cases} 
7y - 6, & 1 \leq y \leq 2, \\
6y - 5, & 3 \leq y \leq n,
\end{cases} \)

\( \chi(x^{(y)}) = \begin{cases} 
3, & y = 1, \\
6y - 3, & 2 \leq y \leq n.
\end{cases} \) \( \tag{25} \)

Therefore,
\[ \chi^*(u_t u_{t+1}) = -2 + 6 \gamma, \quad \text{for } 1 \leq \gamma \leq n - 1, \]
\[ \chi^*(v_t v_{t+1}) = 6 \gamma, \quad \text{for } 1 \leq \gamma \leq n - 1, \]
\[ \chi^*(u_t v_{t+1}) = 6 \gamma - 1, \quad \text{for } 1 \leq \gamma \leq n - 1, \]
\[ \chi^*(u_t v_{t+1}) = \begin{cases} 2, & \gamma = 1, \\ 6 \gamma - 4, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi^*(v_t v_{t+1}) = 6 \gamma - 5, \quad \text{for } 1 \leq \gamma \leq n, \]
\[ \chi^*(v_t x_{t+1}) = \begin{cases} 3, & \gamma = 1, \\ 6 \gamma - 3, & 2 \leq \gamma \leq n. \end{cases} \]

Case (ii). \( m = 2 \).

Construct a mapping \( \chi \) from \( V(L_n \ast S_2) \) to \( \{1, 2, 3, \ldots, 8n - 2\} \):
\[ \chi(u_t) = \begin{cases} 2, & \gamma = 1, \\ 8 \gamma - 3, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi(v_t) = \begin{cases} 6, & \gamma = 1, \\ 8 \gamma - 5, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi(w_1) = \begin{cases} 1, & \gamma = 1, \\ 8 \gamma - 4, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi(w_2) = \begin{cases} 3, & \gamma = 1, \\ 8 \gamma - 2, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi(x_1) = \begin{cases} 4, & \gamma = 1, \\ 8 \gamma - 7, & 2 \leq \gamma \leq n, \end{cases} \]
\[ \chi(x_2) = \begin{cases} 9, & \gamma = 1, \\ 8 \gamma - 6, & 2 \leq \gamma \leq n. \end{cases} \]

Hence, for \( m \leq 2 \) and \( n \geq 2 \), the graph \( TL_n \ast S_m \) is a classical mean graph.

A classical mean labeling of \( TL_2 \ast S_1 \) is shown in Figure 9.
A classical mean labeling of \( TL_3 \ast S_2 \) is shown in Figure 10.

**Theorem 7.** For \( n \geq 2 \), the slanting ladder graph \( SL_n \) is a classical mean graph.

**Proof.** Construct a mapping \( \chi \) from \( V(SL_n) \) to \( \{1, 2, 3, \ldots, 3n - 2\} \):
\[ \chi(u_t) = 1, \]
\[ \chi(v_t) = -4 + 3 \gamma, \quad \text{for } 2 \leq \gamma \leq n, \]
\[ \chi(v_{t+1}) = 3 \gamma, \quad \text{for } 1 \leq \gamma \leq n - 1 + n, \]
\[ \chi(v_n) = -2 + 3n. \]

Therefore,
\[ \chi^*(u_t u_{t+1}) = \begin{cases} 1, & \gamma = 1, \\ 3 \gamma - 3, & 2 \leq \gamma \leq n - 1, \end{cases} \]
\[ \chi^*(v_t v_{t+1}) = 3 \gamma + 1, \quad \text{for } 1 \leq \gamma \leq n - 2, \]
\[ \chi^*(v_n v_{t+1}) = 3n - 3, \]
\[ \chi^*(v_n v_{t+1}) = 3 \gamma - 1, \quad \text{for } 1 \leq \gamma \leq n - 1. \]

Hence, for \( n \geq 2 \), the slanting ladder graph \( SL_n \) is a classical mean graph.
A classical mean labeling of \( SL_8 \) is shown in Figure 11.

**Theorem 8.** For \( m \leq 2 \) and \( n \geq 2 \), the graph \( SL_n \ast S_m \) is a classical mean graph.

**Proof.** Let \( E(SL_n \ast S_m) = \{u_t u_{t+1}, \ v_t v_{t+1} : 1 \leq \gamma \leq \gamma_{t+1} - 1 \cup \{u_t v_{t+1} : 2 \leq \gamma \leq n\} \cup \{u_t w_1 : 1 \leq \gamma \leq n, 1 \leq \delta \leq m\} \cup \{v_{t+1} v_n : 1 \leq \gamma \leq n, 1 \leq \delta \leq m\} \}

Case (i). \( n \geq 3 \) and \( m = 1 \).

Construct a mapping \( \chi \) from \( V(SL_n \ast S_1) \) to \( \{1, 2, 3, \ldots, 3n - 2\} \):
\[ \chi(u_t) = \begin{cases} 1 + \gamma, & 1 \leq \gamma \leq 2, \\ -6 + 5 \gamma, & 3 \leq \gamma \leq n, \end{cases} \]
\[ \chi(v_t) = \begin{cases} 6, & \gamma = 1, \\ 5 \gamma, & 2 \leq \gamma \leq n - 1, \\ -2 + 5n, & \gamma = n, \end{cases} \]
\[ \chi(w_1) = \begin{cases} -2 + 3 \gamma, & 1 \leq \gamma \leq 2, \\ -7 + 5 \gamma, & 3 \leq \gamma \leq n, \\ 7, & \gamma = 1, \\ -3 + 5n, & \gamma = n. \end{cases} \]

Therefore,
Figure 8: A classical mean labeling of $TL_n$.

Figure 9: A classical mean labeling of $TL_5 \circ S_1$.

Figure 10: A classical mean labeling of $TL_4 \circ S_2$.

Figure 11: A classical mean labeling of $SL_n$. 

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Construct a mapping $\chi$ from $V(\text{SL}_n \times S_2)$ to \{1, 2, 3, …, 7n−2\}

\[
\begin{align*}
\chi(u) &= \begin{cases}
2 + 7y, & 1 \leq y \leq 2, \\
-9 + 7y, & y \text{ odd and } 3 \leq y \leq n - 1, \\
-6 + 7y, & y \text{ even and } 3 \leq y \leq n - 1, \\
-10 + 7n, & n \text{ even and } y = n, \\
-9 + 7n, & n \text{ odd and } y = n,
\end{cases} \\
\chi(v) &= \begin{cases}
8, & y = 1, \\
-1 + 7y, & y \text{ odd and } 2 \leq y \leq n - 3, \\
2 + 7y, & y \text{ even and } 2 \leq y \leq n - 3, \\
-13 + 7n, & n \text{ even and } y = n - 2, \\
-15 + 7n, & n \text{ odd and } y = n - 2, \\
7n - 5, & y = n - 1, \\
7n - 3, & y = n,
\end{cases} \\
\chi(w_1) &= \begin{cases}
1, & y = 1, \\
-5 + 5y, & 2 \leq y \leq 3, \\
-7 + 7y, & y \text{ even and } 4 \leq y \leq n - 1, \\
-10 + 7y, & y \text{ odd and } 4 \leq y \leq n - 1, \\
-11 + 7n, & n \text{ even and } y = n, \\
-10 + 7n, & n \text{ odd and } y = n, \\
-5 + 7y, & 1 \leq y \leq 2, \\
-5 + 7y, & y \text{ even and } 3 \leq y \leq n - 1, \\
-8 + 7y, & y \text{ odd and } 3 \leq y \leq n - 1, \\
-7 + 7n, & n \text{ even and } y = n, \\
-8 + 7n, & n \text{ odd and } y = n,
\end{cases} \\
\chi(w_2) &= \begin{cases}
9, & y = 1, \\
7y, & 2 \leq y \leq n - 3 \text{ and } y \text{ is even}, \\
-3 + 7y, & 2 \leq y \leq n - 3 \text{ and } y \text{ is odd}, \\
-12 + 7n, & n \text{ is even and } y = n - 2, \\
-17 + 7n, & n \text{ is odd and } y = n - 2, \\
-8 + 7n, & n \text{ is even and } y = n - 1, \\
-7 + 7n, & n \text{ is odd and } y = n - 1, \\
-4 + 7n, & y = n,
\end{cases}
\end{align*}
\]

Case (ii). $n \geq 3$ and $m = 2$.

\[
\chi(x_1) = \begin{cases}
-1 + 4y, & 1 \leq y \leq 2, \\
-5 + 7y, & 3 \leq y \leq n - 2, \\
-14 + 7n, & n \text{ is even and } y = n - 1, \\
-12 + 7n, & n \text{ is odd and } y = n - 1, \\
11, & y = 1, \\
3 + 7y, & 2 \leq y \leq n - 3, \\
-2 + 7y, & y \text{ is odd and } 2 \leq y \leq n - 3, \\
-9 + 7n, & n \text{ is even and } y = n - 2, \\
-16 + 7n, & n \text{ is odd and } y = n - 2, \\
-6 + 7n, & y = n - 1, \\
-2 + 7n, & y = n.
\end{cases}
\]

Therefore,

\[
\chi(x_2) = \begin{cases}
5, & y = 1, \\
-1 + 7y, & 2 \leq y \leq n - 1, \\
1, & y = 1, \\
-8 + 6y, & 2 \leq y \leq 3, \\
-7 + 7y, & y \text{ even and } 4 \leq y \leq n - 1, \\
-10 + 7y, & y \text{ odd and } 4 \leq y \leq n - 1, \\
-11 + 7n, & n \text{ even and } y = n, \\
-10 + 7n, & n \text{ odd and } y = n, \\
-2 + 4y, & 1 \leq y \leq 2, \\
-6 + 7y, & y \text{ even and } 3 \leq y \leq n - 1, \\
-9 + 7y, & y \text{ odd and } 3 \leq y \leq n - 1, \\
-9 + 7n, & y = n,
\end{cases}
\]

\[
\chi^*(u_1u_{r+1}) = \begin{cases}
2 + 5y, & 1 \leq y \leq n - 2, \\
-4 + 5y, & 3 \leq y \leq n - 1, \\
2 + 5y, & 1 \leq y \leq n - 2, \\
-4 + 5n, & y = n - 1, \\
-1 + 5y, & \text{for } 1 \leq y \leq n - 1, \\
1, & y = 1, \\
-7 + 5y, & 2 \leq y \leq n, \\
6, & y = 1, \\
5y, & 2 \leq y \leq n - 1, \\
-3 + 5n, & y = n.
\end{cases}
\]
For Theorem 9.

Figure 13. A classical mean labeling of $SL_6$ is shown in Figure 15.

**Theorem 10.** For $n \geq 1$, the diamond ladder graph $Dl_n$ is a classical mean graph.

**Proof.** $V(Dl_n) = \{x_y, y_z : 1 \leq y \leq 2n\} \cup \{z_y : 1 \leq y \leq 2n\}$ and $E(Dl_n) = \{x_y y_z, y_z y_{z+1} : 1 \leq y \leq n - 1\} \cup \{x_y z_{y+1}, x_{y+1} z_{y+1} : 1 \leq y \leq n\}$.

Construct a mapping $\chi$ from $V(Dl_n)$ to $\{1, 2, 3, \ldots, 2n\}$:

\[
\chi(x_y) = -5 + 8y, \quad \text{for } 1 \leq y \leq n,
\]

\[
\chi(y_z) = -3 + 8y, \quad \text{for } 1 \leq y \leq n,
\]

Hence, for $n \geq 1$, the diamond ladder graph $Dl_n$ is a classical mean graph.

**Theorem 11.** For $n \geq 4$, the latitude ladder graph $LL_n$ is a classical mean graph.

**Proof.** Here, $V(LL_n) = \{u_y, u_{y+1}, u_{y+n-1} : 1 \leq y \leq n - 1\} \cup \{u_y u_{y+1} : 1 \leq y \leq n - 1\} \cup \{u_y u_{y+n-1} : 2 \leq y \leq (n/2)\}$.

Construct a mapping $\chi$ from $V(LL_n)$ to $\{1, 2, 3, \ldots, (3n/2)\}$:

\[
\chi(u_y) = -2 + 8y, \quad \text{for } 1 \leq y \leq n - 1,
\]

\[
\chi(u_{y+1}) = -5 + 8y, \quad \text{for } 1 \leq y \leq n - 1,
\]

\[
\chi(u_{y+n-1}) = -3 + 8y, \quad \text{for } 1 \leq y \leq n - 1.
\]

Hence, for $n \geq 1$, the latitude ladder graph $LL_n$ is a classical mean graph.
Figure 12: A classical mean labeling of $SL_2 \circ S_1$ and $SL_2 \circ S_2$.

Figure 13: A classical mean labeling of $SL_6 \circ S_1$.

Figure 14: A classical mean labeling of $SL_9 \circ S_2$.

Figure 15: A classical mean labeling of $D^*_4$. 
5. Conclusion

The Cartesian product is one among graph operations. Based on this operation, the classical mean labeling of various graphs such as the one-sided step graph $S_{2n}$, double-sided step graph $2S_{2n}$, graph $P_m \times P_n$, ladder graph $L_n$, graph $L_n \circ S_n$, triangular ladder graph $TL_n$, graph $TL_n \circ S_n$, slanting ladder graph $SL_n$, graph $SL_n \circ S_n$, graph $D^*_n$, diamond ladder graph $DL_n$, and longitude ladder graph $LL_n$ are established. It would be very interesting to analyze that the classical meanness of various ladder-related graphs. Investigating classical mean labeling of other classes of graphs is still open and this is for future work. One can also explore the exclusive applications of classical mean labeling in real-life problems.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


