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Retraction

Retracted: Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] T. M. Al-shami and A. Mhemdi, "Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications," *Journal of Mathematics*, vol. 2021, Article ID 9940301, 12 pages, 2021.

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Research Article

Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications

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We aim through this paper to achieve two goals: first, we define some types of belong and nonbelong relations between ordinary points and double-framed soft sets. These relations are one of the distinguishing characteristics of double-framed soft sets and are somewhat expression of the degrees of membership and nonmembership. We explore their main properties and determine the conditions under which some of them are equivalent. Also, we introduce the concept of soft mappings between two classes of double-framed soft sets and investigate the relationship between an ordinary point and its image and preimage with respect to the different types of belong and nonbelong relations. By the notions presented herein, many concepts can be studied on double-framed soft topology such as soft separation axioms and cover properties. Second, we give an educational application of optimal choices using the idea of double-framed soft sets. We provide an algorithm of this application with an example to show how this algorithm is carried out.

1. Introduction

The (crisp) set theory is a main mathematical approach to deal with a class of problems that are characterized by precision, exactness, specificity, perfection, and certainty. However, many problems in the real-life inherently involve inconsistency, imprecision, ambiguity, and uncertainties. In particular, such classes of problems arise in engineering, economics, medical sciences, environmental sciences, social sciences, and many different scopes. The crisp (classical) mathematical tools fail to model or solve these types of problems.

In the course of time, mathematicians, engineers, and scientists, particularly those who focus on artificial intelligence, are seeking for alternative mathematical approaches to solve the problems that contain uncertainty or vagueness. They initiated several set theories such as probability theory, fuzzy set [1], intuitionistic fuzzy set [2], and rough set [3].

In 1999, Molodtsov [4] proposed the concept of soft sets as a new mathematical tool to cope with uncertainties. He investigated the efficiency of soft sets to deal with complicated problems compared with the probability theory and fuzzy set theory. After Molodtsov's work, many researchers have studied several operations and relations between soft sets (see, for example, [5–10]). Soft sets were applied in various domains such as algebraic structures (see, for example, [11–13]), soft topological spaces (see, for example, [14–16]), and decision-making problems (see, for example, [17–25]). Also, the relationship among soft sets, rough sets, and fuzzy sets was the goal of some papers such as [17, 26, 27].

In the last few years, a number of scholars have extensively studied some extensions of soft set. These studies go into two ways: the first one is initiated by giving some generalizations of the structure of soft sets. This leads to define binary soft set [28], N-soft set [29], double-framed soft set [30], and bipolar soft set [31] (several relations between bipolar soft sets and ordinary points were presented in [32]). The second one is coming from the combination of soft set (or its updating forms) with rough set or fuzzy set or both. This leads to define fuzzy soft set [33], fuzzy bipolar soft set [34], bipolar fuzzy soft set [35], soft rough set [26], bipolar soft rough set [36], and modified rough bipolar soft set [37].

Soft set was formulated over an initial universal set X by using a map from a set of parameters A into the power set of X. However, we need sometimes to define two maps from A into the power set of X; for example, if we schedule students' results in **n** subjects, we define **n** different maps over the same sets X and A. For this purpose, Jun and Ahn [30] initiated the notion of double-framed soft sets and applied in BCK/BCI algebras. In 2014, Muhiuddin and Al-Roqi [38] studied the concept of double-framed soft hypervector spaces, and in 2015, Naz [39] revealed some algebraic properties of double-framed soft set. In 2017, Khana et al. [40] introduced the concept of double-framed soft LA-semigroups. In the same year, Shabir and Samreena [41] made use of a double-framed soft set to define a new soft structure called a double-framed soft topological space. They initiated its basic notions such as DFS open and closed sets and DFS neighborhoods. In 2018, Iftikhar and Mahmood [42] presented some results on lattice-ordered double-framed soft semirings; and Park [43] discussed double-framed soft deductive system of subtraction algebras. Bordbar et al. [44] applied double-framed soft set theory to hyper-*BCK* algebras. Saeed et al. [45] formulated the concepts of N-framed soft set and then defined the soft union and intersection of two double-framed soft sets. They also provided an example to elucidate an application of N-framed soft set.

The motivation for this work is to define new types of belong and nonbelong relations between ordinary points and double-framed soft sets which create new degrees of membership and nonmembership for the ordinary points. In fact, this leads to initiate novel concepts on double-framed soft topology, in particular in the areas of soft separation axioms and cover properties.

We organize the rest of this paper as follows. Section 2 recalls some operations between double-framed soft sets. In Section 3, we formulate four types of belong relations between ordinary points and double-framed soft sets called weakly partial belong, strongly partial belong, weakly total belong, and strongly total belong relations and formulate four types of nonbelong relations between ordinary points and double-framed soft sets called weakly partial nonbelong, strongly partial nonbelong, weakly total nonbelong, and strongly total nonbelong relations. Then, we examine their behaviours under the operations of soft intersection and union. Also, we study soft mappings with respect to the classes of double-framed soft sets and prob the relationships between ordinary points and their images and preimages. In Section 4, we propose a method of optimum choice based on double-framed soft sets. We provide an example to illustrate how this method can be applied to model some real-life problems. Finally, we summarize the main obtained results and present some future works in Section 5.

2. Preliminaries

In this part, we mention some definitions and results of double-framed soft sets.

In this article, the sets of parameters are denoted by A, B, C, D, E, M, N; the initial universal sets are denoted by X, Y; and the power set of X is denoted by 2^X.

Definition 1 (see [4]). A soft set over X, denoted by (h, A), is a map h from A to 2^X. We call X an initial universal set and A a set of parameters.

Usually, we write (h, A) as a set of ordered pairs:

$$(h, A) = \left\{(a, h(a)) \colon a \in A \,\text{and}\, h(a) \in 2^X\right\}. \tag{1}$$

Definition 2 (see [30]). Let h, k be two mappings from A to 2^X. A double-framed soft set over X, determined by h and k, is the set $\{(a, h(a), k(a)) \colon a \in A\}$.

We will denote this double-framed soft set by (h, k, A). The set X is called the initial universal set, and the set A is called the set of parameters.

A class of all double-framed soft sets defined over X with all parameters subsets of A is denoted by $C(X_A)$.

In a similar way, one define the concepts of triple-framed soft set, quadruple-framed soft set, quintuple-framed soft set, sextuple-framed soft set, septuple-framed soft set,..., and N-framed soft set.

Definition 3 (see [45]). $(h_1, h_2, \ldots, h_n, A)$ is said to be an N-framed soft set over a nonempty set X, where h_i is a map from A into 2^X for $i = 1, 2, \ldots, n$, X is an initial universal set, and A is a set of parameters.

An N-framed soft set is expressed as follows:

$$(h_1, h_2, \ldots, h_n, A) = \left\{(a, h_1(a), h_2(a), \ldots, h_n(a)) \colon a \in A \,\text{and}\, h_i(a) \in 2^X \text{for each}\, i = 1, 2, \ldots, n\right\}. \tag{2}$$

Henceforth, we assume that the initial universal set of every double-framed soft set in this paper is nonempty.

Example 1. Let $X = \{x_1, x_2, \ldots, x_{50}\}$ be the universal set of third graders and $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters, where a_1 represents the students holding first rank, a_2 represents the students holding second rank, a_3 represents the students holding third rank, and a_4 represents the students holding fourth rank.

Let h: $A \longrightarrow 2^X$ be a map of ranking students in mathematics subject and k: $A \longrightarrow 2^X$ be a map of ranking students in physics subject.

Suppose that h and k are given as follows:

$$\begin{aligned}
h(a_1) &= \{x_{14}\},\\
k(a_1) &= \{x_3, x_{14}\},\\
h(a_2) &= \{x_{19}\},\\
k(a_2) &= \{x_7\},\\
h(a_3) &= \{x_7, x_{21}, x_{26}\},\\
k(a_3) &= \{x_{35}\},\\
h(a_4) &= \{x_2\},\\
k(a_4) &= \{x_2, x_{43}\}.
\end{aligned} \tag{3}$$

Now, we can describe this system using a double-framed soft set as follows:

$$(h,k,A) = \{(a_1, \{x_{14}\}, \{x_3, x_{14}\}), (a_2, \{x_{19}\}, \{x_7\}), (a_3, \{x_7, x_{21}, x_{26}\}, \{x_{35}\}), (a_4, \{x_2\}, \{x_2, x_{43}\})\}. \tag{4}$$

If there are three maps of subjects, a system is described using a triple-framed soft set; and if there are four maps of subjects, a system is described using a quadruple-framed soft set and so on.

Definition 4 (see [41]). Let (h,k,A) be a double-framed soft set and $x \in X$. We say that $x \in (h,k,A)$ if $x \in h(a)$ and $x \in k(a)$ for all $a \in A$ and $x \notin (h,k,A)$ if $x \notin h(a)$ or $x \notin k(a')$ for some $a, a' \in A$.

Definition 5 (see [39]). A double-framed soft set (h,k,A) is said to be a null double-framed soft set (resp., an absolute double-framed soft set) if $h(a), k(a)$ equals to the empty (resp., universal) set for each $a \in A$.

Henceforth, the null and absolute double-framed soft sets are symbolized by $(\widetilde{\Phi_A}, \widetilde{\Phi_A})$ and $(\widetilde{X_A}, \widetilde{X_A})$, respectively.

Definition 6 (see [45]). The intersection of two double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) is a double-framed soft set (f_1, f_2, C) such that $C = A \cap B \neq \varnothing$ and $f_1\colon C \longrightarrow 2^X$ and $f_2\colon C \longrightarrow 2^X$ are defined by $f_1(c) = h_1(c) \cap k_1(c)$ and $f_2(c) = h_2(c) \cap k_2(c)$.

It is symbolized by $(h_1, h_2, A) \widetilde{\cap} (k_1, k_2, B)$.

Definition 7 (see [45]). The soft union of two double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) is a double-framed soft set (f_1, f_2, C), where $C = A \cup B$ and $f_1\colon C \longrightarrow 2^X$ and $f_2\colon C \longrightarrow 2^X$ are defined by

$$f_i(c) = \begin{cases} h_i(c), & : c \in A - B,\\ k_i(c), & : c \in B - A,\\ h_i(c) \cup k_i(c), & : c \in A \cap B. \end{cases} \tag{5}$$

It is symbolized by $(h_1, h_2, A) \widetilde{\cup} (k_1, k_2, B)$.

Definition 8 (see [30]). A double-framed soft set (h_1, h_2, A) is called a subset of a double-framed soft set (k_1, k_2, B), denoted by $(h_1, h_2, A) \widetilde{\subseteq} (k_1, k_2, B)$, if $A \subseteq B$, and $h_1(a) \subseteq k_1(a)$ and $h_2(a) \subseteq k_2(a)$ holds true for all $a \in A$.

The double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) are called equal if $(h_1, h_2, A) \widetilde{\subseteq} (k_1, k_2, B)$ and $(k_1, k_2, B) \widetilde{\subseteq} (h_1, h_2, A)$.

Definition 9 (see [39]). The relative complement of a double-framed soft set (h,k,A) is a double-framed soft set $(h,k,A)^c = (h^c, k^c, A)$, where h^c and k^c are two maps from A to 2^X defined as follows:

$$\begin{aligned}
h^c(a) &= X - h(a),\\
k^c(a) &= X - k(a).
\end{aligned} \tag{6}$$

Proposition 1 (see [39]). *ie operations of soft union and soft intersection of double-framed soft sets are commutative and associative.*

Proposition 2 (see [39]). *We have the following results for two double-framed soft sets:*

(i) $[(h,k,A) \widetilde{\cup} (p,t,A)]^c = (h,k,A)^c \widetilde{\cap} (p,t,A)^c$.

(ii) $[(h,k,A) \widetilde{\cap} (p,t,A)]^c = (h,k,A)^c \widetilde{\cup} (p,t,A)^c$.

3. Belong and Nonbelong Relations on Double-Framed Soft Sets

We dedicate this section to establish four types of memberships and four types of nonmemberships between an ordinary point and double-framed soft set and lay the foundations of them. We obtain some results that concern the soft intersection and union operators, the product of double-framed soft sets and soft mappings.

Definition 10. Let (h,k,A) be a double-framed soft set and $\delta \in X$. We say that

(i) $\delta \Subset_w (h,k,A)$, reading as δ weakly partial belongs to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$.

(ii) $\delta \Subset_s (h,k,A)$, reading as δ strongly partial belongs to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$.

(iii) $\delta \in_w (h,k,A)$, reading as δ weakly total belongs to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a)$ for all $a \in A$.

(iv) $\delta \in_s (h,k,A)$, reading as δ strongly total belongs to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a)$ for all $a \in A$.

Definition 11. Let (h,k,A) be a double-framed soft set and $\delta \in X$. We say that

(i) $\delta \not\Subset_w (h,k,A)$, reading as δ weakly partial belong to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$.

(ii) $\delta \not\Subset_s (h,k,A)$, reading as δ strongly partial belong to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$.

(iii) $\delta \notin_w (h,k,A)$, reading as δ does not weakly total belong to (h,k,A), if $\delta \notin h(a)$ or $\delta \notin k(a)$ for all $a \in A$.

(iv) $\delta \notin_s (h,k,A)$, reading as δ does not strongly total belong to (h,k,A), if $\delta \notin h(a)$ and $\delta \notin k(a)$ for all $a \in A$.

Remark 1. The relations of strongly total belong and weakly partial nonbelong were introduced in [41] (see Definition 4).

Proposition 3. *For a double-framed soft set (h,k,A) and $\delta \in X$, we have the following results:*

(i) $\delta \Subset_w (h,k,A)$ iff $\delta \not\Subset_w (h^c,k^c,A)$.

(ii) $\delta \Subset_s (h,k,A)$ iff $\delta \not\Subset_s (h^c,k^c,A)$.

(iii) $\delta \in_w (h,k,A)$ iff $\delta \notin_w (h^c,k^c,A)$.

(iv) $\delta \in_s (h,k,A)$ iff $\delta \notin_s (h^c,k^c,A)$.

Proof. We will just prove (i) and (iv).

(i) $\delta \Subset_w (h,k,A) \Leftrightarrow \delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A \Leftrightarrow \delta \notin X - h(a) = h^c(a)$ or $\delta \notin X - k(a') = k^c(a')$ for some $a, a' \in A \Leftrightarrow \delta \not\Subset_w (h^c,k^c,A)$.

(ii) $\delta \in_s (h,k,A) \Leftrightarrow \delta \in h(a)$ and $\delta \in k(a)$ for all $a \in A \Leftrightarrow \delta \notin X - h(a) = h^c(a)$ and $\delta \notin X - k(a') = k^c(a')$ for all $a \in A \Leftrightarrow \delta \notin_s (h^c,k^c,A)$.

The following proposition is a direct result of Definition 10. □

Proposition 4. *Let (h,k,A) be a double-framed soft set and $\delta \in X$. Then,*

(i) $\delta \in_s (h,k,A) \Rightarrow \delta \in_w (h,k,A) \Rightarrow \delta \Subset_w (h,k,A)$.

(ii) $\delta \in_s (h,k,A) \Rightarrow \delta \Subset_s (h,k,A) \Rightarrow \delta \Subset_w (h,k,A)$.

(iii) $\delta \notin_s (h,k,A) \Rightarrow \delta \notin_w (h,k,A) \Rightarrow \delta \not\Subset_w (h,k,A)$.

(iv) $\delta \notin_s (h,k,A) \Rightarrow \delta \not\Subset_s (h,k,A) \Rightarrow \delta \not\Subset_w (h,k,A)$.

Example below is given to clarify that the converse of Proposition 4 fails. Also, it shows that the relations of strongly partial belong and weakly total belong (the relations of weakly total nonbelong and strongly partial nonbelong) are independent of each other.

Example 2. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters and (h,k,A) double-framed soft set over $X = \{x_1, x_2, \ldots, x_{10}\}$ be defined as follows:

$$(h,k,A) = \{(a_1, \{x_1\}, \{x_2, x_4, x_{10}\}), (a_2, \varnothing, \{x_4, x_{10}\}), (a_3, \{x_2, x_3\}, \{x_4, x_5, x_{10}\})\}. \tag{7}$$

We find the next relations:

(i) $x_1 \Subset_w (h,k,A)$, but $x_1 \in_w (h,k,A)$ and $x_1 \Subset_s (h,k,A)$ do not hold.

(ii) $x_4 \in_w (h,k,A)$, but $x_4 \in_s (h,k,A)$ does not hold.

(iii) $x_2 \Subset_s (h,k,A)$, but $x_2 \in_s (h,k,A)$ does not hold.

(iv) $x_2 \not\Subset_s (h,k,A)$, but $x_2 \notin_w (h,k,A)$ and $x_2 \notin_s (h,k,A)$ do not hold.

(v) $x_4 \in_w (h,k,A)$, but $x_4 \Subset_s (h,k,A)$ does not hold. Also, $x_2 \Subset_s (h,k,A)$, but $x_2 \in_w (h,k,A)$ does not hold.

(vi) $x_3 \notin_w (h,k,A)$, but $x_2 \notin_s (h,k,A)$ does not hold.

Remark 2. It is well-known in the Quantum physics the possibility of existence and nonexistence of an electron in the same place. This matter also occurs here with respect to weakly partial belong and weakly partial nonbelong relations; strongly partial belong and strongly partial nonbelong relations; and weakly total belong and weakly total nonbelong relations. To illustrate that it can be seen from Example 2 that

$$\begin{gathered} x_5 \Subset_w (h,k,A), \\ x_5 \not\Subset_w (h,k,A), \\ x_2 \Subset_s (h,k,A), \\ x_2 \not\Subset_s (h,k,A), \\ x_{10} \in_w (h,k,A), \\ x_{10} \notin_w (h,k,A). \end{gathered} \tag{8}$$

Proposition 5. *Let (h,k,A) and (p,t,A) be double-framed soft sets such that $(h,k,A) \widetilde{\subseteq} (p,t,A)$. Then,*

(i) If $\delta \Subset_w (h,k,A)$ (resp., $\delta \Subset_s (h,k,A)$, $\delta \in_w (h,k,A)$, $\delta \in_s (h,k,A)$), then $\delta \Subset_w (p,t,A)$ (resp., $\delta \Subset_s (p,t,A)$, $\delta \in_w (p,t,A)$, $\delta \in_s (p,t,A)$).

(ii) *If* $\delta \not\Subset_w (p,t,A)$ *(resp.,* $\delta \not\Subset_s (p,t,A)$, $\delta \notin_w (p,t,A)$, $\delta \notin_s (p,t,A)$*), then* $\delta \not\Subset_w (h,k,A)$ *(resp.,* $\delta \not\Subset_s (h,k,A)$, $\delta \notin_w (h,k,A)$, $\delta \notin_s (h,k,A)$*).*

Proof. Straightforward. □

Remark 3. Note that satisfying the two conditions (i) and (ii) of the above proposition does not imply $(h,k,A) \widetilde{\subseteq} (p,t,A)$. To illustrate this fact, consider Example 2 and let $(p,t,A) = \{(a_1, \{x_2, x_4, x_{10}\}, \{x_1\}), (a_2 \varnothing, \{x_4, x_{10}\}), (a_3, \{x_4, x_5, x_{10}\}\{x_2, x_3\})\}$. It is clear that $\delta \Subset_w (h,k,A)$ (resp., $\delta \Subset_s (h,k,A)$, $\delta \in_w (h,k,A)$, $\delta \in_s (h,k,A)$) if and only if $\delta \Subset_w (p,t,A)$ (resp., $\delta \Subset_s (p,t,A)$, $\delta \in_w (p,t,A)$, $\delta \in_s (p,t,A)$). However, $(h,k,A) \widetilde{\nsubseteq} (p,t,A)$ and $(p,t,A) \widetilde{\nsubseteq} (h,k,A)$.

Proposition 6. *For two double-framed soft sets* (h,k,A) *and* (p,t,A) *and* $\delta \in X$*, we have the following results:*

(i) $\delta \Subset_w (h,k,A)$ *or* $\delta \Subset_w (p,t,A) \Leftrightarrow \delta \Subset_w (h,k,A) \widetilde{\cup} (p,t,A)$.

(ii) $\delta \Subset_s (h,k,A)$ *or* $\delta \Subset_s (p,t,A) \Rightarrow \delta \Subset_s (h,k,A) \widetilde{\cup} (p,t,A)$.

(iii) $\delta \in_w (h,k,A)$ *or* $\delta \in_w (p,t,A) \Rightarrow \delta \in_w (h,k,A) \widetilde{\cup} (p,t,A)$.

(iv) $\delta \in_s (h,k,A)$ *or* $\delta \in_s (p,t,A) \Rightarrow \delta \in_s (h,k,A) \widetilde{\cup} (p,t,A)$.

(v) $\delta \Subset_w (h,k,A) \widetilde{\cap} (p,t,A) \Rightarrow \delta \Subset_w (h,k,A)$ *and* $\delta \Subset_w (p,t,A)$.

(vi) $\delta \Subset_s (h,k,A) \widetilde{\cap} (p,t,A) \Rightarrow \delta \Subset_s (h,k,A)$ *and* $\delta \Subset_s (p,t,A)$.

(vii) $\delta \in_w (h,k,A) \widetilde{\cap} (p,t,A) \Rightarrow \delta \in_w (h,k,A)$ *and* $\delta \in_w (p,t,A)$.

(viii) $\delta \in_s (h,k,A) \widetilde{\cap} (p,t,A) \Leftrightarrow \delta \in_s (h,k,A)$ *and* $\delta \in_s (p,t,A)$.

Proof. Since (h,k,A) and (p,t,A) are subsets of $(h,k,A) \widetilde{\cup} (p,t,A)$, then the necessary parts of (i) to (iv) hold; and since $(h,k,A) \widetilde{\cap} (p,t,A)$ are subsets of (h,k,A) and (p,t,A), then the necessary parts of (v) to (viii) hold.

To prove the sufficient part of (i), let $\delta \Subset_w (h,k,A) \widetilde{\cup} (p,t,A)$. Then, $\delta \in h(a) \cup p(a)$ or $\delta \in k(a') \cup t(a')$ for some $a, a' \in A$. Say $\delta \in h(a) \cup p(a)$ for some $a \in A$. Therefore, $\delta \in h(a)$ or $p(a)$ for some $a \in A$, and hence, $\delta \Subset_w (h,k,A)$ or $\delta \Subset_w (p,t,A)$.

To prove the sufficient part of (viii), let $\delta \in_s (h,k,A)$ and $\delta \in_s (p,t,A)$. Then, for all $a \in A$, we have $\delta \in h(a)$ and $\delta \in k(a)$ and $\delta \in p(a)$ and $\delta \in t(a)$. Therefore, $\delta \in h(a) \cap p(a)$ and $\delta \in k(a) \cap t(a)$ for all $a \in A$, and hence, $\delta \in_s (h,k,A) \widetilde{\cap} (p,t,A)$.

Example below is given to clarify that the converse of the results (ii) to (iv) and (v) to (vii) of Proposition 6 fails. □

Example 3. Let $A = \{a_1, a_2\}$ be a set of parameters and (h,k,A), (p,t,A) double-framed soft sets over $X = \{x_1, x_2, x_3, x_4, x_5\}$ defined as follows:

$$(h,k,A) = \{(a_1, \{x_1, x_3\}, \varnothing), (a_2, \{x_3, x_4\}, \{x_4, x_5\})\},$$
$$(p,t,A) = \{(a_1, \{x_4\}, \{x_3, x_4, x_5\}), (a_2, \{x_2\}, \{x_1, x_3\})\}. \tag{9}$$

Then, $(h,k,A) \widetilde{\cup} (p,t,A) = \{(a_1, \{x_1, x_3, x_4\}, \{x_3, x_4, x_5\}), (a_2, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4, x_5\})\}$ and $(h,k,A) \widetilde{\cap} (p,t,A) = \widetilde{\Phi}$.

We note the following:

(i) $x_1 \Subset_s (h,k,A) \widetilde{\cup} (p,t,A)$, but $x_1 \Subset_s (h,k,A)$ or $x_1 \Subset_s (p,t,A)$ does not hold.

(ii) $x_5 \in_w (h,k,A) \widetilde{\cup} (p,t,A)$, but $x_5 \in_w (h,k,A)$ or $x_5 \in_w (p,t,A)$ does not hold.

(iii) $x_4 \in_s (h,k,A) \widetilde{\cup} (p,t,A)$, but $x_4 \in_s (h,k,A)$ or $x_4 \in_s (p,t,A)$ does not hold.

(iv) $x_4 \Subset_w (h,k,A)$ and $x_4 \Subset_w (p,t,A)$, but $x_4 \Subset_w (h,k,A) \widetilde{\cap} (p,t,A)$ does not hold.

(v) $x_4 \Subset_s (h,k,A)$ and $x_4 \Subset_s (p,t,A)$, but $x_4 \Subset_s (h,k,A) \widetilde{\cap} (p,t,A)$ does not hold.

(vi) $x_3 \in_w (h,k,A)$ and $x_3 \in_w (p,t,A)$, but $x_3 \in_w (h,k,A) \widetilde{\cap} (p,t,A)$ does not hold.

Similarly, it can be proved the following result.

Proposition 7. *For two double-framed soft sets* (h,k,A) *and* (p,t,A) *over* X *and* $\delta \in X$*, we have the following results:*

(i) $\delta \not\Subset_w (h,k,A) \widetilde{\cup} (p,t,A) \Rightarrow \delta \not\Subset_w (h,k,A)$ *and* $\delta \not\Subset_w (p,t,A)$.

(ii) $\delta \not\Subset_s (h,k,A) \widetilde{\cup} (p,t,A) \Rightarrow \delta \not\Subset_s (h,k,A)$ *and* $\delta \not\Subset_s (p,t,A)$.

(iii) $\delta \notin_w (h,k,A) \widetilde{\cup} (p,t,A) \Rightarrow \delta \notin_w (h,k,A)$ *and* $\delta \notin_w (p,t,A)$.

(iv) $\delta \notin_s (h,k,A) \widetilde{\cup} (p,t,A) \Leftrightarrow \delta \notin_s (h,k,A)$ *and* $\delta \notin_s (p,t,A)$.

(v) $\delta \not\Subset_w (h,k,A)$ *or* $\delta \not\Subset_w (p,t,A) \Rightarrow \delta \not\Subset_w (h,k,A) \widetilde{\cap} (p,t,A)$.

(vi) $\delta \not\Subset_s (h,k,A)$ *or* $\delta \not\Subset_s (p,t,A) \Rightarrow \delta \not\Subset_s (h,k,A) \widetilde{\cap} (p,t,A)$.

(vii) $\delta \notin_w (h,k,A)$ *or* $\delta \notin_w (p,t,A) \Rightarrow \delta \notin_w (h,k,A) \widetilde{\cap} (p,t,A)$.

(viii) $\delta \notin_s (h,k,A)$ *or* $\delta \notin_s (p,t,A) \Rightarrow \delta \notin_s (h,k,A) \widetilde{\cap} (p,t,A)$.

Definition 12. A double-framed soft set (h,k,A) is said to be 2-stable if $h(a) = U \subseteq X$ and $k(a) = V \subseteq X$ for each $a \in A$. If $U = V$, then (h,k,A) is said to be 1-stable.

Obviously, a 1-stable double-framed soft set is 2-stable, but the converse is not always true.

Proposition 8. *Let* (h,k,A) *be a 1-stable double-framed soft set. Then,* $\delta \Subset_w (h,k,A) \Leftrightarrow \delta \Subset_s (h,k,A) \Leftrightarrow \delta \in_w (h,k,A) \Leftrightarrow \delta \in_s (h,k,A)$.

Proof. Since (h,k,A) is a 1-stable double-framed soft set, there is a subset U of X such that $h(a) = k(a) = U$ for each

$a \in A$. This means that $\delta \in h(a)$ or $\delta \in k(a)$ for some $a \in A$ iff $\delta \in h(a)$ and $\delta \in k(a)$ for each $a \in A$. Hence, the desired result is proved. □

Corollary 1. *Let (h, k, A) be a 1-stable double-framed soft set. Then, $\delta \not\Subset_w (h, k, A) \Leftrightarrow \delta \not\Subset_s (h, k, A) \Leftrightarrow \delta \notin_w (h, k, A) \Leftrightarrow \delta \notin_s (h, k, A)$.*

Proposition 9. *Let (h, k, A) be a 2-stable double-framed soft set. Then,*

(i) $\delta \Subset_w (h, k, A) \Leftrightarrow \delta \in_w (h, k, A)$.

(ii) $\delta \Subset_s (h, k, A) \Leftrightarrow \delta \in_s (h, k, A)$.

Proof. Since (h, k, A) is a 2-stable double-framed soft set, there exist two subsets U, V of X such that $h(a) = U$ and $k(a) = V$ for each $a \in A$. Now, we have the following two cases:

Case 1: $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$ if and only if $\delta \in h(a)$ or $\delta \in k(a')$ for all $a, a' \in A$.

Case 2: $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$ if and only if $\delta \in h(a)$ and $\delta \in k(a')$ for all $a, a' \in A$.

Hence, the desired results are proved. □

Corollary 2. *Let (h, k, A) be a 2-stable double-framed soft set. Then,*

(i) $\delta \not\Subset_w (h, k, A) \Leftrightarrow \delta \notin_w (h, k, A)$.

(ii) $\delta \not\Subset_s (h, k, A) \Leftrightarrow \delta \notin_s (h, k, A)$.

Definition 14. The Cartesian product of two double-framed soft sets (h, k, A) and (p, t, B), denoted by $(h \times p, k \times t, A \times B)$, is defined as $(h \times p)(e, e') = h(e) \times p(e')$ and $(k \times t)(e, e') = k(e) \times t(e')$ for each $(e, e') \in A \times B$.

Proposition 10.

(i) $(\delta, \zeta) \Subset_s (h, k, A) \times (p, t, B)$ if and only if $\delta \Subset_s (h, k, A)$ and $\xi \Subset_s (p, t, B)$.

(ii) If $(\delta, \zeta) \Subset_w (h, k, A) \times (p, t, B)$, then $\delta \Subset_w (h, k, A)$ and $\xi \Subset_w (p, t, B)$.

(iii) $(\delta, \zeta) \in_s (h, k, A) \times (p, t, B)$ if and only if $\delta \in_s (h, k, A)$ and $\xi \in_s (p, t, B)$.

(iv) If $(\delta, \zeta) \in_w (h, k, A) \times (p, t, B)$, then $\delta \in_w (h, k, A)$ and $\xi \in_w (p, t, B)$.

Proof. (i) $(\delta, \zeta) \Subset_s (h, k, A) \times (p, t, B) = (h \times p, k \times t, A \times B)$.

$\Leftrightarrow (\delta, \zeta) \in (h \times p)(a, b) = h(a) \times p(b)$ and $(\delta, \zeta) \in (k \times t)(a', b') = k(a') \times t(b')$ for some $(a, b), (a', b') \in A \times B$.

$\Leftrightarrow \delta \in h(a)$ and $\xi \in p(b)$ for some $a \in A$ and $b \in B$ and $\delta \in k(a')$ and $\xi \in t(b')$ for some $a' \in A$ and $b' \in B$.

$\Leftrightarrow \delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$ and $\xi \in p(b)$ and $\xi \in t(b')$ for some $b, b' \in B$.

$\Leftrightarrow \delta \Subset_s (h, k, A)$ and $\xi \Subset_s (p, t, B)$.

The other cases can be achieved similarly.

The following example explains that the converses of (ii) and (iv) of the above proposition fail. □

Example 4. Let $A = \{a_1, a_2\}$ be a set of parameters and (h, k, A), (p, t, A) double-framed soft sets over $X = \{x_1, x_2, x_3, x_4\}$ defined as follows:

$$\begin{aligned} (h, k, A) &= \{(a_1, \{x_1, x_2\}, \varnothing), (a_2, \{x_2\}, \{x_4\})\}, \\ (p, t, A) &= \{(a_1, \{x_1\}, \{x_3\}), (a_2, \{x_4\}, \{x_3\})\}. \end{aligned} \tag{10}$$

Then, $(h, k, A) \times (p, t, A) = \{((a_1, a_1), \{(x_1, x_1), (x_2, x_1)\}, \varnothing), ((a_1, a_2), \{(x_1, x_4), (x_2, x_4)\}, \varnothing), ((a_2, a_1), \{(x_2, x_1)\}, \{(x_4, x_3)\}), ((a_2, a_2), \{(x_2, x_4)\}, \{(x_4, x_3)\})\}$.

We find the following relations:

(i) $x_1 \Subset_w (h, k, A)$ and $x_3 \Subset_w (p, t, B)$; however, $(x_1, x_3) \Subset_w (h, k, A) \times (p, t, B)$ does not hold true.

(ii) $x_2 \in_w (h, k, A)$ and $x_3 \in_w (p, t, B)$; however, $(x_2, x_3) \in_w (h, k, A) \times (p, t, B)$ does not hold true.

Definition 15. A soft mapping π_φ from $C(X_A)$ into $C(Y_B)$ is a pair (π, φ) of crisp mappings such that $\pi: X \longrightarrow Y$ and $\varphi: A \longrightarrow B$ and is defined as follows: the image of a double-framed soft set (f_1, f_2, M) in $C(X_A)$ is a double-framed soft set $\pi_\varphi(f_1, f_2, U) = (\pi_{f_1}, \pi_{f_2}, E)$ in $C(Y_B)$ such that $E = \varphi(\beta) \subseteq B$ and π_{f_1} and π_{f_2} are two maps defined as

$$\pi_{f_i}(e) = \pi\left(\bigcup_{\varepsilon \in \varphi^{-1}(e) \cap \beta} f_i(\varepsilon)\right), \tag{11}$$

for each $e \in E$ and $i = 1, 2$.

Definition 16. A soft map $\pi_\varphi: C(X_A) \longrightarrow C(Y_B)$ is said to be injective (resp., surjective and bijective) if π and φ are injective (resp., surjective and bijective).

Definition 17. Let $\pi_\varphi: C(X_A) \longrightarrow C(Y_B)$ be a soft mapping. Then, the preimage of a double-framed soft set (g_1, g_2, N) in $C(Y_B)$ is a double-framed soft set $\pi_\varphi^{-1}(g_1, g_2, N) = (\pi_{g_1}^{-1}, \pi_{g_2}^{-1}, D)$ in $C(X_A)$ such that $D = \varphi^{-1}(N) \subseteq A$ and $\pi_{g_1}^{-1}$ and $\pi_{g_2}^{-1}$ are two maps defined as

$$\pi_{g_i}^{-1}(d) = \pi^{-1}(g_i \varphi(d)), \tag{12}$$

for each $d \in D$ and $i = 1, 2$.

Proposition 11. *Let $\pi_\varphi: C(X_A) \longrightarrow C(Y_B)$ be a soft mapping, and let (f_1, f_2, β) and (h_1, h_2, β') be two double-framed soft sets in $C(X_A)$. Then,*

(i) $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) \widetilde{\subseteq} (\widetilde{\Phi_B}, \widetilde{\Phi_B})$. The equality holds if φ is surjective.

(ii) $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) \widetilde{\subseteq} (\widetilde{Y_B}, \widetilde{Y_B})$. The equality holds if π and φ are surjective.

(iii) If $(f_1, f_2, \beta) \widetilde{\subseteq} (h_1, h_2, \beta')$, then $\pi_\varphi(f_1, f_2, \beta) \widetilde{\subseteq} \pi_\varphi(h_1, h_2, \beta')$.

(iv) $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cup}(h_1, h_2, \beta')] = \pi_\varphi(f_1, f_2, \beta)\widetilde{\cup}\pi_\varphi(h_1, h_2, \beta')$.

(v) $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cap}(h_1, h_2, \beta')]\widetilde{\subseteq}\pi_\varphi(f_1, f_2, \beta)\widetilde{\cap}\pi_\varphi(h_1, h_2, \beta')$.

The equality holds if π and φ are injective.

Proof. To prove (i), let $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = \pi_\varphi(u, u, A) = (v, v, E)$, where $u(a) = \varnothing$ for each $a \in A$ and $E = \varphi(A)$. Then, $v(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)}u(\varepsilon)) = \pi(\varnothing) = \varnothing$ for each $e \in E$. Therefore, $(v, v, E) = (\widetilde{\Phi_E}, \widetilde{\Phi_E})$. Since $E\subseteq B$, then $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = (\widetilde{\Phi_E}, \widetilde{\Phi_E})\widetilde{\subseteq}(\widetilde{\Phi_B}, \widetilde{\Phi_B})$.

If φ is surjective, then $E = \varphi(A) = B$. Hence, $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = (\widetilde{\Phi_E}, \widetilde{\Phi_E}) = (\widetilde{\Phi_B}, \widetilde{\Phi_B})$.

To prove (ii), let $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) = \pi_\varphi(u, u, A) = (u, u, E)$, where $u(a) = X$ for each $a \in A$ and $E = \varphi(A)$. Then, $v(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)}u(\varepsilon)) = \pi(X)\subseteq Y$ for each $e \in E$. Therefore, $(v, v, E)\widetilde{\subseteq}(\widetilde{Y_B}, \widetilde{Y_B})$.

If φ and π are surjective, then $E = \varphi(A) = B$ and $\pi(X) = Y$. Hence, $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) = (\widetilde{Y_B}, \widetilde{Y_B})$.

One can prove (iii) easily.

To prove (iv), first, let $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cup}(h_1, h_2, \beta')] = \pi_\varphi(u_1, u_2, \beta\cup\beta') = (v_1, v_2, E)$, where $E = \varphi(\beta\cup\beta')$. Now, for each $e \in E$, we have $v_i(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap E}u_i(\varepsilon))$. Since

$$u_i(\varepsilon) = \begin{cases} f_i(\varepsilon), & : \varepsilon \in \beta - \beta', \\ h_i(\varepsilon), & : \varepsilon \in \beta' - \beta, \\ f_i(\varepsilon)\cup h_i(\varepsilon), & : \varepsilon \in \beta\cap\beta', \end{cases} \tag{13}$$

then

$$\pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(e)\cap E}u_i(\varepsilon)\right) = \pi\left(\cup\begin{cases} f_i(\varepsilon) & : \varepsilon \in (\beta-\beta')\cap\varphi^{-1}(e) \\ h_i(\varepsilon) & : \varepsilon \in (\beta'-\beta)\cap\varphi^{-1}(e) \\ f_i(\varepsilon)\cup h_i(\varepsilon) & : \varepsilon \in (\beta\cap\beta')\cap\varphi^{-1}(e) \end{cases}\right). \tag{14}$$

Second, let $\pi_\varphi(f_1, f_2, \beta)\widetilde{\cup}\pi_\varphi(h_1, h_2, \beta') = (w_1, w_2, N)$, where $N = \varphi(\beta)\cup\varphi(\beta')$. Now, for each $n \in N$, we have

$$\begin{aligned} w_i(n) &= \pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} f_i(\varepsilon)\right)\cup\pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} h_i(\varepsilon)\right) \\ &= \pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} f_i(\varepsilon)\cup\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} h_i(\varepsilon)\right) \\ &= \pi\left(\cup\begin{cases} f_i(\varepsilon) & : \varepsilon \in (\beta-\beta')\cap\varphi^{-1}(n) \\ h_i(\varepsilon) & : \varepsilon \in (\beta'-\beta)\cap\varphi^{-1}(n) \\ f_i(\varepsilon)\cup h_i(\varepsilon) & : \varepsilon \in (\beta\cap\beta')\cap\varphi^{-1}(n) \end{cases}\right). \end{aligned} \tag{15}$$

Since $\varphi(\beta\cup\beta') = \varphi(\beta)\cup\varphi(\beta')$, then $E = N$. Thus, $v_i(e) = w_i(e)$ for each $e \in E = N$. Hence, we obtain the desired result.

One can prove (v) similarly.

By using a similar technique, one can prove the following result. □

Proposition 12. *Let π_φ: $C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (g_1, g_2, N) and (l_1, l_2, N') be two double-framed soft sets in $C(Y_B)$. Then, we have the following results:*

(i) $\pi_\varphi^{-1}(\widetilde{\Phi_B}, \widetilde{\Phi_B}) = (\widetilde{\Phi_A}, \widetilde{\Phi_A})$.

(ii) $\pi_\varphi^{-1}(\widetilde{Y_B}, \widetilde{Y_B}) = (\widetilde{X_A}, \widetilde{X_A})$.

(iii) If $(g_1, g_2, N)\widetilde{\subseteq}(l_1, l_2, N')$, then $\pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\subseteq}\pi_\varphi^{-1}(l_1, l_2, N')$.

(iv) $\pi_\varphi^{-1}[(g_1, g_2, N)\widetilde{\cup}(l_1, l_2, N')] = \pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\cup}\pi_\varphi^{-1}(l_1, l_2, N')$.

(v) $\pi_\varphi^{-1}[(g_1, g_2, N)\widetilde{\cap}(l_1, l_2, N')] = \pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\cap}\pi_\varphi^{-1}(l_1, l_2, N')$.

Proposition 13. *Let π_φ: $C(X_A) \longrightarrow C(Y_B)$ be a soft mapping, and let (h, k, M) be a double-framed soft set in $C(X_A)$. Then, we have the following results:*

(i) If $\delta\Subset_w(h, k, M)$, then $\pi(\delta)\Subset_w\pi_\varphi(h, k, M)$.

(ii) If $\delta\Subset_s(h, k, M)$, then $\pi(\delta)\Subset_s\pi_\varphi(h, k, M)$.

(iii) If $\delta\in_w(h, k, M)$, then $\pi(\delta)\in_w\pi_\varphi(h, k, M)$.

(iv) If $\delta\in_s(h, k, M)$, then $\pi(\delta)\in_s\pi_\varphi(h, k, M)$.

(v) If $\delta\not\Subset_w(h, k, M)$ and φ is injective, then $\pi(\delta)\not\Subset_w\pi_\varphi(h, k, M)$.

(vi) If $\delta\not\Subset_s(h, k, M)$ and φ is injective, then $\pi(\delta)\not\Subset_s\pi_\varphi(h, k, M)$.

(vii) If $\delta \notin_w (h, k, M)$, then $\pi(\delta) \notin_w\pi_\varphi(h, k, M)$.

(viii) If $\delta \notin_s (h, k, M)$, then $\pi(\delta) \notin_s\pi_\varphi(h, k, M)$.

Proof. We only prove (i), (ii), (v), and (viii). The other cases can be made similarly.

To prove (i), let $\delta\Subset_w(h, k, M)$, then there exist parameters $a, a' \in M\subseteq A$ such that $\delta \in h(a)$ or $\delta \in k(a')$. Without loss of generality, consider $\delta \in h(a)$. Now, there is a parameter $b \in \varphi(M)\subseteq B$ such that $a \in \varphi^{-1}(b)$. Obviously, $a \in \varphi^{-1}(b)\cap M$, so that it follows from Definition 15 that $\pi(\delta) \in \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M}h(\varepsilon))$. Therefore, $\pi(\delta)\Subset_w(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (ii), let $\delta\Subset_s(h, k, M)$. Then, there exist parameters $a, a' \in M\subseteq A$ such that $\delta \in h(a)$ and $\delta \in k(a')$. Without loss of generality, suppose that there exist two distinct parameters $b, b' \in \varphi(M)\subseteq B$ such that $a \in \varphi^{-1}(b)$ and $a' \in \varphi^{-1}(b')$. Obviously, $a \in \varphi^{-1}(b)\cap M$ and $a' \in \varphi^{-1}(b')\cap M$ so that it follows from Definition 15 that $\pi(\delta) \in \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M}h(\varepsilon))$ and $\pi(\delta) \in \pi_k(b') = \pi(\cup_{\varepsilon\in\varphi^{-1}(b')\cap M}k(\varepsilon))$. Therefore, $\pi(\delta)\Subset_s(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (v), let $\delta\not\Subset_w(h, k, M)$. Then, there exist parameters $a, a' \in M\subseteq A$ such that $\delta \notin h(a)$ or $\delta \notin k(a')$. Say $\delta \notin h(a)$. Then, there is a parameter $b \in \varphi(M)\subseteq B$ such that $a \in \varphi^{-1}(b)$. Since φ is injective, then $a = \varphi^{-1}(b)$. This means that $\{a\} = \varphi^{-1}(b)\cap M$. Therefore, $\pi(\delta) \notin \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M}h(\varepsilon)) = \pi(h(a))$. Therefore, $\pi(\delta)\not\Subset_w(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (viii), let $\delta \notin_s (h, k, M)$. Then, $\delta \notin h(a)$ and $\delta \notin k(a)$ for all $a \in M\subseteq A$. Therefore, for each parameter $b \in \varphi(M)\subseteq B$, there is $a \in M$ such that $a \in \varphi^{-1}(b)$. Thus, for each $b \in \varphi(M)$, we obtain $\pi(\delta) \notin \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M} h(\varepsilon)) = \pi(h(a))$ and $\pi(\delta) \notin \pi_k(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M} k(\varepsilon)) = \pi(h(a))$. Hence, $\pi(\delta) \notin_s (\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h,k,M)$, as required. □

Proposition 14. *Let $\pi_\varphi\colon C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (p,t,N) be a double-framed soft set in $C(Y_B)$. If φ is surjective, then we have the following results:*

(i) If $\xi \Subset_w (p,t,N)$, then $\delta \Subset_w \pi_\varphi^{-1}(p,t,N)$ for each $\delta \in \pi^{-1}(\xi)$.

(ii) If $\xi \Subset_s (p,t,N)$, then $\delta \Subset_s \pi_\varphi^{-1}(p,t,N)$ for each $\delta \in \pi^{-1}(\xi)$.

(iii) If $\xi \in_w (p,t,N)$, then $\delta \in_w \pi_\varphi^{-1}(p,t,N)$ for each $\delta \in \pi^{-1}(\xi)$.

(iv) If $\xi \in_s (p,t,N)$, then $\delta \in_s \pi_\varphi^{-1}(p,t,N)$ for each $\delta \in \pi^{-1}(\xi)$.

(v) If $\xi \not\Subset_w (p,t,N)$ such that π is injective, then $\pi^{-1}(\xi) \not\Subset_w \pi_\varphi^{-1}(p,t,N)$.

(vi) If $\xi \not\Subset_s (p,t,N)$ such that π is injective, then $\pi^{-1}(\xi) \not\Subset_s \pi_\varphi^{-1}(p,t,N)$.

(vii) If $\xi \notin_w (p,t,N)$ such that π is injective, then $\pi^{-1}(\xi) \notin_w \pi_\varphi^{-1}(p,t,N)$.

(viii) If $\xi \notin_s (p,t,N)$ such that π is injective, then $\pi^{-1}(\xi) \notin_s \pi_\varphi^{-1}(p,t,N)$.

Proof. We only prove (i), (ii), (v), and (viii). The other cases can be made similarly.

To prove (i), let $\xi \Subset_w (p,t,N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \in p(b)$ or $\xi \in t(b')$. Without loss of generality, consider $\xi \in p(b)$. Since φ is surjective, then there is a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$. Now, for each $\delta \in \pi^{-1}(\xi)$, we obtain $\delta \Subset_w (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p,t,N)$, as required.

To prove (ii), let $\xi \Subset_s (p,t,N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \in p(b)$ and $\xi \in t(b')$. Since φ is surjective, then there are two parameters $a, a' \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$ and $\varphi(a') = b'$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$ and $\pi_l^{-1}(a') = \pi^{-1}(t\varphi(a')) = \pi^{-1}(t(b'))$. Now, for each $\delta \in \pi^{-1}(\xi)$, we obtain $\delta \Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p,t,N)$, as required.

To prove (v), let $\xi \not\Subset_w (p,t,N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \notin p(b)$ or $\xi \notin t(b')$. Say $\xi \notin p(b)$. Since φ is surjective, then there exists a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$. Since π is injective, then $\pi^{-1}(\xi) \not\Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p,t,N)$, as required.

To prove (viii), let $\xi \notin_s (p,t,N)$. Then, $\xi \notin p(b)$ and $\xi \notin t(b)$ for all $b \in N \subseteq B$. Since φ is surjective, then there exists a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$ and $\pi_l^{-1}(a) = \pi^{-1}(t\varphi(a)) = \pi^{-1}(t(b))$. Since π is injective, then $\pi^{-1}(\xi) \not\Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p,t,N)$, as required. □

Proposition 15. *Let $\pi_\varphi\colon C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (h,k,M) and (p,t,N) be two double-framed soft sets in $C(X_A)$ and $C(Y_B)$, respectively. Then, the following holds:*

(i) If π is bijective, then $\pi_\varphi((h,k,M)^c) = [\pi_\varphi(h,k,M)]^c$.

(ii) $\pi_\varphi^{-1}((p,t,N)^c) = [\pi_\varphi^{-1}(p,t,N)]^c$.

Proof. We only prove (i).

It is clear that $\pi_\varphi((h,k,M)^c) = \pi_\varphi(h^c, k^c, M)$, where $\pi_{h^c}(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} h^c(\varepsilon))$ and $\pi_{k^c}(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} k^c(\varepsilon))$ for each $e \in \varphi(M)$. Since π is bijective, then $\pi_{h^c}(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} h^c(\varepsilon)) = (\pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} h(\varepsilon)))^c$ and $\pi_{k^c}(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} k^c(\varepsilon)) = (\pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap M} k(\varepsilon)))^c$ □

4. Application of Double-Framed Soft Sets

In this section, we present an application of optimal choices using the idea of double-framed soft sets. The idea of this application is based on the evaluation of rank of the applicants in the different disciplines under study, not on the total summation of marks obtained by the applicant. The philosophy of this method is based on comprehensive evaluation, in other words, confirming the ability of applicants of satisfying high levels for all testing criteria.

Now, we provide an example to demonstrate: how we make optimal choices? Then, we construct an algorithm of this method.

Example 5. Ministry of education advertises of five scholarships supported from the government for the students who finished secondary stage. The trade-off between applicants is based on the examinations of two subjects: maths and physics.

Twenty students $S = \{s_i: i = 1, 2, \ldots, 20\}$ applied to compete with each other to gain one of these scholarships. They carried out the examination of the two subjects. Then, we input subjects' marks of all students in Table 1.

Now, we determine the ranks of the students for each subject. In fact, this step will depend on the content of the application or the desire of those in charge of work. Regarding our example, we put a set $A = \{a_i: i = 1, 2, \ldots, 10\}$ expressing ten levels of ranks:

a_1 stands for the students with the first rank.

a_2 stands for the students with the second rank.

⋮

a_n stands for the students with the n-th rank.

From Table 1, we complete Table 2 by constructing a double-framed soft set $(f_{\text{Maths}}, f_{\text{Physics}}, A)$ over S, where the maps f_{Maths} and f_{Physics} from A into the power set of S are given by $f_{\text{Maths}}(a_i)$ = the set of students who rank are a_i in maths subject and $f_{\text{Physics}}(a_i)$ = the set of students who rank are a_i in physics subject.

Table 1: Subjects' marks of twenty students.

Student	Subjects: Maths	Physics
s_1	35	31
s_2	28	25
s_3	42	48
s_4	22	19
s_5	49	47
s_6	33	36
s_7	18	23
s_8	34	34
s_9	50	37
s_{10}	21	25
s_{11}	20	18
s_{12}	27	32
s_{13}	11	17
s_{14}	30	25
s_{15}	49	40
s_{16}	50	41
s_{17}	36	44
s_{18}	14	16
s_{19}	16	25
s_{20}	46	38

Table 2: Maps of subjects.

A	Maps: f_{Maths}	f_{Physics}
a_1	$\{s_9, s_{16}\}$	$\{s_3\}$
a_2	$\{s_5, s_{15}\}$	$\{s_5\}$
a_3	$\{s_1\}$	$\{s_{20}, s_{17}\}$
a_4	$\{s_3\}$	$\{s_{16}\}$
a_5	$\{s_{17}\}$	$\{s_{15}\}$
a_6	$\{s_{20}\}$	$\{s_{19}\}$
a_7	$\{s_8\}$	$\{s_9\}$
a_8	$\{s_6\}$	$\{s_6\}$
a_9	$\{s_{14}\}$	$\{s_8\}$
a_{10}	$\{s_2\}$	$\{s_{12}\}$

Finally, we give each rank a standard score. Regarding our example, we consider the following standard score of each rank a_i:

Rank a_1 takes 10 standard scores of each subject.
Rank a_2 takes 9 standard scores of each subject.
⋮
Rank a_{10} takes 1 standard score of each subject.

Any rank a_m such that $m > 10$ takes standard zero score of each subject.

For each map f_j of a double-framed soft set $(f_{\text{Maths}}, f_{\text{Physics}}, A)$ and each student $s_i \in S$, we calculate the value of each pair (s_i, f_j) of Table 3 by the following rule:

$$\left(s_i, f_j\right) = \begin{cases} \text{the standard score } a_m, & s_i \in f_j\left(a_m\right), \\ 0, & s_i \notin \left(f_j, A\right). \end{cases} \tag{16}$$

We sum the standard scores of all subjects for each student and then decide the student's rank depending on the summation of his/her standard scores.

Table 3 illustrates this step.

Table 3: Students' rank.

Student	f_j: f_{Maths}	f_{Physics}	Total	Rank
s_1	5	0	5	7^{th}
s_2	1	0	1	9^{th}
s_3	7	10	17	2^{nd}
s_4	0	0	0	10^{th}
s_5	9	9	18	1^{st}
s_6	3	3	6	6^{th}
s_7	0	5	5	7^{th}
s_8	4	2	6	6^{th}
s_9	10	4	14	4^{th}
s_{10}	0	0	0	10^{th}
s_{11}	0	0	0	10^{th}
s_{12}	0	1	1	8^{th}
s_{13}	0	0	0	13^{th}
s_{14}	2	0	2	8^{th}
s_{15}	9	6	15	3^{rd}
s_{16}	10	7	17	2^{nd}
s_{17}	6	8	14	4^{th}
s_{18}	0	0	0	10^{th}
s_{19}	0	5	5	7^{th}
s_{20}	5	8	13	5^{th}

One can note from the above table that we can decide four wining students: s_5 is the first, s_3 and s_{16} are the second, and s_{15} is the third. However, the last wining student is chosen from the set $\{s_9, s_{17}\}$. The method of choosing them can be done by ways such as interview, total marks, or random lottery.

In the following, we present an algorithm of determining the wining students.

On the contrary, if the subjects f_j are not of equal significance, that is, Ministry of education imposes weights on the subjects, i.e., corresponding to each subject f_j, there is a weight $w_i \in 0, 1$.

Step 1. Examine the applicants in the specified subjects.
Step 2. Input the marks of each applicant in the specified subjects (see Table 1).
Step 3. Determine the range of rank a_i: $i = 1, 2, \ldots, n$
Step 4. Classify the students according to the proposed range rank of each subject (see Table 2).
Step 5. Give each rank a standard score.
Step 6. Sum the standard scores of all subjects for each student (see Table 3).
Step 7. Order the column of the total standard scores in descending order.
Step 8. Choose the first students according to the permissible range, if there are more than one student in the last chosen rank, then you can compare between them by interview, or total marks, or random lottery.

Algorithm 1: Algorithm of determining the winning students in the case of equal significance.

Step 1. Repeat Step 1–Step 5 of Algorithm 1.
Step 2. Find a weighted table of the subjects f_j according to the weights decided by the organizer of the competition, and the weights are denoted by w_i: $i = 1, 2, \ldots, m$.
Step 3. Multiple each standard score with its corresponding weight (see Table 4).
Step 4. Sum the weight standard scores of all subjects for each student.
Step 5. Order the column of the total standard scores in descending order.
Step 6. Choose the first students according to the permissible range, if there are more than one student in the last chosen rank, then you can compare between them by interview, or total marks, or random lottery.

Algorithm 2: Algorithm of determining the winning students in the case of different significance.

Table 4: Students' weight rank.

	f_j, w_i			
S	$f_{\text{Maths}}, W_1 = 30\%$	$H_{\text{Physics}}, W_2 = 70\%$	Total	Rank
s_1	1.5	0	1.5	11^{th}
s_2	0.3	0	0.3	15^{th}
s_3	2.1	7	9.1	1^{st}
s_4	0	0	0	16^{th}
s_5	2.7	6.3	9	2^{nd}
s_6	0.9	2.1	3	9^{th}
s_7	0	3.5	3.5	8^{th}
s_8	1.2	1.4	2.6	10^{th}
s_9	3	2.8	5.8	7^{th}
s_{10}	0	0	0	16^{th}
s_{11}	0	0	2.6	10^{th}
s_{12}	0	0.7	0.7	12^{th}
s_{13}	0	0	0	16^{th}
s_{14}	0.6	0	0.6	13th
s_{15}	2.7	4.2	6.9	6^{th}
s_{16}	3	4.9	7.9	3^{rd}
s_{17}	1.8	5.6	7.4	4^{th}
s_{18}	0	0	0.4	14^{th}
s_{19}	0	0	3.5	8^{th}
s_{20}	1.5	5.6	7.1	5^{th}

In this case, we modify the previous algorithm to be convenient for weighted selection.

With respect to our example (Algorithm 2), suppose that the weights 30% and 70% are, respectively, corresponding to maths and physics subjects. Then, we update Table 3 to be as follows.

Now, one can note from the above table that the five wining students are as follows: s_3 is the first, s_5 is the second, s_{16} is the third, s_{17} is the fourth, and s_{20} is the fifth.

5. Conclusions

In this article, we have initiated four types of belong relations and four types of nonbelong relations between an ordinary point and double-framed soft sets. These relations are primary indicator of the degree of membership and nonmembership of an element. Then, we have defined soft mappings between two classes of double-framed soft sets and determine the conditions under which an ordinary point and its image and preimage are preserved with respect to the different types of belong and nonbelong relations. In the end, we have exploited the idea of double-framed soft sets to investigate an educational application of choosing the best students in terms of their performance rank in all testing criteria. An algorithm of the application was explained with the aid of an illustrative example.

We draw attention to that the different types of belong and nonbelong relations classify the relationships between elements and double-framed soft sets into eight levels as well as classify the stability into two levels. One of the unique properties of these relations is the possibility of belonging and nonbelonging of the element to the same double-framed soft sets with respect to weakly partial belong and weakly partial nonbelong relations, strongly partial belong and strongly partial nonbelong relations, and weakly total belong and weakly total nonbelong relations. This matter leads to new relations between belonging and nonbelonging of the ordinary points and the soft intersection and union of double-framed soft sets.

As future works, we shall apply the relations presented in this work to formulate several types of soft separation axioms and compact spaces on double-framed soft topological spaces. To simplify and clarify this idea, we define four types of covers of a double-framed soft topological space using weakly partial belong, strongly partial belong, weakly total belong, and strongly total belong relations. In addition, we try to model some natural phenomena using the idea of N-framed soft set. It is worthy to note that one can extend this work by studying the belong and nonbelong relations introduced herein with respect to N-framed soft sets, where $N = 3, 4, \ldots$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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