Research Article

Divisor Problems Related to Hecke Eigenvalues in Three Dimensions

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In this paper, we consider divisor problems related to Hecke eigenvalues in three dimensions. We establish upper bounds and asymptotic formulas for these problems on average.

1. Introduction

Let $\Gamma = SL_2(\mathbb{Z})$ be the full modular group. Let H_k^* denote the set of primitive holomorphic forms with even integral weight $k \geq 2$ for Γ . Then H_k^* consists of common eigenfunctions f of all Hecke operators T_n . The Hecke eigenfunction f has the following Fourier series expansion:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}, \quad (\Im z > 0), \quad (1)$$

where $\lambda_f(n)$ denotes the n -th normalized eigenvalue. It is known that $\lambda_f(n)$, as a function of n , is real-valued and multiplicative. Moreover, for all integers $n \geq 1$, Deligne [1] showed that

$$|\lambda_f(n)| \leq d(n), \quad (2)$$

where $d(n)$ is the divisor function. Also, for every prime p ,

$$\begin{aligned} \alpha_f(p) &= \alpha_f(p) + \beta_f(p), \\ \alpha_f(p)\beta_f(p) &= |\alpha_f(p)| = |\beta_f(p)| = 1. \end{aligned} \quad (3)$$

Now we introduce some specific automorphic L -functions. Define the Hecke L -function attached to f as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left(1 - \frac{\alpha_f(p)^{-1}}{p^s} \right)^{-1}, \quad (4)$$

for $\Re s > 1$. Moreover, the Rankin-Selberg L -function attached to f can be defined as

$$\begin{aligned} L(s, f \times f) &= \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s} \right)^{-1} \left(1 - \frac{1}{p^s} \right)^{-2} \\ &\quad \cdot \left(1 - \frac{\alpha_f(p)^{-2}}{p^s} \right)^{-1}, \end{aligned} \quad (5)$$

$\Re s > 1$.

Then, $L(s, f \times f)$ can be rewritten as

$$L(s, f \times f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n^s} = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}(n)}{n^s}. \quad (6)$$

As usual, $\zeta(s)$ denotes the Riemann zeta-function. The symmetric square L -function attached to f can be defined as, for $\Re s > 1$,

$$L(s, \text{sym}^2 f) = \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)^{-2}}{p^s}\right)^{-1}, \quad (7)$$

which can also be expressed in the Dirichlet series

$$\begin{aligned} L(s, \text{sym}^2 f) &= \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\text{sym}^2 f}(p)}{p^s} + \frac{\lambda_{\text{sym}^2 f}(p^2)}{p^{2s}} + \frac{\lambda_{\text{sym}^2 f}(p^3)}{p^{3s}} + \dots\right), \quad \Re s > 1. \end{aligned} \quad (8)$$

The symmetric square L -function $L(s, \text{sym}^2 f)$ has a functional equation and could be analytic continued to an entire function over the whole complex plane. We refer to works of Hecke in [2], Gelbert and Jacquet [3], Kim [4], and Kim and Shahidi [5,6] for these properties. Therefore, the symmetric square L -function $L(s, \text{sym}^2 f)$ could be seen as a general L -function in the sense of Perelli [7].

In number theory, considering the properties and average behaviors of the Fourier coefficients is meaningful and interesting. Some classical problems concern mean values of these Fourier coefficients and related problems with the corresponding automorphic L -functions (see [1, 8–29], etc.). Here, we just give a brief history for general divisor problems related to these Fourier coefficients.

Let ω be an integer greater than one, and

$$\begin{aligned} \lambda_{\omega, f}(n) &= \sum_{n=n_1 n_2, \dots, n_\omega} \lambda_f(n_1) \lambda_f(n_2), \dots, \lambda_f(n_\omega), \\ \lambda_{\omega, f \times f}(n) &= \sum_{n=n_1 n_2, \dots, n_\omega} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2), \dots, \lambda_{f \times f}(n_\omega). \end{aligned} \quad (9)$$

In particular, we have $\lambda_{1, f}(n) = \lambda_f(n)$ and $\lambda_{1, f \times f}(n) = \lambda_{f \times f}(n)$. In 1927, Hecke [30] showed that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{(1/2)}. \quad (10)$$

Subsequently, this upper bound was improved by many scholars (for example, see [12, 21, 24]). In this direction, the best result so far was obtained by Wu [24] who showed that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{(1/3)} (\log x)^{\rho_{(1/2)}^+}, \quad (11)$$

where

$$\begin{aligned} \rho_{(1/2)}^+ &= \frac{102 + 7\sqrt{21}}{210} \left(\frac{6 - \sqrt{21}}{5}\right)^{(1/2)} \\ &\quad + \frac{102 - 7\sqrt{21}}{210} \left(\frac{6 + \sqrt{21}}{5}\right)^{(1/2)} - \frac{33}{55}, \\ &= -0.118 \dots \end{aligned} \quad (12)$$

Rankin [20] and Selberg [22] established

$$\sum_{n \leq x} \lambda_{f \times f}(n) = C_f x + O_f(x^{(3/5)}), \quad (13)$$

where C_f is a positive constant depending on f . Later, Kanemitsu, Sankaranarayanan, and Tanigawa [31] studied general divisor problems and proved

$$\begin{aligned} \sum_{n \leq x} \lambda_{\omega, f}(n) &\ll x^{1 - (3/2k+2)+\varepsilon}, \\ \sum_{n \leq x} \lambda_{\omega, f \times f}(n) &= M_\omega(x) + O(x^{1 - (1/2k)+\varepsilon}), \end{aligned} \quad (14)$$

where ω is an integer and $\omega \geq 2$; $M_\omega(x)$ comes from a residue and takes the form $xP_{\omega-1}(\log x)$; $P_{\omega-1}(t)$ denotes a polynomial of t with degree $\omega - 1$. Fomenko [32] also proved the same result for the sum $\sum_{n \leq x} \lambda_{\omega, f}(n)$. After that, Kanemitsu, Sankaranarayanan, and Tanigawa's result was improved by Lü [33], and some general results were obtained (see [34–37], etc.).

In this paper, we consider divisor problems related to Hecke eigenvalues in three dimensions motivated by the above results and Ivić's work on three-dimensional divisor problems (see, e.g., [38]). We first introduce some notation. For any fixed integer $1 < a < b < c$, we define

$$\begin{aligned} \lambda_f^{a,b,c}(n) &:= \sum_{n=n_1^a n_2^b n_3^c} \lambda_f(n_1) \lambda_f(n_2) \lambda_f(n_3), \\ \lambda_{f \times f}^{a,b,c}(n) &:= \sum_{n=n_1^a n_2^b n_3^c} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2) \lambda_{f \times f}(n_3). \end{aligned} \quad (15)$$

We are interested in studying the average behaviors of sums

$$\begin{aligned} S_f(a, b, c; x) &:= \sum_{n \leq x} \lambda_f^{a,b,c}(n), \\ S_{f \times f}(a, b, c; x) &:= \sum_{n \leq x} \lambda_{f \times f}^{a,b,c}(n), \end{aligned} \quad (16)$$

which can be seen as divisor problems related to Hecke eigenvalues in three dimensions. We establish the following results.

Theorem 1. Let a, b , and c be fixed integers satisfying $1 < a < b < c$. Then, for any $\epsilon > 0$, one has

$$S_f(a, b, c; x) \ll \begin{cases} x^{(1/a) - (3/2)(7a-b-c) + \epsilon}, & \text{if } c \leq 2a, \\ x^{(1/a) - (3/2)(5a-b) + \epsilon}, & \text{if } b < 2a < c, \\ x^{(1/2a) + \epsilon}, & \text{if } 2a \leq b. \end{cases} \quad (17)$$

Theorem 2. Let a, b , and c be fixed integers satisfying $1 < a < b < c$. Then, for any $\epsilon > 0$, one has

$$S_{f \times f}(a, b, c; x) = \begin{cases} M_1 x^{(1/a)} + M_2 x^{(1/b)} + M_3 x^{(1/c)} + O(x^{(1/a) - (84/748a - 131(b+c)) + \epsilon}), & \text{if } c \leq 2a, \\ M_1 x^{(1/a)} + M_2 x^{(1/b)} + O(x^{(1/a) - (84/486a - 131b) + \epsilon}), & \text{if } b < 2a < c, \\ M_1 x^{(1/a)} + O(x^{(5/8a) + \epsilon}), & \text{if } 2a \leq b, \end{cases} \quad (18)$$

where

$$\begin{aligned} M_1 &= L\left(\frac{b}{a}, f \times f\right) L\left(\frac{c}{a}, f \times f\right) L(1, \text{sym}^2 f), \\ M_2 &= L\left(\frac{a}{b}, f \times f\right) L\left(\frac{c}{b}, f \times f\right) L(1, \text{sym}^2 f), \\ M_3 &= L\left(\frac{a}{c}, f \times f\right) L\left(\frac{b}{c}, f \times f\right) L(1, \text{sym}^2 f). \end{aligned} \quad (19)$$

2. Some Lemmas

To prove our theorems, we need to quote some lemmas, which include the individual and averaged subconvexity bounds for the Riemann zeta-function and the symmetric square L -function. And from the following Lemma 1, we know that the Rankin–Selberg L -function $L(s, f \times f)$ could be decomposed into the product of the Riemann zeta-function and the corresponding symmetric square L -function.

Lemma 1. For $\Re s > 1$, one has

$$L(s, f \times f) = \zeta(s) L(s, \text{sym}^2 f). \quad (20)$$

Proof. By comparing the Euler products of two sides of (20) and applying (3), we can get this lemma. This lemma can also be found in [33]. $\square$

Lemma 2. For any $\epsilon > 0$, one has

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \sim CT \log T, \quad (21)$$

uniformly for $T \geq 1$, and the subconvexity bound in the critical strip

$$L(\sigma + it, f) \ll \begin{cases} (1 + |t|)^{(2(1-\sigma)/3) + \epsilon}, & \text{if } \frac{1}{2} < \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (22)$$

where $|t| \geq 1$.

Proof. These results are proved by Good [11]. $\square$

Lemma 3. For any $\epsilon > 0$, one has

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt \ll T^{2+\epsilon}, \quad (23)$$

uniformly for $T \geq 1$, and the subconvexity bound in the critical strip

$$\zeta(\sigma + it) \ll \begin{cases} (1 + |t|)^{(13/42)(1-\sigma) + \epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (24)$$

where $|t| \geq 1$.

Proof. The twelfth mean value estimate (23) is due to Heath-Brown [39]. The subconvexity bound (24) is due to Bourgain [40]. $\square$

Lemma 4. For any $\epsilon > 0$, one has

$$\int_1^T \left| L(\sigma + it, \text{sym}^2 f) \right|^2 dt \ll T^{3(1-\sigma) + \epsilon}, \quad (25)$$

uniformly for $T \geq 1$, and the subconvexity bound in the critical strip

$$L(\sigma + it, \text{sym}^2 f) \ll \begin{cases} (1 + |t|)^{(5/4)(1-\sigma) + \epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (26)$$

where $|t| \geq 1$.

Proof. The first result follows from Perelli's mean value theorem with $L(s, \text{sym}^2 f)$ (see [7]), and the second one is due to Nunes [19]. $\square$

3. Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. Note that

$$L(as, f)L(bs, f)L(cs, f) = \sum_{n=1}^{\infty} \frac{\lambda_f^{a,b,c}(n)}{n^s}. \quad (27)$$

Then, by (27) and Perron's formula (see Proposition 5.54 in [41]), we can obtain

$$S_f(a, b, c; x) = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} L(as, f)L(bs, f)L(cs, f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right), \quad (28)$$

where $s = \sigma + it$, $\eta = (1/a) + \varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

We shift the line of the integral of (28) to the line $\Re s = (1/2a)$. Then, Cauchy's residue theorem shows that

$$\begin{aligned} S_f(a, b, c; x) &= \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta+iT}^{(1/2a)-iT} \right\} \\ &\quad L(as, f)L(bs, f)L(cs, f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\ &:= G_1 + G_2 + G_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right). \end{aligned} \quad (29)$$

The following work is to estimate G_1, G_2 , and G_3 . The estimates for the integrals over the horizontal segments are similar, so we handle G_2 and G_3 first. To get this goal, we consider three cases $c \leq 2a$, $b < 2a < c$, and $2a \leq b$.

We first consider the case $c \leq 2a$. To estimate G_2 and G_3 , we divide the integral interval into the following eight intervals $I_1, \dots, I_4$, some of which may be empty, and use Lemma 2.

Interval I_1 :

$$\begin{aligned} I_1 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{2a} \leq \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (30)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_1} x^\sigma |L(as + iat, f)L(bs + ibt, f)L(cs + ict, f)| d\sigma \\ \ll \max_{(1/2a) \leq \sigma \leq (1/c)} x^\sigma T^{(2/3)(1-a\sigma)+(2/3)(1-b\sigma)+(2/3)(1-c\sigma)+\varepsilon} T^{-1} \\ \ll \max_{(1/2a) \leq \sigma \leq (1/c)} T^{1+\varepsilon} \left(\frac{x}{T^{(2/3)(a+b+c)}} \right)^\sigma \\ \ll x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon} + x^{(1/2a)} T^{(2/3)-(b+c/3a)+\varepsilon}. \end{aligned} \quad (31)$$

Interval I_2 :

$$\begin{aligned} I_2 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{c} < \sigma \leq \frac{1}{b} \right\}. \end{aligned} \quad (32)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_2} x^\sigma |L(as + iat, f)L(bs + ibt, f)L(cs + ict, f)| d\sigma \\ \ll \max_{(1/c) < \sigma \leq (1/b)} x^\sigma T^{(2/3)(1-a\sigma)+(2/3)(1-b\sigma)+\varepsilon} T^{-1} \\ \ll \max_{(1/c) < \sigma \leq (1/b)} T^{(1/3)+\varepsilon} \left(\frac{x}{T^{(2/3)(a+b)}} \right)^\sigma \\ \ll x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} + x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon}. \end{aligned} \quad (33)$$

Interval I_3 :

$$\begin{aligned} I_3 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (34)$$

This interval is an empty set noting that $(1/b) > (1/c)$.

Interval I_4 :

$$\begin{aligned} I_4 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\}. \end{aligned} \quad (35)$$

In this interval, we have

$$T^{-1} \times \int_{I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma$$

Thus, by (31)–(36), we have

$$\begin{aligned} &\ll \max_{(1/b) < \sigma \leq (1/a)} x^\sigma T^{(2/3)(1-a\sigma)+\varepsilon} T^{-1} \\ &\ll x^{(1/a)} T^{-1+\varepsilon} + x^{(1/b)} T^{-(1/3)-\frac{2a}{3b}+\varepsilon}. \end{aligned} \quad (36)$$

$$\begin{aligned} |G_2 + G_3| &\ll T^{-1} \int_{(1/2a)}^\eta x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma \\ &= T^{-1} \int_{I_1 \cup \dots \cup I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma \\ &\ll x^{(1/2a)} T^{(2/3)-(b+c/3a)+\varepsilon} + x^{(1/b)} T^{-(1/3)-(2a/3b)+\varepsilon} + x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (37)$$

Next, we handle G_1 . We have

$$\begin{aligned} |G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\ &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a}. \end{aligned} \quad (38)$$

Then, by Lemma 2 and applying Cauchy's inequality, we can obtain

$$\begin{aligned} |G_1| &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} T_1^{(2/3)(1-(b/2a))} T_1^{(2/3)(1-(c/2a))} \\ &\quad \cdot \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \times \left(\int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\ &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{(4/3)-(b+c/3a)+\varepsilon} \\ &\ll x^{1/2a} T^{(4/3)-(b+c/3a)+\varepsilon}. \end{aligned} \quad (39)$$

According to (29), (37), and (39), we have

$$\begin{aligned} S_f(a, b, c; x) &\ll x^{1/2a} T^{(4/3)-(b+c/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)} \\ &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (40)$$

By taking $T = x^{(3/2)(7a-b-c)}$ in (40), we can obtain

$$S_f(a, b, c; x) \ll x^{(1/a)-(3/2)(7a-b-c)+\varepsilon}, \quad (41)$$

which proves the first result of Theorem 1.

For the case $b < 2a < c$, to estimate G_2 and G_3 , we still divide the integral interval into four corresponding short intervals $I_1'', \dots, I_4''$, which are different from ones for the case $c \leq 2a$. In fact, the corresponding short intervals I_1'', I_3'' become empty sets in the current situation. However, we still can estimate $G_2 + G_3$ similar to the corresponding parts of the case $c \leq 2a$ and get

$$\begin{aligned} |G_2 + G_3| &\ll x^{1/2a} T^{-(b/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} \\ &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (42)$$

The estimate of G_1 becomes the following at the current case by noting $(c/2a) > 1$.

$$\begin{aligned}
|G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1+(2/3)(1-(b/2a))} \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \left(\int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1+(2/3)(1-(b/2a))+(1/2)+(1/2)+\varepsilon} + x^{1/2a} \\
&\ll x^{1/2a} T^{(2/3)-(b/3a)+\varepsilon}.
\end{aligned} \tag{43}$$

Recalling (29), we have

$$\begin{aligned}
S_f(a, b, c; x) &\ll x^{1/2a} T^{(2/3)-(b/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} \\
&\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
\end{aligned} \tag{44}$$

By taking $T = x^{(3/2)(5a-b)}$ in (44), we can get

$$S_f(a, b, c; x) \ll x^{(1/a)-(3/2)(5a-b)+\varepsilon}, \tag{45}$$

which proves the second result of Theorem 1.

For the case $2a \leq b$, to estimate G_2 and G_3 , we also divide the integral interval into four corresponding short intervals $I'_1, \dots, I'_4$, which are different from ones for the case $c \leq 2a$. In fact, the corresponding short intervals I'_1, I'_2, I'_3 become empty sets in the current situation. However, we still can estimate $G_2 + G_3$ similarly to the corresponding parts of the case $c \leq 2a$ and get

$$|G_2 + G_3| \ll x^{1/2a} T^{-(2/3)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{46}$$

The estimate of G_1 becomes the following by noting $(c/2a) > (b/2a) > 1$.

$$\begin{aligned}
|G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \left(\int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+(1/2)+(1/2)+\varepsilon} + x^{1/2a} \\
&\ll x^{1/2a} T^\varepsilon.
\end{aligned} \tag{47}$$

Recalling (29), we have

$$S_f(a, b, c; x) \ll x^{1/2a} T^\varepsilon + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{48}$$

By taking $T = x^{1/2a}$ in (48), we can get

$$S_f(a, b, c; x) \ll x^{(1/2a)+\varepsilon}, \tag{49}$$

which proves the third result of Theorem 1.

4. Proof of Theorem 2

In this section, we shall prove Theorem 2, the process of which is more complicated than Theorem 1. Note that

$$L(as, f \times f) L(bs, f \times f) L(cs, f \times f) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}^{a,b,c}(n)}{n^s}. \tag{50}$$

Then, by (50) and Perron's formula (see Proposition 5.54 in [41]), we have

\openup3

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} L(as, f \times f) L(bs, f \times f) L \\
&\quad \cdot (cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right),
\end{aligned} \tag{51}$$

where $s = \sigma + it$, $\eta = (1/a) + \varepsilon$ and $1 \leq T \leq x$ is a parameter which will be decided later.

In view of (20), the points $s = (1/a)$, $s = (1/b)$ and $s = (1/c)$ are the only three possible simple poles of the integrand in the rectangle $I_T = \{s = \sigma + it: (1/2a) \leq \sigma \leq \eta, |t| \leq T\}$. The corresponding possible residues at $s = (1/a)$, $s = (1/b)$, and $s = (1/c)$ are equal to

$$\begin{aligned} M_1 &:= L\left(\frac{b}{a}, f \times f\right) L\left(\frac{c}{a}, f \times f\right) L(1, \text{sym}^2 f), \\ M_2 &:= L\left(\frac{a}{b}, f \times f\right) L\left(\frac{c}{b}, f \times f\right) L(1, \text{sym}^2 f), \\ M_3 &:= L\left(\frac{a}{c}, f \times f\right) L\left(\frac{b}{c}, f \times f\right) L(1, \text{sym}^2 f), \end{aligned} \quad (52)$$

respectively.

We move the integral line of the integral in (28) to the parallel segment with $\Re s = (1/2a)$. We first consider the case $c \leq 2a$. In this situation, the points $s = (1/a)$, $s = (1/b)$, and $s = (1/c)$ are all poles of the integrand in the rectangle I_T . Therefore, by Cauchy's residue theorem, we can obtain

$$\begin{aligned} S_{f \times f}(a, b, c; x) &= \left\{ \text{Res}_{s=(1/a)} + \text{Res}_{s=(1/b)} + \text{Res}_{s=(1/c)} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} \\ &\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta+iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\ &:= M_1 x^{(1/a)} + M_2 x^{(1/b)} + M_3 x^{(1/c)} + J_1 + J_2 + J_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right). \end{aligned} \quad (53)$$

Now, the remaining work is to handle these three terms J_1 , J_2 , and J_3 . In addition, the estimates of these integrals on the horizontal parts are analogous, and so we deal with J_2 and J_3 firstly. To estimate J_2 and J_3 , similarly to the method estimating G_2 and G_3 , we still divide the integral interval into the following four short intervals $I_1^*, \dots, I_4^*$ and apply Lemmas 3 and 4.

Interval I_1^* :

$$\begin{aligned} I_1^* &:= \left\{ s = \sigma + iT: \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT: \frac{1}{2a} \leq \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (54)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_1^*} x^\sigma &\left| \zeta(a\sigma + iat) L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\ &\ll \max_{(1/2a) \leq \sigma \leq (1/c)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{((13/42)+(5/4))(1-b\sigma)} T^{((13/42)+(5/4))(1-c\sigma)} T^{-1+\varepsilon} \\ &\ll \max_{(1/2a) \leq \sigma \leq (1/c)} T^{(309/84)+\varepsilon} \left(\frac{x}{T^{(131/84)(a+b+c)}} \right)^\sigma \\ &\ll x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon} + x^{1/2a} T^{(487/168)-(131(b+c)/168a)+\varepsilon}. \end{aligned} \quad (55)$$

Interval I_2^* :

$$\begin{aligned} I_2^* &:= \left\{ s = \sigma + iT: \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT: \frac{1}{c} < \sigma \leq \frac{1}{b} \right\}. \end{aligned} \quad (56)$$

In this interval, we have

$$\begin{aligned}
 & T^{-1} \times \int_{I_2^*} x^\sigma \left| \zeta(a\sigma + iat) \left(L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll \max_{(1/c) < \sigma \leq (1/b)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{((13/42)+(5/4))(1-b\sigma)} T^{-1+\varepsilon} \\
 & \ll \max_{(1/c) < \sigma \leq (1/b)} T^{(89/42)+\varepsilon} \left(\frac{x}{T^{(131/84)(a+b)}} \right)^\sigma \\
 & \ll x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon}.
 \end{aligned} \tag{57}$$

Interval I_3^* :

$$\begin{aligned}
 I_3^* &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\
 &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{c} \right\}.
 \end{aligned} \tag{58}$$

This interval is an empty set noting that $(1/b) > (1/c)$.

Interval I_4^* :

$$\begin{aligned}
 I_4^* &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, 1 < c\sigma \leq c\eta \right\} \\
 &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\}.
 \end{aligned} \tag{59}$$

In this interval, we have

$$\begin{aligned}
 & T^{-1} \times \int_{I_4^*} x^\sigma \left| \zeta(a\sigma + iat) \left(L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll \max_{(1/b) < \sigma \leq (1/a)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{-1+\varepsilon} \\
 & \ll x^{1/a} T^{-1+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon}.
 \end{aligned} \tag{60}$$

From (55)–(60), we can obtain

$$\begin{aligned}
 & |J_2 + J_3| \\
 & \ll T^{-1} \int_{1/2a}^\eta x^\sigma \left| \zeta(a\sigma + iat) \left(L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & = T^{-1} \int_{I_1^* \cup \dots \cup I_4^*} x^\sigma \left| \zeta(a\sigma + iat) \left(L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll x^{1/2a} T^{(487/168)-(131(b+c)/168a)+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
 \end{aligned} \tag{61}$$

Now, we turn to estimate J_1 , and we have

$$\begin{aligned}
 |J_1| & \ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f \times f\right) L\left(\frac{b}{2a} + ibt, f \times f\right) L\left(\frac{c}{2a} + ict, f \times f\right) \right| t^{-1} dt + x^{1/2a} \\
 & \ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| \\
 & \quad \cdot \left| \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| dt.
 \end{aligned} \tag{62}$$

Then, using Lemmas 3 and 4 and Hölder's inequality, we obtain

$$\begin{aligned}
 |J_1| &\ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} T_1^{(13/42)(1-(b/2a))} T_1^{(5/4)(1-(b/2a))} T_1^{(13/42)(1-(c/2a))} T_1^{(5/4)(1-(c/2a))} \\
 &\quad \times \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt \\
 &\ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(13/42)(1-(b/2a))} T_1^{(5/4)(1-(b/2a))} T_1^{(13/42)(1-(c/2a))} T_1^{(5/4)(1-(c/2a))} \\
 &\quad \times \left(\int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left(\int_{T_1/2}^{T_1} 1 dt \right)^{5/12} \\
 &\ll x^{(1/2a)+\varepsilon} T^{(145/42)-(131(b+c)/168a)+\varepsilon}.
 \end{aligned} \tag{63}$$

By putting (53), (61), and (63) together, we have

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} + M_3 x^{1/c} + O\left(x^\varepsilon \left(x^{1/2a} T^{(145/42)-(131(b+c)/168a)} + x^{1/b} T^{(47/84)-(131a/84b)} + x^{1/c} T^{(89/42)-(131(a+b)/84c)} + x^{1/a} T^{-1+\varepsilon}\right)\right).
 \end{aligned} \tag{64}$$

By taking $T = x^{(84/748a - 131(b+c))}$ in (64), we can obtain

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} + M_3 x^{1/c} \\
 &\quad + O\left(x^{(1/a)-(84/748a - 131(b+c))+\varepsilon}\right),
 \end{aligned} \tag{65}$$

which proves the first result in Theorem 2.

For the case $b < 2a < c$, we use a similar argument to the first corresponding case. In this case, the points $s = (1/a)$ and $s = (1/b)$ are the two simple poles of the integrand in the rectangle I_T . Then, by Cauchy's residue theorem we have

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= \left\{ \text{Res}_{s=(1/a)} + \text{Res}_{s=(1/b)} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} \\
 &\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta-iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\
 &= M_1 x^{(1/a)} + M_2 x^{(1/b)} + J_1'' + J_2'' + J_3'' + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right).
 \end{aligned} \tag{66}$$

To estimate $J_2'' + J_3''$, we still divide the integral interval into four short intervals $I_1^{**}, \dots, I_4^{**}$, which are different from ones for the case $c \leq 2a$. In fact, the corresponding short intervals I_1^{**}, I_3^{**} become empty sets in this situation. However, we still can estimate $J_2'' + J_3''$ by following a similar argument to the corresponding parts of the case $c \leq 2a$ and get

$$\begin{aligned}
 |J_2'' + J_3''| &\ll x^{1/2a} T^{(75/56)-(131b/168a)+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} \\
 &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
 \end{aligned} \tag{67}$$

The estimate of J_1'' becomes the following at the current case by noting $(c/2a) > 1 > (b/2a)$.

$$\begin{aligned}
|J'_1| &\ll x^{1/2a} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \right| \cdot \left| \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(131/84)(1-(b/2a))} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(131/84)(1-(b/2a))} \left(\int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \times \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left(\int_{T_1/2}^{T_1} 1 dt \right)^{5/12} + x^{1/2a} \\
&\ll x^{1/2a} T^{(159/84)-(131b/168a)+\varepsilon}.
\end{aligned} \tag{68}$$

Thus, recalling (67), we have

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} \\
&\quad + O\left(x^{1/2a} T^{(159/84)-(131b/168a)+\varepsilon}\right. \\
&\quad \left.+ x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}\right).
\end{aligned} \tag{69}$$

By taking $T = x^{(84/486a-131b)}$ in (69), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + M_2 x^{1/b} + O\left(x^{1/a-(84/486a-131b)+\varepsilon}\right), \tag{70}$$

which proves the second result of Theorem 2.

For the case $2a \leq b$, we use a similar argument to the first corresponding case. In this situation, the point $s = (1/a)$ is the only simple pole of the integrand in the rectangle I_T . Then, Cauchy's residue theorem shows

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= \text{Res}_{s=(1/a)} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\
&\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta-iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds \\
&= M_1 x^{(1/a)} + J'_1 + J'_2 + J'_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right).
\end{aligned} \tag{71}$$

To estimate $J'_2 + J'_3$, we still divide the integral interval into four short intervals $I_1^{***}, \dots, I_4^{***}$, which are different from ones for the case $c \leq 2a$. In fact, the corresponding short intervals $I_1^{***}, I_2^{***}, I_3^{***}$ become empty sets in this situation. However, we still can estimate $J'_2 + J'_3$ by following

a similar argument to the corresponding parts of the case $c \leq 2a$ and get

$$|J'_2 + J'_3| \ll x^{1/2a} T^{-(37/168)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{72}$$

The estimate of J_1' becomes the following by noting $(c/2a) > (b/2a) > 1$.

$$\begin{aligned}
 |J_1'| &\ll x^{1/2a} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \right| \cdot \left| \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| t^{-1} dt + x^{1/2a} \\
 &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt + x^{1/2a} \\
 &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} \left(\int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \\
 &\quad \times \left(\int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left(\int_{T_1/2}^{T_1} 1 dt \right)^{5/12} + x^{1/2a} \\
 &\ll x^{1/2a} T^{(1/3)+\varepsilon}.
 \end{aligned} \tag{73}$$

Thus, recalling (71), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + O\left(x^{1/2a} T^{(1/3)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1}\right). \tag{74}$$

By taking $T = x^{(3/8a)}$ in (74), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + O\left(x^{(5/8a)+\varepsilon}\right), \tag{75}$$

which proves the third result of Theorem 2.

Data Availability

The data supporting the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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