Research Article

On Some Complete Monotonicity of Functions Related to Generalized k – Gamma Function

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Received 30 March 2021; Accepted 10 May 2021; Published 24 May 2021

Academic Editor: Nasser Saad

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In this paper, we presented two completely monotonic functions involving the generalized k –gamma function $\Gamma_k(x)$ and its logarithmic derivative $\psi_k(x)$, and established some upper and lower bounds for $\Gamma_k(x)$ in terms of $\psi_k(x)$.

1. Introduction

The ordinary gamma function is given by the following equation [1]:

$$\Gamma(u) = \lim_{n \rightarrow \infty} \frac{n! n^{u-1}}{u(u+1)(u+2)\dots(u+(n-1))}, \quad u > 0, \quad (1)$$

which was discovered by Euler when he generalized the factorial function to noninteger values. Lots of mathematicians studied the gamma function because of its great importance; for complete studies of the gamma function, please refer to [2, 3]. The digamma function is the logarithmic derivative of the gamma function and is given by [4]:

$$\psi(u) = -\gamma - \frac{1}{u} + \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{u+i} \right), \quad u > 0, \quad (2)$$

where $\gamma = \lim_{n \rightarrow \infty} (\sum_{s=1}^n (1/s) - \log n)$ is Euler–Mascheroni's constant. A function $L(x)$ defined on an interval I is said to be completely monotonic if it possesses derivatives $L^{(n)}(u)$ for all $n \in \mathbb{N}$ such that

$$(-1)^n L^{(n)}(u) \geq 0, \quad u \in I; n \in \mathbb{N}. \quad (3)$$

Completely monotonic functions have remarkable applications in different fields such as in the theory of special functions, numerical and asymptotic analysis, probability, and physics. Some important properties of these functions were collected in [5], and for more information about this topic, we refer the reader to [6, 7].

In 2007, Alzer and Batir [8] proved that the function

$$M_\alpha(u) = \ln \Gamma(u) - u \ln u + u - \ln \sqrt{2\pi} + \frac{1}{2} \psi(u + \alpha), \quad \alpha \geq 0, \quad (4)$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \geq (1/3)$, and they also proved the function $-M_\alpha(u)$ is completely monotonic on $(0, \infty)$ if and only if $\alpha = 0$. As a consequence, they introduced the following sharp bounds for Γ function in terms of ψ function:

$$\exp \left[-\frac{1}{2} \psi(u+c) \right] < \frac{\Gamma(u)}{(\sqrt{2\pi} u^u e^{-u})} < \exp \left[-\frac{1}{2} \psi(u+d) \right], \quad u > 0, \quad (5)$$

with the best possible constants $c = (1/3)$ and $d = 0$. In order to refine inequality (5), Batir [9] modified the function $M_0(u)$ and proved that the function

$$T_\delta(u) = M_0(u) + \frac{1}{6(u-\delta)}, \quad (6)$$

which is completely monotonic on $(0, \infty)$ if and only if $\delta \geq (1/4)$, and the function $-T_\delta(u)$ is completely monotonic

$$\exp\left[-\frac{1}{2}\psi(u) - \frac{1}{6(u-a)}\right] < \frac{\Gamma(u)}{(\sqrt{2\pi}u^u e^{-u})} < \exp\left[-\frac{1}{2}\psi(u) - \frac{1}{6(u-b)}\right], \quad u > 0, \quad (7)$$

with the best possible constants $a = (1/4)$ and $b = 0$.

In many contexts, such as the combinatorics of creation, the annihilation operators, and the perturbative computation of Feynman integrals, the following symbol appears [10–12]:

$$(u)_{n,k} = u(u+k)(u+2k), \dots, (u+(n-1)k), \quad u \text{ and } k > 0; n \in \mathbb{N}, \quad (8)$$

which is a generalization of the ordinary Pochhammer symbol $(u)_n = ((\Gamma(u+n))/(\Gamma(u)))$, when $k = 1$. Diaz and Pariguan [13] were motivated by the importance of $(u)_{n,k}$, and they called it “the Pochhammer k -symbol,” and they introduced the k -gamma function by

$$\Gamma_k(u) = \lim_{n \rightarrow \infty} \frac{n!k^n (nk)^{(u/k)-1}}{(u)_{n,k}} = \int_0^\infty v^{u-1} e^{-(v^k/k)} dv, \quad u \text{ and } k > 0, \quad (9)$$

which satisfies the functional equation:

$$\begin{aligned} \Gamma_k(u+k) &= u\Gamma_k(u), \\ \Gamma_k(k) &= 1, \quad u, k > 0. \end{aligned} \quad (10)$$

As special cases, $\Gamma_1(u)$ is the ordinary gamma function and the case $k = 2$ is of particular interest since $\Gamma_2(u) = \int_0^\infty v^{u-1} e^{-(v^2/2)} dv$ is the Gaussian integral. The k -gamma and ordinary gamma functions are related by the relation

$$\Gamma_k(u) = k^{(u/k)-1} \Gamma\left(\frac{u}{k}\right), \quad u \text{ and } k > 0. \quad (11)$$

The k -analogue of the digamma function is given by [14]:

$$\psi_k(u) = \frac{\ln k - \gamma}{k} - \frac{1}{u} + \sum_{n=1}^{\infty} \left[\frac{1}{nk} - \frac{1}{nk+u} \right], \quad u \text{ and } k > 0, \quad (12)$$

and it satisfies the following relations for u and $k > 0$ and $m = 0, 1, 2, \dots$:

$$\begin{aligned} \psi_k^{(m)}(u+k) &= \frac{(-1)^m m!}{u^{m+1}} + \psi_k^{(m)}(u), \\ \psi_k(k) &= \frac{\ln k - \gamma}{k}. \end{aligned} \quad (13)$$

In 2018, Nantomah et al. [15] presented the following integral representations:

on $(0, \infty)$ if and only if $\delta \leq 0$. As a consequence, he deduced the following refinement of the inequality (5):

$$\psi_k(u) = \int_0^\infty \left[\frac{2e^{-t} - e^{-kt}}{kt} - \frac{e^{-ut}}{1 - e^{-kt}} \right] dt, \quad u \text{ and } k > 0, \quad (14)$$

$$\psi_k^{(m)}(u) = (-1)^{m+1} \int_0^\infty t^m \left(\frac{e^{-ut}}{1 - e^{-kt}} \right) dt, \quad u \text{ and } k > 0; m \in \mathbb{N}. \quad (15)$$

Ege and Yildirim [16] obtained an inequality involving $\psi'_k(u+k)$ and $\psi''_k(u+k)$ which can be written, by using (13), as

$$-u\psi''_k(u) - \frac{1}{u^2} - \frac{1}{ku} < \psi'_k(u) - \frac{1}{ku} < -u\psi''_k(u) - \frac{1}{2u^2} - \frac{1}{ku}, \quad k \text{ and } u > 0. \quad (16)$$

In 2020, Yildirim [17] presented some monotonicity properties for $\psi_k^{(m)}$ functions, and he deduced the following inequalities:

$$\frac{1}{2u} < \frac{1}{k} \ln u - \psi_k(u) < \frac{1}{2u} + \frac{k}{12u^2}, \quad u \text{ and } k > 0, \quad (17)$$

$$\frac{1}{2u^2} + \frac{k}{6u^3} - \frac{k^3}{30u^5} < \psi'_k(u) - \frac{1}{ku} < \frac{1}{2u^2} + \frac{k}{6u^3}, \quad u \text{ and } k > 0, \quad (18)$$

$$\frac{1}{u^3} < -\psi''_k(u) - \frac{1}{ku^2} < \frac{1}{u^3} + \frac{k}{2u^4}, \quad u \text{ and } k > 0. \quad (19)$$

For more information about Γ_k and ψ_k functions, we refer the reader to [18–20] and the related references therein.

Nantomah et al. [21] introduced the following two-parameter deformation of the gamma and digamma functions for u and $k > 0$ and $p \in \mathbb{N}$:

$$\begin{aligned} \Gamma_{p,k}(u) &= \frac{(p+1)!k^{p+1}(pk)^{(u/k)-1}}{u(u+k)(u+2k), \dots, (u+pk)} = \int_0^p v^{u-1} \left(1 - \frac{v^k}{pk} \right)^p dv, \\ \psi_{p,k}(u) &= \frac{1}{k} \ln(pk) - \sum_{i=0}^p \frac{1}{u+ik} = \frac{1}{k} \ln(pk) - \int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} \right) e^{-ut} dt. \end{aligned} \quad (20)$$

Nantomah et al. [22] introduced some complete monotonicity properties for $\Gamma_{p,k}$, and they deduced the following inequalities:

$$0 \leq \frac{1}{k} \ln \left(\frac{pku}{u + pk + k} \right) - \psi_{p,k}(u) \leq \frac{1}{u} - \frac{1}{(u + pk + k)}, \quad u \text{ and } k > 0; p \in \mathbb{N}, \quad (21)$$

$$0 \leq \psi_{p,k}'(u) - \frac{1}{k} \left[\frac{1}{u} - \frac{1}{u + pk + k} \right] \leq \frac{1}{u^2} - \frac{1}{(u + pk + k)^2}, \quad u \text{ and } k > 0; p \in \mathbb{N}. \quad (22)$$

Motivated by these results, we will present a k -analogue of the functions $M_\alpha(u)$ and $T_\delta(u)$ and study their monotonicity. As a consequence, we deduce some inequalities for Γ_k and $\psi_k^{(m)}$ functions.

2. Preliminary

Using relation (11), we have

$$\psi_k(x) = \frac{1}{k} \ln k + \frac{1}{k} \psi \left(\frac{x}{k} \right), \quad x \text{ and } k > 0, \quad (23)$$

$$\psi_k^{(m)}(x) = \frac{1}{k^{m+1}} \psi^{(m)} \left(\frac{x}{k} \right), \quad x \text{ and } k > 0; m \in \mathbb{N}.$$

Also, using the following asymptotic formula of Γ function [4]

$$\ln \Gamma(y) \sim \left(y - \frac{1}{2} \right) \ln y - y + \ln \sqrt{2\pi} \quad (24)$$

$$+ \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)(2i-1)y^{2i-1}}, \quad y \longrightarrow \infty,$$

with relation (11), we conclude for $k > 0$ that

$$\ln \Gamma_k(x) \sim \ln \left(\sqrt{\frac{2\pi}{k}} \right) + \left(\frac{x}{k} - \frac{1}{2} \right) \ln x - \frac{x}{k} + \sum_{i=1}^{\infty} \frac{k^{2i-1} B_{2i}}{(2i)(2i-1)x^{2i-1}}, \quad x \longrightarrow \infty, \quad (25)$$

$$\psi_k(x) \sim \frac{1}{k} \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{k^{2i-1} B_{2i}}{(2i)x^{2i}}, \quad x \longrightarrow \infty, \quad (26)$$

and for $s \in \mathbb{N}$,

$$\begin{aligned} \psi_k^{(s)}(x) &\sim \frac{(-1)^{s-1} (s-1)!}{kx^s} - \frac{(-1)^s s!}{2x^{s+1}} \\ &+ (-1)^{s+1} \sum_{i=1}^{\infty} \frac{(s+2i-1)! k^{2i-1} B_{2i}}{(2i)! x^{2i+s}}, \quad x \longrightarrow \infty. \end{aligned} \quad (27)$$

In the following, we will use the following important result [23].

Corollary 1. Let S be a real-valued function defined on $x > x_0$; $x_0 \in \mathbb{R}$ with $S(x)$ tends to zero as $x \longrightarrow \infty$. Then, for $r \in \mathbb{R}^+$, $S(x) > 0$, if $S(x+r) < S(x)$ for all $x > x_0$ and $S(x) < 0$, if $S(x+r) > S(x)$ for all $x > x_0$.

Now, we will prove the following auxiliary results.

Lemma 1. For $k > 0$, we have

$$\begin{aligned} \psi_k' \left(x + \frac{k}{3} \right) &> \frac{1}{kx}, \quad \forall x > \frac{k}{3}, \\ \psi_k'' \left(x + \frac{k}{3} \right) &< \frac{2x}{k} \psi_k''(x) + \frac{2}{kx^2} + \frac{2}{k^2 x}, \quad \forall x \geq \frac{83k}{100}, \\ \psi_k''' \left(x + \frac{k}{3} \right) &> \frac{2}{kx^3}, \quad \forall x \geq \frac{76k}{100}. \end{aligned} \quad (28)$$

Proof. Consider the function $F_k(x) = \psi_k'(x + (k/3)) - (1/kx)$, and using (13), we get

$$\begin{aligned} F_k(x+k) - F_k(x) &= \frac{-1}{(x + (k/3))^2} + \frac{1}{x(x+k)} \\ &= \frac{-k(3x-k)}{x(x+k)(3x+k)^2} < 0, \quad \forall x > \frac{k}{3}, \end{aligned} \quad (29)$$

and using asymptotic expansion (27), we have $\lim_{x \rightarrow \infty} F_k(x) = 0$. Then, Corollary 1 gives us that $F_k(x) > 0$

for all $x > (k/3)$. Now, we let $H_k(x) = -(k/2)\psi_k''(x + (k/3)) + x\psi_k''(x) + (1/x^2) + (1/kx)$ and use (13) to get

$$H_k(x+k) - H_k(x) = k\psi_k''(x+k) + \frac{k^5 + 10k^4x + 10k^3x^2 + 9k^2x^3 + 27kx^4 + 27x^5}{x^2(k+x)^2(k+3x)^3},$$

$$\doteq N_k(x), \quad (30)$$

$$N_k(x+k) - N_k(x) = \frac{k^2h_k(x-0.83k)}{x^2(k+x)^3(2k+x)^2(k+3x)^3(4k+3x)^3},$$

where

$$\begin{aligned} 10^{18}h_k(x) = & 507528366752760409361k^9 + 159366156531359440200300k^8x \\ & + 10^4(76615622621545898964)k^7x^2 + 10^6(1604508702355386252)k^6x^3 \\ & + 10^8(19056153366876366)k^5x^4 + 10^{10}(140271547017402)k^4x^5 \\ & + 10^{12}(654834692196)k^3x^6 + 10^{14}(1892006748)k^2x^7 \\ & + 10^{16}(3091689)kx^8 + 10^{18}(2187)x^9 > 0, \quad x \geq 0. \end{aligned} \quad (31)$$

Then, $N_k(x+k) - N_k(x) > 0$ for all $x \geq 0.83k$ with $\lim_{x \rightarrow \infty} N_k(x) = 0$; hence, we have $N_k(x) < 0$ for all $x \geq 0.83k$. It follows that $H_k(x+k) - H_k(x) < 0$ for all $x \geq 0.83k$ with $\lim_{x \rightarrow \infty} H_k(x) = 0$. Then, $H_k(x) > 0$ for all $x \geq 0.83k$. Finally, consider the function $L_k(x) = (k/2)\psi_k''(x + (k/3)) - (1/x^3)$ and use (13) to obtain

$$L_k(x+k) - L_k(x) = \frac{-k^2f_k(x-0.76k)}{x^3(k+x)^3(k+3x)^4}, \quad (32)$$

where

$$\begin{aligned} 9765625f_k(x) = & 13183088k^5 + 2823864000k^4x \\ & + 10116487500k^3x^2 + 13331953125k^2x^3 \\ & + 7593750000kx^4 + 1582031250x^5 > 0, \end{aligned} \quad (33)$$

for $x > 0$. Then, $L_k(x+k) - L_k(x) < 0$ with $\lim_{x \rightarrow \infty} L_k(x) = 0$. Hence, $L_k(x) > 0$ for all $x \geq 0.76k$.

Using relation (12) of ψ_k , we conclude the following corollary which will be useful in the next part. $\square$

Corollary 2. Let $x \in (0, \infty)$ and $k > 0$. Then, the following limits are valid:

$$\lim_{x \rightarrow 0} x^{m+1} \psi_k^{(m)}(x) = (-1)^{m+1} m!, \quad m = 0, 1, 2, \dots, \quad (34)$$

and

$$\lim_{x \rightarrow 0} x^m \psi_k^{(m)}(x + kb) = 0, \quad m \in \mathbb{N} \text{ and } b > 0. \quad (35)$$

3. Two Completely Monotonic Functions Involving $\Gamma_k(x)$ and $\psi_k(x)$ Functions

Theorem 1. The function

$$M_{\lambda,k}(x) = \ln \Gamma_k(x) - \ln \left(\sqrt{\frac{2\pi}{k}} \right) - \frac{x}{k} \ln x$$

$$+ \frac{x}{k} + \frac{k}{2} \psi_k(x + \lambda k), \quad \lambda \geq 0; k > 0, \quad (36)$$

is completely monotonic on $(0, \infty)$ if and only if $\lambda \geq (1/3)$, and the function $-M_{\lambda,k}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda = 0$.

Proof. Using relation (15), we get

$$M_{\lambda,k}''(x) = \psi_k'(x) - \frac{1}{kx} + \frac{k}{2} \psi_k''(x + \lambda k) = \int_0^\infty \frac{e^{-xt}}{k(e^{kt} - 1)} \Phi(t) dt, \quad (37)$$

where

$$\Phi(t) = 1 - e^{kt} + kte^{kt} - \frac{(kt)^2}{2} e^{(1-\lambda)kt}. \quad (38)$$

Let $\lambda \geq (1/3)$, then we obtain

$$\begin{aligned} \Phi(t) \geq & 1 - e^{kt} + kte^{kt} - \frac{(kt)^2}{2} e^{(2/3)kt} \\ = & \sum_{r=2}^{\infty} \left[\sum_{l=2}^r \binom{r}{l} 2^{r-l} \right] \frac{(kt)^{r+2} (r+1)}{(r+2)! 3^r} > 0. \end{aligned} \quad (39)$$

Consequently, $M_{\lambda,k}''(x)$ is completely monotonic on $(0, \infty)$; hence, $M_{\lambda,k}'(x)$ is increasing on $(0, \infty)$. Using asymptotic expansions (26) and (27), we have $\lim_{x \rightarrow \infty} M_{\lambda,k}'(x) = 0$, and then, $M_{\lambda,k}'(x) < 0$. Hence, $M_{\lambda,k}(x)$ is decreasing on $(0, \infty)$. Using asymptotic

expansions (25) and (26), we have $\lim_{x \rightarrow \infty} M_{\lambda,k}(x) = 0$; hence, $M_{\lambda,k}(x) > 0$. Therefore, $M_{\lambda,k}(x)$ is completely monotonic on $(0, \infty)$. Conversely, if $M_{\lambda,k}(x)$ is completely monotonic, then we get

$$\frac{x}{k} M_{\lambda,k}(x) = \frac{x}{k} \left[\ln \left(\frac{\Gamma_k(x)}{\sqrt{(2\pi/k)} x^{(x/k)-(1/2)} e^{-(x/k)}} \right) - \frac{1}{2} \ln x + \frac{k}{2} \psi_k(x + \lambda k) \right] > 0, \quad x \text{ and } k > 0. \quad (40)$$

From (25), we have

$$\lim_{x \rightarrow \infty} \frac{x}{k} \ln \left(\frac{\Gamma_k(x)}{\sqrt{(2\pi/k)} x^{(x/k)-(1/2)} e^{-(x/k)}} \right) = \frac{1}{12}. \quad (41)$$

Using asymptotic expansion (26), we get $\lim_{x \rightarrow \infty} (x/k)[k\psi_k(x + \lambda k) - \ln x] = \lambda - (1/2)$. Then, from (40), we conclude that $(1/12) + (1/2)(\lambda - (1/2)) \geq 0$; hence, $\lambda \geq (1/3)$. Now, for $\lambda = 0$, we have

$$M_{0,k}''(x) = \int_0^\infty \frac{e^{-xt}}{k(e^{kt} - 1)} \left(\sum_{r=1}^\infty \left[-\frac{r(r+1)}{2} \right] \frac{(kt)^{r+2}}{(r+2)!} \right) dt, \quad (42)$$

and therefore, $-M_{0,k}''(x)$ is completely monotonic on $(0, \infty)$. Thus, $M_{0,k}'(x)$ is decreasing on $(0, \infty)$ with $\lim_{x \rightarrow \infty} M_{0,k}'(x) = 0$; then, $M_{0,k}'(x) > 0$. Thus, $M_{0,k}(x)$ is increasing on $(0, \infty)$ with $\lim_{x \rightarrow \infty} M_{0,k}(x) = 0$; hence, $M_{0,k}(x) < 0$. Then, $-M_{0,k}(x)$ is completely monotonic on $(0, \infty)$. Conversely, we assume that $-M_{\lambda,k}(x)$ (with $\lambda > 0$) is completely monotonic on $(0, \infty)$; then, $M_{\lambda,k}(x)$ is negative on $(0, \infty)$. But this contradicts that $\lim_{x \rightarrow 0} M_{\lambda,k}(x) = \infty$; hence, $\lambda = 0$. $\square$

Theorem 2. The function

$$U_{\beta,k}(x) = \ln \Gamma_k(x) - \ln \left(\sqrt{\frac{2\pi}{k}} \right) - \frac{x}{k} \ln x + \frac{x}{k} + \frac{k}{2} \psi_k(x) + \frac{k}{6(x - \beta k)}, \quad x \text{ and } k > 0, \quad (43)$$

is completely monotonic on $(0, \infty)$ if and only if $\beta \geq (1/4)$. Also, $-U_{\beta,k}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \leq 0$.

Proof. Using relation (15), we get

$$U_{\beta,k}''(x) = \psi_k'(x) - \frac{1}{kx} + \frac{k}{2} \psi_k''(x) + \frac{k}{3(x - \beta k)^3} = \int_0^\infty \frac{e^{-xt}}{k(e^{kt} - 1)} \varphi(t) dt, \quad (44)$$

where

$$\varphi(t) = 1 - e^{kt} + kte^{kt} - \frac{(kt)^2}{2} e^{kt} + \frac{(kt)^2}{6} e^{\beta kt} (e^{kt} - 1). \quad (45)$$

Let $\beta \geq (1/4)$, then we obtain

$$\varphi(t) \geq \frac{11(kt)^5}{2880} + \frac{7(kt)^6}{2304} + \sum_{r=5}^\infty \left(\frac{f(r)}{(r+2)!} (kt)^r \right) > 0, \quad t > 0, \quad (46)$$

where

$$f(r) = \frac{(r+1)}{6} \left[r \left(\left(\frac{5}{4} \right)^r - 3 \right) + \frac{1}{4^r} \left(7r + 2 \sum_{l=2}^r \binom{r}{l} 4^l \right) \right] > 0, \quad r \geq 5. \quad (47)$$

Then, $U_{\beta,k}''(x)$ is completely monotonic on $(0, \infty)$. Thus, $U_{\beta,k}'(x)$ is an increasing function on $(0, \infty)$; using asymptotic expansions (26) and (27), we get that $\lim_{x \rightarrow \infty} U_{\beta,k}'(x) = 0$. Then, $U_{\beta,k}(x)$ is a decreasing function with $\lim_{x \rightarrow \infty} U_{\beta,k}(x) = 0$; hence, $U_{\beta,k}(x) > 0$. Thus, we deduce that $U_{\beta,k}(x)$ is a completely monotonic function on $(0, \infty)$. Conversely, if $U_{\beta,k}(x)$ is a completely monotonic function, then it is positive and we obtain

$$\beta \geq \lim_{x \rightarrow \infty} \left[\frac{x}{k} + \frac{1}{6(\ln \Gamma_k(x) - \ln(\sqrt{(2\pi/k)}) - (x/k) \ln x + (x/k) + (k/2) \psi_k(x))} \right], \quad k > 0. \quad (48)$$

Using asymptotic expansions (25) and (26), we have

$$\ln \Gamma_k(x) - \ln \left(\sqrt{\frac{2\pi}{k}} \right) - \frac{x}{k} \ln x + \frac{x}{k} + \frac{k}{2} \psi_k(x) = \frac{-k(4x+k)}{24x^2} + O\left(\frac{1}{x^3}\right). \quad (49)$$

Then,

$$\beta \geq \lim_{x \rightarrow \infty} \left[\frac{x}{k} - \frac{4x^2}{k(4x+k)} \right] = \frac{1}{4}. \quad (50)$$

Now, for $\beta \leq 0$, then

$$\begin{aligned} \varphi(t) &\leq 1 - e^{kt} + kte^{kt} - \frac{(kt)^2}{3}e^{kt} - \frac{(kt)^2}{6} \\ &= -\frac{1}{3} \sum_{r=2}^{\infty} \frac{(r^2-1)(kt)^{r+2}}{(r+2)!} < 0, \end{aligned} \quad (51)$$

and hence, $-U_{\beta,k}''(x)$ is a completely monotonic function on $(0, \infty)$ for $\beta \leq 0$. Now, $U_{\beta,k}'(x)$ is a decreasing function on $(0, \infty)$ with $\lim_{x \rightarrow \infty} U_{\beta,k}'(x) = 0$, and then, $U_{\beta,k}'(x) > 0$. Thus, $U_{\beta,k}(x)$ is an increasing function on $(0, \infty)$ with $\lim_{x \rightarrow \infty} U_{\beta,k}(x) = 0$; hence, $U_{\beta,k}(x) < 0$. Therefore, we deduce that $-U_{\beta,k}(x)$ is a completely monotonic function on $(0, \infty)$ for $\beta \leq 0$. Conversely, if $-U_{\beta,k}(x)$ is a completely monotonic function, then $U_{\beta,k}(x) < 0$ and

$$\beta \leq \lim_{x \rightarrow 0} \left[\frac{x}{k} + \frac{1}{6 \ln \Gamma_k(x) - 6 \ln(\sqrt{(2\pi/k)}) - (6x/k) \ln x + (6x/k) + 3k\psi_k(x)} \right], \quad k > 0. \quad (52)$$

Using functional relations (10) and (13), we get $\beta \leq \lim_{x \rightarrow 0} [(x/A_k(x))] = 0$, where

$$\begin{aligned} A_k(x) &= 6x \ln \Gamma_k(x+k) - 6x \ln \left(\sqrt{\frac{2\pi}{k}} \right) \\ &\quad - 6 \left(x + \frac{x^2}{k} \right) \ln x + \frac{6x^2}{k} + 3kx\psi_k(x+k) - 3k. \end{aligned} \quad (53)$$

3.1. Some Inequalities for $\Gamma_k(x)$, $\psi_k(x)$, and $\psi_k^{(m)}(x)$ Functions. As a consequence of the completely monotonicity properties of the two functions $M_{\lambda,k}(x)$ and $U_{\beta,k}(x)$, we conclude the following results.

Corollary 3. For a and $b \geq 0$, the following inequality holds:

$$\begin{aligned} \exp \left[\frac{k}{2} \psi_k(x+ka) \right] &< \frac{\Gamma_k(x)}{\left(\sqrt{(2\pi/k)} x^{(x/k)} e^{-(x/k)} \right)} \\ &< \exp \left[\frac{k}{2} \psi_k(x+kb) \right], \quad x \text{ and } k > 0, \end{aligned} \quad (54)$$

with the best possible constants $a = (1/3)$ and $b = 0$.

Proof. The left-hand side of inequality (54) is equivalent $(x/k)M_{a,k}(x) > 0$ which leads to $a \geq (1/3)$ as mentioned in the proof of Theorem 1. Using the increasing property of $\psi_k(x)$ on $(0, \infty)$, we get $e^{-(k/2)\psi_k(x+(k/3))} \geq e^{-(k/2)\psi_k(x+ka)}$ for

$a \geq (1/3)$, and then, $a = (1/3)$ is the best possible constant in (54). Also, Theorem 1 gives the right-hand side of inequality (54) for $b = 0$. If there exists $b > 0$ such that the right-hand side of (54) is valid for $x \in (0, \infty)$, then we would have

$$\lim_{x \rightarrow 0} \Gamma_k(x) < \sqrt{\frac{2\pi}{k}} \exp \left[\frac{k}{2} \psi_k(kb) \right], \quad (55)$$

which contradicts that $\lim_{x \rightarrow 0} \Gamma_k(x) = \infty$. Hence, $b = 0$ is the best possible constant in (54). $\square$

Remark 1. If we put $k = 1$ in inequality (54), then we get inequality (5).

Corollary 4. For c and $d \geq 0$, the following inequality holds:

$$\frac{k}{2} \psi_k'(x+kc) < \frac{1}{k} \ln x - \psi_k(x) < \frac{k}{2} \psi_k'(x+kd), \quad x \text{ and } k > 0, \quad (56)$$

with the best possible constants $c = (1/3)$ and $d = 0$.

Proof. The left-hand side of inequality (56) is equivalent that

$$\frac{x^2}{k} M_{c,k}'(x) = \frac{x^2}{k} S_k(x) + \frac{x^2}{2k} \left[k\psi_k'(x+kc) - \frac{1}{x} - \frac{k}{6x^2} \right] < 0, \quad (57)$$

where $S_k(x) = \psi_k(x) - (1/k) \ln x + (1/2x) + (k/12x^2)$. Using asymptotic expansions (26) and (27), we have $\lim_{x \rightarrow \infty} (x^2/k) S_k(x) = 0$ and

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2}{2k} \left[k\psi_k'(x+kc) - \frac{1}{x} - \frac{k}{6x^2} \right] &= \lim_{x \rightarrow \infty} \frac{x^2}{2k} \left[\frac{k(2x^2 - 2kcx - c^2k^2)}{6x^2(x+ck)^2} \right. \\ &\quad \left. - \frac{ck}{x(x+ck)} + O\left(\frac{1}{x^3}\right) \right] = \frac{1}{2} \left(-c + \frac{1}{3} \right). \end{aligned} \quad (58)$$

Hence, we conclude from (57) that $(1/2)(-c + (1/3)) \leq 0$, and then, $c \geq (1/3)$. Using the decreasing property of $\psi'_k(x)$ on $(0, \infty)$, we get that $c = (1/3)$ is the best possible constant in (56). Also, Theorem 1 gives the right-hand side of inequality (56) for $d = 0$. If there exists $d > 0$ such that the right-hand side of (56) is valid for $x \in (0, \infty)$, then we would have

$$-\lim_{x \rightarrow 0} x\psi_k(x) < \frac{k}{2} \lim_{x \rightarrow 0} x\psi'_k(x + kd). \quad (59)$$

From (34) and (35), we have $\lim_{x \rightarrow 0} x\psi_k(x) = -1$ and $\lim_{x \rightarrow 0} x\psi'_k(x + kd) = 0$ which contradict with (59). Hence, the best possible constant in (56) is $d = 0$. $\square$

Remark 2. The lower bound of (56) improves the lower bound of (21), when $p \rightarrow \infty$, for all $x > 0$.

Remark 3. From Lemma 1, we conclude that the lower bound of (56) improves the lower bound of (17) for all $x > (k/3)$.

Corollary 5. For g and $h \geq 0$, x and $k > 0$, and $m = 2, 3, 4, \dots$, the following inequality holds:

$$\begin{aligned} -\frac{(-1)^m k}{2} \psi_k^{(m)}(x + gk) &< (-1)^m \psi_k^{(m-1)}(x) - \frac{(m-2)!}{kx^{m-1}} \\ &< -\frac{(-1)^m k}{2} \psi_k^{(m)}(x + hk), \end{aligned} \quad (60)$$

with the best possible constants $g = (1/3)$ and $h = 0$.

Proof. For $m = 2, 3, \dots$, the left-hand side of (60) is equivalent that

$$\begin{aligned} \frac{(-1)^m x^{m+1}}{k} M_{c,k}^{(m)}(x) &= \frac{(-1)^m x^{m+1}}{k} V_k(x) + \frac{x^{m+1}}{2k} \left[(-1)^m k \psi_k^{(m)}(x + gk) \right. \\ &\quad \left. + \frac{(m-1)!}{x^m} + \frac{km!}{6x^{m+1}} \right] > 0, \end{aligned} \quad (61)$$

where

$$(-1)^m V_k(x) = (-1)^m \psi_k^{(m-1)}(x) - \frac{(m-2)!}{kx^{m-1}} - \frac{(m-1)!}{2x^m} - \frac{km!}{12x^{m+1}}. \quad (62)$$

Using asymptotic expansion (27), we have $\lim_{x \rightarrow \infty} ((-1)^m x^{m+1}/k) V_k(x) = 0$ and

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^{m+1}}{2k} \left[(-1)^m k \psi_k^{(m)}(x + gk) + \frac{(m-1)!}{x^m} + \frac{km!}{6x^{m+1}} \right] \\ = \lim_{x \rightarrow \infty} \frac{x^{m+1}}{2k} \left[\frac{(m-1)! g k \sum_{i=0}^{m-1} x^{m-1-i} (x + gk)^i}{x^m (x + gk)^m} + \frac{km!}{6x^{m+1}} \right. \\ \left. - \frac{km!}{2(x + gk)^{m+1}} + O\left(\frac{1}{x^{m+2}}\right) \right], \\ = \frac{m!}{2} \left(g - \frac{1}{3} \right). \end{aligned} \quad (63)$$

Hence, we conclude from (61) that $(m!/2)(g - (1/3)) \geq 0$, and this leads to $g \geq (1/3)$. Since $\psi'_k(x)$ is strictly completely monotonic on $(0, \infty)$, then the function $(-1)^m \psi_k^{(m)}(x)$ is increasing on $(0, \infty)$ for $m = 0, 1, 2, \dots$, and then, $g = (1/3)$ is the best possible constant in (60). Also, Theorem 1 gives the right-hand side of inequality (60) for $h = 0$. If there exists $h > 0$ such that the right-hand side of (60) is valid for $x \in (0, \infty)$, then we would have

$$(-1)^m \lim_{x \rightarrow 0} x^m \psi_k^{(m-1)}(x) < \frac{(-1)^{m+1} k}{2} \lim_{x \rightarrow 0} x^m \psi_k^{(m)}(x + hk), \quad m = 2, 3, \dots, h > 0. \quad (64)$$

From (34) and (35), we have $\lim_{x \rightarrow 0} x^m \psi_k^{(m-1)}(x) = (-1)^m (m-1)!$ and also $\lim_{x \rightarrow 0} x^m \psi_k^{(m)}(x + hk) = 0$, and

this contradicts with (64). Then, the best possible constant in (60) is $h = 0$. $\square$

Remark 4. For $m = 2$, the lower bound of (60) improves the lower bound of (22), when $p \rightarrow \infty$, for all $x > 0$.

Remark 5. From Lemma 1, we conclude that the lower bound of (60) for $m = 2$ improves the lower bound of (16) for all $x \geq 0.83k$.

Remark 6. From Lemma 1, we conclude that the lower bound of (60) for $m = 3$ improves the lower bound of (19) for all $x \geq 0.76k$.

Corollary 6. The following inequality holds:

$$\exp\left[-\frac{k}{2}\psi_k(x) - \frac{k}{6(x-k/4)}\right] < \frac{\Gamma_k(x)}{(\sqrt{(2\pi/k)}x^{(x/k)}e^{-(x/k)})} < \exp\left[-\frac{k}{2}\psi_k(x) - \frac{k}{6x}\right], \quad x \text{ and } k > 0. \quad (65)$$

Proof. The left-hand side of inequality (65) is equivalent $U_{(1/4),k}(x) > 0$, and the right-hand side of inequality (65) is equivalent $U_{0,k}(x) < 0$ in Theorem 2. $\square$

Remark 7. If we let $k = 1$ in inequality (65), then we obtain inequality (7).

Remark 8. The upper bound of inequality (65) improves the upper bound of inequality (54) for all $x > 0$.

Corollary 7. The following inequality holds:

$$\frac{k}{2}\psi'_k(x) - \frac{k}{6(x-k/4)^2} < \frac{1}{k} \ln x - \psi_k(x) < \frac{k}{2}\psi'_k(x) - \frac{k}{6x^2}, \quad x \text{ and } k > 0. \quad (66)$$

Proof. The left-hand side of inequality (66) is equivalent $U_{(1/4),k}'(x) < 0$, and the right-hand side of inequality (66) is equivalent $U_{0,k}'(x) > 0$ in Theorem 2. $\square$

Remark 9. The upper bound of inequality (66) improves the upper bound of inequality (56) for all $x > 0$.

Corollary 8. The following inequality holds:

$$\begin{aligned} & -\frac{(-1)^m k}{2} \psi_k^{(m)}(x) - \frac{km!}{6(x-k/4)^{m+1}} < (-1)^m \psi_k^{(m-1)}(x) - \frac{(m-2)!}{kx^{m-1}} \\ & < -\frac{(-1)^m k}{2} \psi_k^{(m)}(x) - \frac{km!}{6x^{m+1}}, \quad x \text{ and } k > 0; m = 2, 3, \dots, \end{aligned} \quad (67)$$

Proof. The left-hand side of inequality (67) is equivalent $(-1)^m U_{(1/4),k}^{(m)}(x) > 0$, and the right-hand side of inequality (67) is equivalent $(-1)^m U_{0,k}^{(m)}(x) < 0$ in Theorem 2. $\square$

Remark 10. The upper bound of inequality (67) improves the upper bound of inequality (60) for all $x > 0$.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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