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Research Article

Common α-Fuzzy Fixed Point Results with Applications to Volterra Integral Inclusions

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The purpose of this paper is to establish some common α-fuzzy fixed point theorems for a pair fuzzy mappings and obtain some results of literature for multivalued mappings. For it, we define the notion of generalized Θ-contractions in the context of b-metric spaces. As applications, we investigate the solutions of Volterra integral inclusions by our established results.

1. Introduction and Preliminaries

Among all the impressive and inspiring generalizations of metric spaces, b-metric space has an integral place. Czerwik [1] in 1993 extended the notion of metric space by introducing the conception of b-metric space in this way.

Definition 1. Let $\mathscr{M} \neq \emptyset$. A mapping $d_b: \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}_0^+$ is called b-metric if it satisfies these assertions:

$(b_1) d_b(\omega, \varpi) = 0 \Leftrightarrow \omega = \varpi$

$(b_2) d_b(\omega, \varpi) = d_b(\varpi, \omega)$

$(b_3) d_b(\omega, \upsilon) \leq s(d_b(\omega, \varpi) + d_b(\varpi, \upsilon))$

for all $\omega, \varpi, \upsilon \in \mathscr{M}$, where $s \geq 1$.

Then, $(\mathscr{M}, d_b, s)$ is called a b-metric space. A standard example of b-metric space which is not metric space is the following:

$\mathscr{M} = \mathbb{R}$ and $d_b: \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ defined by

$$d_b(\omega, \varpi) = |\omega - \varpi|^2, \tag{1}$$

for all $\omega, \varpi \in \mathscr{M}$ with $s = 2$.

Let $P_{cb}(\mathscr{M})$ represent the class of all nonempty, bounded, and closed subsets of $\mathscr{M}$. For $\Xi_1, \Xi_2, \Xi_3 \in P_{cb}(\mathscr{M})$, we define $H_b: P_{cb}(\mathscr{M}) \times P_{cb}(\mathscr{M}) \longrightarrow \mathbb{R}^+$ by

$$H_b(\Xi_1, \Xi_2) = \max\{\delta_b(\Xi_1, \Xi_2), \delta_b(\Xi_2, \Xi_1),\} \tag{2}$$

where

$$\begin{aligned} \delta_b(\Xi_1, \Xi_2) &= \sup\{d_b(\omega, \varpi): \omega \in \Xi_1, \varpi \in \Xi_2\}, \\ D_b(\omega, \Xi_3) &= D_b(\{\omega\}, \Xi_3) = \inf\{d_b(\omega, \upsilon): \omega \in \Xi_1, \upsilon \in \Xi_3\}. \end{aligned} \tag{3}$$

Note that H_b is called the Hausdorff b-metric induced by the b-metric d_b. We recall the following properties from [1–3].

Lemma 1 (see [2]). *Let $(\mathscr{M}, d_b, s)$ be a b-metric space. For any $\Xi_1, \Xi_2, \Xi_3 \in P_{cb}(\mathscr{M})$ and any $\omega, \varpi \in \mathscr{M}$, we have the following:*

(i) $D_b(\omega, \Xi_2) \leq d_b(\omega, b)$ *for any* $b \in \Xi_2$

(ii) $\delta_b(\Xi_1, \Xi_2) \leq H_b(\Xi_1, \Xi_2)$

(iii) $D_b(\omega, \Xi_2) \leq H_b(\Xi_1, \Xi_2)$ *for any* $\omega \in \Xi_1$

(iv) $H_b(\Xi_1, \Xi_2) = 0$

(v) $H_b(\Xi_1, \Xi_2) = H_b(\Xi_2, \Xi_1)$

(vi) $H_b(\Xi_1, \Xi_3) \leq s[H_b(\Xi_1, \Xi_2) + H_b(\Xi_2, \Xi_3)]$

(vii) $D_b(\omega, \Xi_1) \leq s[d_b(\omega, \varpi) + d_b(\varpi, \Xi_1)]$.

Later on, many authors (see [4–7]) worked in this way. Recently Jleli and Samet [8] gave the notion of Θ-contractions and proved a contemporary result for such contractions in generalized metric spaces. Afterwards, Hancer et al. [9] revised the foregoing definitions by including a broad condition (Θ_4). Inspired by Jleli and Samet [8] and Hancer et al. [9], Alamri et al. [10] initiated the above notions in the context of b-metric spaces and introduced a more general condition (Θ_5) along with above axioms.

Definition 2 (see [10]). We represent by $\Omega_s\,(s\geq 1)$ the family of all mappings $\Theta\colon \mathbb{R}^+ \longrightarrow (1,\infty)$ satisfying these properties:

(Θ_1) $0<\varrho_1<\varrho_2 \longrightarrow \Theta(\varrho_1)\leq\Theta(\varrho_2)$

(Θ_2) for $\{\varrho_n\}\subseteq\mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\Theta(\varrho_n)=1$ if and only if $\lim_{n\longrightarrow\infty}(\varrho_n)=0$

(Θ_3) there exists $h\in(0,1)$ and $\mathfrak{q}\in(0,\infty]$ such that $\lim_{\varrho\longrightarrow 0^+}(\Theta(\varrho)-1/\varrho^h)=\mathfrak{q}$

(Θ_4) $\Theta(\inf\Xi)=\inf\Theta(\Xi)$ for all $\Xi\subset(0,\infty)$ with $\inf\Xi>0$

(Θ_5) for all $\{\varrho_n\}\subseteq\mathbb{R}^+$ such that $\Theta(s\varrho_n)\leq[\Theta(\varrho_{n-1})]^k$, $\forall\, n\in\mathbb{N}$ and some $k\in(0,1)$, then $\Theta(s_n^n\varrho)\leq[\Theta(s^{n-1}\varrho_{n-1})]^k$, for all $n\in\mathbb{N}$

They supported this condition by the following nontrivial example.

Example 1 (see [10]). Let $\Theta\colon (0,\infty)\longrightarrow(1,\infty)$ be given by $\theta(\eta)=e^{\sqrt{\eta e^{\eta}}}$. Clearly, Θ satisfies (Θ_1)-(Θ_5). Here we show only (Θ_5). Assume that, for all $n\in\mathbb{N}$ and some $k\in(0,1)$, we have $\theta(s\varrho_n)\leq[\theta(\varrho_{n-1})]^k$, which implies that

$$\begin{aligned} e^{\sqrt{s\varrho_n e^{s\varrho_n}}} &\leq \left[e^{\sqrt{\varrho_{n-1}e^{\varrho_{n-1}}}}\right]^k,\\ \sqrt{s\varrho_n e^{s\varrho_n}} &\leq k\sqrt{\varrho_{n-1}e^{\varrho_{n-1}}}. \end{aligned}\tag{4}$$

This implies that

$$\sqrt{s\varrho_n e^{s\varrho_n-\varrho_{n-1}}}\leq k\sqrt{\varrho_{n-1}}.\tag{5}$$

As $\theta(s\varrho_n)\leq[\theta(\varrho_{n-1})]^k\leq\theta(\varrho_{n-1})$. Also θ is nondecreasing, so $s\varrho_n\leq\varrho_{n-1}$ and $s\varrho_n-\varrho_{n-1}\leq 0$ implies $e^{s^{n-1}(s\varrho_n-\varrho_{n-1})}\leq e^{s\varrho_n-\varrho_{n-1}}$. Therefore, (5) implies

$$\begin{aligned} &\sqrt{s\varrho_n e^{s^{n-1}(s\varrho_n-\varrho_{n-1})}}\leq k\sqrt{\varrho_{n-1}}\Rightarrow\sqrt{\frac{s\varrho_n e^{s^n\varrho_n}}{e^{s^{n-1}\varrho_{n-1}}}}\leq k\sqrt{\varrho_{n-1}}\\ &\Rightarrow\sqrt{s\varrho_n e^{s^n\varrho_n}}\leq k\sqrt{\varrho_{n-1}e^{s^{n-1}\varrho_{n-1}}}\Rightarrow\sqrt{s^n\varrho_n e^{s^n\varrho_n}}\leq k\sqrt{s^{n-1}\varrho_{n-1}e^{s^{n-1}\varrho_{n-1}}}\\ &\Rightarrow e^{\sqrt{s^n\varrho_n e^{s^n\varrho_n}}}\leq e^{k\sqrt{s^{n-1}\varrho_{n-1}e^{s^{n-1}\varrho_{n-1}}}}\Rightarrow\theta(s^n\varrho_n)\leq\left[\theta\left(s^{n-1}\varrho_{n-1}\right)\right]^k, \end{aligned}\tag{6}$$

and hence (Θ_5) holds.

On the other side, Kumam et al. [11] utilized the concept of b-metric space and obtained common α- fuzzy fixed points for fuzzy mappings under generalized rational contractions. For more details in the direction of fixed point results for fuzzy mappings, we refer [9–20] to the readers.

We need the following lemma of Czerwik [2].

Lemma 2 (see [11]). *Let $(\mathscr{M},d_b,s)$ be a b-metric space and $\Xi_1,\Xi_2\in CB(\mathscr{M})$, then $\forall\,\omega\in\Xi_1$,*

$$d_b(\omega,\Xi_2)\leq H_b(\Xi_1,\Xi_2).\tag{7}$$

In this paper, we obtain common α-fuzzy fixed point results for a pair of fuzzy mappings and establish some theorems to generalize some results from the literature. We solve the Volterra integral inclusions as application of our established results.

2. Main Results

In this way, we state our main result.

Theorem 1. *Let $(\mathscr{M},d_b,s)$ be a complete b-metric space with coefficient $s\geq 1$ such that d_b is continuous. Assume that $\mathscr{O}_1,\mathscr{O}_2\colon \mathscr{M}\longrightarrow\mathscr{F}(\mathscr{M})$ and for each $\omega,\varpi\in\mathscr{M}$, there exist $\alpha_{\mathscr{O}_1}(\omega),\alpha_{\mathscr{O}_2}(\varpi)\in(0,1]$ such that $[\mathscr{O}_1\omega]_{\alpha_{\mathscr{O}_1}(\omega)},[\mathscr{O}_2\varpi]_{\alpha_{\mathscr{O}_2}(\varpi)}\in P_{cb}(\mathscr{M})$. If there exist $\Theta\in\Omega_s$ and $k\in(0,1)$ such that*

$$\Theta\left(sH_b\left([\mathscr{O}_1\omega]_{\alpha_{\mathscr{O}_1}(\omega)},[\mathscr{O}_2\varpi]_{\alpha_{\mathscr{O}_2}(\varpi)}\right)\right)\leq[\Theta(d_b(\omega,\varpi))]^k,\tag{8}$$

for all $\omega,\varpi\in\mathscr{M}$ with $H_b([\mathscr{O}_1\omega]_{\alpha_{\mathscr{O}_1}(\omega)},[\mathscr{O}_2\varpi]_{\alpha_{\mathscr{O}_2}(\varpi)})>0$, then there exists $\omega^*\in\mathscr{M}$ such that $\omega^*\in[\mathscr{O}_1\omega^*]_{\alpha_{\mathscr{O}_1}(\omega^*)}\cap[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$.

Proof.

Let $\omega_0\in\mathscr{M}$, then by assumption there exists $\alpha_{\mathscr{O}_1}(\omega_0)\in(0,1]$ such that $[\mathscr{O}_1\omega_0]_{\alpha_{\mathscr{O}_1}(\omega_0)}\in CB(\mathscr{M})$. Let $\omega_1\in[\mathscr{O}_1\omega_0]_{\alpha_{\mathscr{O}_1}(\omega_0)}$. For this ω_1, there exists $\alpha_{\mathscr{O}_2}(\omega_1)\in(0,1]$ such that $[\mathscr{O}_2\omega_1]_{\alpha_{\mathscr{O}_2}(\omega_1)}\in P_{cb}(\mathscr{M})$. By Lemma 2, ($\Theta_1$) and (8), we have

$$\Theta\left(s\,d\left(\omega_1,[\mathscr{O}_2\omega_1]_{\alpha_{\mathscr{O}_2}(\omega_1)}\right)\right)\leq\Theta\left(sH_b\left([\mathscr{O}_1\omega_0]_{\alpha_{\mathscr{O}_1}(\omega_0)},[\mathscr{O}_2\omega_1]_{\alpha_{\mathscr{O}_2}(\omega_1)}\right)\right)\leq[\Theta(d_b(\omega_0,\omega_1))]^k.\tag{9}$$

Thus,

$$\Theta\left(s\,d\left(\omega_1,[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\right)\right)\le[\Theta(d_b(\omega_0,\omega_1))]^k. \tag{10}$$

From (Θ_4), we know that

$$\Theta\left(s\,d\left(\omega_1,[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\right)\right)=\inf_{y\in[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}}\Theta(sd_b(\omega_1,y)). \tag{11}$$

Thus from (10), we get

$$\inf_{y\in[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}}\Theta(sd_b(\omega_1,y))\le[\Theta(d_b(\omega_0,\omega_1))]^k. \tag{12}$$

Then, from (12), there exists $\omega_2\in[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}$ (obviously, $\omega_2\ne\omega_1$) such that

$$\Theta(sd_b(\omega_1,\omega_2))\le[\Theta(d_b(\omega_0,\omega_1))]^k. \tag{13}$$

For this ω_2, there exists $\alpha_{\mathcal{O}_1}(\omega_2)\in(0,1]$ such that $[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\in P_{cb}(\mathcal{M})$. By Lemma 2, (Θ_1), and (8), we have

$$\Theta\left(sd_b\left(\omega_2,[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\right)\right)\le\Theta\left(sH_b\left([\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)},[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\right)\right)\le[\Theta(d_b(\omega_1,\omega_2))]^k. \tag{14}$$

Thus,

$$\Theta\left(sd_b\left(\omega_2,[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\right)\right)\le[\Theta(d_b(\omega_1,\omega_2))]^k. \tag{15}$$

From (Θ_4), we know that

$$\Theta\left(sd_b\left(\omega_2,[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\right)\right)=\inf_{y\in[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}}\Theta(sd_b(\omega_2,y)). \tag{16}$$

Thus from (15), we get

$$\inf_{y\in[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}}\Theta(sd_b(\omega_2,y))\le[\Theta(d_b(\omega_1,\omega_2))]^k. \tag{17}$$

Then, from (17), there exists $\omega_3\in[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}$ (obviously, $\omega_3\ne\omega_2$) such that

$$\Theta(sd_b(\omega_2,\omega_3))\le[\Theta(d_b(\omega_1,\omega_2))]^k. \tag{18}$$

So, continuing in the same way, we construct $\{\omega_n\}$ in $\mathcal{M}$ such that

$$\begin{aligned}\omega_{2n+1}&\in[\mathcal{O}_1\omega_{2n}]_{\alpha_{\mathcal{O}_1}(\omega_{2n})},\\ \omega_{2n+2}&\in[\mathcal{O}_2\omega_{2n+1}]_{\alpha_{\mathcal{O}_2}(\omega_{2n+1})},\end{aligned} \tag{19}$$

$$\Theta(sd_b(\omega_{2n+1},\omega_{2n+2}))\le[\Theta(d_b(\omega_{2n},\omega_{2n+1}))]^k, \tag{20}$$

$$\Theta(sd_b(\omega_{2n+2},\omega_{2n+3}))\le[\Theta(d_b(\omega_{2n+1},\omega_{2n+2}))]^k, \tag{21}$$

for all $n\in\mathbb{N}$. From (20) and (21), we get

$$\Theta(sd_b(\omega_n,\omega_{n+1}))\le[\Theta(d_b(\omega_{n-1},\omega_n))]^k, \tag{22}$$

for all $n\in\mathbb{N}$. It follows by (22) and property (Θ_5) that

$$\Theta(s^nd_b(\omega_n,\omega_{n+1}))\le\left[\Theta\left(s^{n-1}d_b(\omega_{n-1},\omega_n)\right)\right]^k, \tag{23}$$

which further implies that

$$\Theta(s^nd_b(\omega_n,\omega_{n+1}))\le\left[\Theta\left(s^{n-1}d_b(\omega_{n-1},\omega_n)\right)\right]^k\le\left[\Theta\left(s^{n-2}d_b(\omega_{n-2},\omega_{n-1})\right)\right]^{k^2}\le\cdots\le[\Theta(d_b(\omega_0,\omega_1))]^{k^n}, \tag{24}$$

for all $n\in\mathbb{N}$. Thus,

$$\Theta(s^nd_b(\omega_n,\omega_{n+1}))\le[\Theta(d_b(\omega_0,\omega_1))]^{k^n}, \tag{25}$$

for all $n\in\mathbb{N}$. Since $\Theta\in\Omega$, so letting $n\longrightarrow\infty$ in (25), we get

$$\lim_{n\longrightarrow\infty}\Theta(s^nd_b(\omega_n,\omega_{n+1}))=1. \tag{26}$$

This implies

$$\lim_{n\longrightarrow\infty}s^nd_b(\omega_n,\omega_{n+1})=0, \tag{27}$$

by (Θ_2). By (Θ_3), there exists $0<r<1$ and $\mathfrak{q}\in(0,\infty]$ so that

$$\lim_{n\longrightarrow\infty}\frac{\Theta(s^nd_b(\omega_n,\omega_{n+1}))-1}{(s^nd_b(\omega_n,\omega_{n+1}))^r}=\mathfrak{q}. \tag{28}$$

Suppose that $\mathfrak{q}<\infty$. For this case, let $\varsigma_2=(\mathfrak{q}/2)>0$. By definition of the limit, there exists $n_0\in\mathbb{N}$ so that

$$\left|\frac{\Theta(s^nd_b(\omega_n,\omega_{n+1}))-1}{(s^nd_b(\omega_n,\omega_{n+1}))^r}-\mathfrak{q}\right|\le\varsigma_2, \tag{29}$$

for all $n>n_0$. This implies that

$$\frac{\Theta(s^nd_b(\omega_n,\omega_{n+1}))-1}{(s^nd_b(\omega_n,\omega_{n+1}))^r}\ge\mathfrak{q}-\varsigma_2=\frac{\mathfrak{q}}{2}=\varsigma_2, \tag{30}$$

for all $n>n_0$. Then

$$n(s^nd_b(\omega_n,\omega_{n+1}))^r\le\varsigma_1n[\Theta(s^nd_b(\omega_n,\omega_{n+1}))-1], \tag{31}$$

for all $n>n_0$, where $\varsigma_1=(1/\varsigma_2)$. Now we assume that $\mathfrak{q}=\infty$. Let $\varsigma_2>0$. From the definition of the limit, there exists $n_0\in\mathbb{N}$ such that

$$\varsigma_2\le\frac{\Theta(s^nd_b(\omega_n,\omega_{n+1}))-1}{(s^nd_b(\omega_n,\omega_{n+1}))^r}, \tag{32}$$

for all $n>n_0$, which implies

$$n\left(s^n d_b(\omega_n,\omega_{n+1})\right)^r \le \varsigma_1 n\left[\Theta\left(s^n d_b(\omega_n,\omega_{n+1})\right)-1\right], \quad (33)$$

for all $n>n_0$, where $\varsigma_1 = (1/\varsigma_2)$. Hence, in all cases, there exists $\varsigma_1>0$ and $n_0\in\mathbb{N}$ such that

$$n\left(s^n d_b(\omega_n,\omega_{n+1})\right)^r \le \varsigma_1 n\left[\Theta\left(s^n d_b(\omega_n,\omega_{n+1})\right)-1\right], \quad (34)$$

for all $n>n_0$. Hence by (25) and (34), we obtain

$$n\left(s^n d_b(\omega_n,\omega_{n+1})\right)^r \le \varsigma_1 n\left(\left[\Theta\left(d_b(\omega_0,\omega_1)\right)\right]^{r^n}-1\right). \quad (35)$$

Taking the limit $n\longrightarrow\infty$, we get

$$\lim_{n\longrightarrow\infty} n\left(s^n d_b(\omega_n,\omega_{n+1})\right)^r = 0. \quad (36)$$

Thus $\lim_{n\longrightarrow\infty} n^{(1/r)} s^n d_b(\omega_n,\omega_{n+1}) = 0$ which implies that $\sum_{n=1}^{\infty} s^n d_b(\omega_n,\omega_{n+1})$ is convergent. Thus $\{\omega_n\}$ is a Cauchy sequence in $\mathscr{M}$. Since $(\mathscr{M},d_b,s)$ is a complete b-metric space, so there exists a $\omega^*\in\mathscr{M}$ such that

$$\lim_{n\longrightarrow\infty}\omega_n = \omega^*. \quad (37)$$

Now, we prove that $\omega^*\in[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$. We suppose on the contrary that $\omega^*\notin[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$, then there exist $n_0\in\mathbb{N}$ and $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $d_b(\omega_{2n_k+1},[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)})>0$, for all $n_k\ge n_0$. Now, using (8) with $\omega=\omega_{2n_k+1}$ and $\varpi=\omega^*$, we obtain

$$\begin{aligned}\Theta\left[d_b\left(\omega_{2n_k+1},[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}\right)\right] &\le \Theta\left[s d_b\left(\omega_{2n_k+1},[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}\right)\right]\\ &\le \Theta\left[sH_b\left(\left[\mathscr{O}_1\omega_{2n_k}\right]_{\alpha_{\mathscr{O}_1}\left(\omega_{2n_k}\right)},[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}\right)\right] \le \left[\Theta\left(d\left(\omega_{2n_k},\omega^*\right)\right)\right]^k.\end{aligned} \quad (38)$$

As $k\in(0,1)$, so by (Θ_1) so we obtain

$$d_b\left(\omega_{2n_k+1},[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}\right) < d_b\left(\omega_{2n_k},\omega^*\right). \quad (39)$$

Letting $n\longrightarrow\infty$, we have

$$d_b\left(\omega^*,[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}\right)\le 0. \quad (40)$$

Hence, $\omega^*\in[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$. Likewise, one can straightforwardly prove that $\omega^*\in[\mathscr{O}_1\omega^*]_{\alpha_{\mathscr{O}_1}(\omega^*)}$. Thus, $\omega^*\in[\mathscr{O}_1\omega^*]_{\alpha_{\mathscr{O}_1}(\omega^*)}\cap[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$.

Note: From now to onwards, we consider d_b as continuous functional and $(\mathscr{M},d_b,s)$ as complete b-metric space.

The following corollary follows from Theorem 1 by considering $\Theta(\eta)=e^{\sqrt{\eta}}$ for $\eta>0$.

Theorem 2. *Let $\mathscr{O}_1,\mathscr{O}_2$: $\mathscr{M}\longrightarrow\mathscr{F}(\mathscr{M})$, and for each $\omega,\varpi\in\mathscr{M}$, $\exists\,\alpha_{\mathscr{O}_1}(\omega),\alpha_{\mathscr{O}_2}(\varpi)\in(0,1]$ such that $[\mathscr{O}_1\omega]_{\alpha_{\mathscr{O}_1}(\omega)}$, $[\mathscr{O}_2\varpi]_{\alpha_{\mathscr{O}_2}(\varpi)}\in P_{cb}(\mathscr{M})$. If $\exists\,k\in(0,1)$ such that*

$$sH_b\left([\mathscr{O}_1\omega]_{\alpha_{\mathscr{O}_1}(\omega)},[\mathscr{O}_2\varpi]_{\alpha_{\mathscr{O}_2}(\varpi)}\right)\le k d_b(\omega,\varpi) \quad (41)$$

for all $\omega,\varpi\in\mathscr{M}$, then there exists $\omega^*\in\mathscr{M}$ such that $\omega^*\in[\mathscr{O}_1\omega^*]_{\alpha_{\mathscr{O}_1}(\omega^*)}\cap[\mathscr{O}_2\omega^*]_{\alpha_{\mathscr{O}_2}(\omega^*)}$.

Theorem 3. *Let $\mathscr{O}$: $\mathscr{M}\longrightarrow\mathscr{F}(\mathscr{M})$, and for each $\omega,\varpi\in\mathscr{M}$, there exist $\alpha_{\mathscr{O}}(\omega),\alpha_{\mathscr{O}}(\varpi)\in(0,1]$ such that $[\mathscr{O}\omega]_{\alpha_{\mathscr{O}}(\omega)}$, $[\mathscr{O}\varpi]_{\alpha_{\mathscr{O}}(\varpi)}\in P_{cb}(\mathscr{M})$. If there exists $k\in(0,1)$ such that*

$$sH_b\left([\mathscr{O}\omega]_{\alpha_{\mathscr{O}}(\omega)},[\mathscr{O}\varpi]_{\alpha_{\mathscr{O}}(\varpi)}\right)\le k d_b(\omega,\varpi), \quad (42)$$

for all $\omega,\varpi\in\mathscr{M}$, then there exists $\omega^*\in\mathscr{M}$ such that $\omega^*\in[\mathscr{O}\omega^*]_{\alpha_{\mathscr{O}}(\omega^*)}$.

Example 2. Let $\mathscr{M}=\{0,1,2\}$ and d_b: $\mathscr{M}\times\mathscr{M}\longrightarrow\mathbb{R}_0^+$ by

$$d_b(\omega,\varpi)=\begin{cases}0, & \text{if } \omega=\varpi,\\ \dfrac{1}{6}, & \text{if } \omega\ne\varpi \text{ and } \omega,\varpi\in\{0,1\},\\ \dfrac{1}{2}, & \text{if } \omega\ne\varpi \text{ and } \omega,\varpi\in\{0,2\},\\ 1, & \text{if } \omega\ne\varpi \text{ and } \omega,\varpi\in\{1,2\}.\end{cases} \quad (43)$$

It is easy to see that $(\mathscr{M},d)$ is a complete b-metric space with coefficient $s=(3/2)$. Define

$$\begin{aligned}(\mathscr{O}0)(\eta)=(\mathscr{O}1)(\eta)&=\begin{cases}\dfrac{1}{2}, & \text{if } \eta=0,\\ 0, & \text{if } \eta=1,2,\end{cases}\\ (\mathscr{O}2)(\eta)&=\begin{cases}0, & \text{if } \eta=0,2,\\ \dfrac{1}{2}, & \text{if } \eta=1.\end{cases}\end{aligned} \quad (44)$$

Define α: $\mathscr{M}\longrightarrow(0,1]$ by $\alpha(\omega)=(1/2)$ for all $\omega\in\mathscr{M}$. Now we obtain that

$$[\mathscr{O}\omega]_{(1/2)}=\begin{cases}\{0\}, & \text{if } \omega=0,1\\ \{1\}, & \text{if } \omega=2.\end{cases} \quad (45)$$

For $\omega, \varpi \in \mathscr{M}$, we get

$$H_b\left([\mathcal{O}1]_{(1/2)}, [\mathcal{O}2]_{(1/2)}\right) = H_b\left([\mathcal{O}1]_{(1/2)}, [\mathcal{O}2]_{(1/2)}\right) = H_b(\{0\},\{1\}) = \frac{1}{6}. \tag{46}$$

Taking $\Theta(\eta) = e^{\sqrt{\eta}}$ for $\eta > 0$ and $k = (1/2)$. Then

$$\Theta\left(sH_b\left([\mathcal{O}0]_{(1/2)}, [\mathcal{O}2]_{(1/2)}\right)\right) = e^{\left(\frac{1}{4}\right)^{(1/2)}} < e^{\left(\frac{1}{2}\right)^{(1/4)}} = \left[\Theta\left(d_b(0,2)\right)\right]^k, \tag{47}$$

also

$$\Theta\left(sH_b\left([\mathcal{O}1]_{(1/2)}, [\mathcal{O}2]_{(1/2)}\right)\right) = e^{\left(\frac{1}{4}\right)^{(1/2)}} < e^{(1)^{(1/4)}} = \left[\Theta\left(d_b(1,2)\right)\right]^k, \tag{48}$$

for all $\omega, \varpi \in \mathscr{M}$. As a result, all assertions of Theorem 6 hold and there exists $0 \in \mathscr{M}$ such that $0 \in [\mathcal{O}0]_{(1/2)}$is an α-fuzzy fixed point of $\mathcal{O}$.

3. Set-Valued Results

Theorem 4. *Let $G_1, G_2\colon X \longrightarrow CB(X)$. Suppose that $\exists k \in (0,1)$ such that*

$$\Theta\left(sH_b\left(G_1\omega, G_2\varpi\right)\right) \leq \left[\Theta\left(d_b(\omega,\varpi)\right)\right]^k, \tag{49}$$

for all $\omega, \varpi \in \mathscr{M}$. Then there exists $\omega^* \in \mathscr{M}$ such that $\omega^* \in G_1\omega^* \cap G_2\omega^*$.

Proof. Define $\alpha\colon \mathscr{M} \longrightarrow [0,1]$ and $\mathcal{O}_1, \mathcal{O}_2\colon \mathscr{M} \longrightarrow \mathscr{F}(\mathscr{M})$ by

$$\begin{aligned}\mathcal{O}_1(\omega)(\eta) &= \begin{cases}\alpha(\omega), & \text{if } \eta \in G_1\omega,\\ 0, & \text{if } \eta \notin G_1\omega,\end{cases}\\ \mathcal{O}_2(\omega)(\eta) &= \begin{cases}\alpha(\omega), & \text{if } \eta \in G_2\omega,\\ 0, & \text{if } \eta \notin G_2\omega.\end{cases}\end{aligned} \tag{50}$$

Then

$$\begin{aligned}\left[\mathcal{O}_1\omega\right]_{\alpha(\omega)} &= \left\{\eta\colon \mathcal{O}_1(\omega)(\eta) \geq \alpha(\omega)\right\} = G_1\omega,\\ \left[\mathcal{O}_2\omega\right]_{\alpha(\omega)} &= \left\{\eta\colon \mathcal{O}_2(\omega)(\eta) \geq \alpha(\omega)\right\} = G_2\omega.\end{aligned} \tag{51}$$

Thus, Theorem 4 can be applied to get $\omega^* \in \mathscr{M}$ such that

$$\omega^* \in \left[\mathcal{O}_1\omega^*\right]_{\alpha(\omega^*)} \cap \left[\mathcal{O}_2\omega^*\right]_{\alpha(\omega^*)} = G_1\omega^* \cap G_2\omega^*. \tag{52}$$

□

Corollary 1. *Let $G\colon X \longrightarrow CB(X)$ be multivalued mapping. Assume that there exists $k \in (0,1)$ such that*

$$\Theta\left(sH_b\left(G\omega, G\varpi\right)\right) \leq \left[\Theta\left(d_b(\omega,\varpi)\right)\right]^k, \tag{53}$$

for all $\omega, \varpi \in \mathscr{M}$. Then there exists $\omega^* \in \mathscr{M}$ such that $\omega^* \in G\omega^*$.

Remark 1. If $s = 1$, then b-metric spaces turns into complete metric space and we obtain some new results for fuzzy mappings as well as multivalued mappings in metric spaces.

4. Applications

Consider the Volterra integral inclusion

$$\omega(\kappa) \in \mathfrak{h}(\kappa) + \int_0^{\kappa} \mathfrak{J}(\kappa,\tau,\omega(\tau))d\tau, \quad \kappa \in [0,1], \tag{54}$$

where $\mathfrak{J}\colon [0,1]\times[0,1]\times\mathbb{R} \longrightarrow \chi_{cv}(\mathbb{R})$ a given set-valued mapping and $\mathfrak{h}, \omega \in C[0,1]$ be such that $\mathfrak{h}$ is given and ω is unknown function.

Now, for $p \geq 1$, consider the b-metric d_b on $C[0,1]$ defined by

$$d_b(\omega,\varpi) = \left(\max_{\kappa\in[0,1]} |\omega(\kappa) - \varpi(\kappa)|\right)^p = \max_{\kappa\in[0,1]} |\omega(\kappa) - \varpi(\kappa)|^p, \tag{55}$$

for all $\omega, \varpi \in C[0,1]$. Then, $(C[0,1], d_b, 2^{p-1})$ is a complete b-metric space.

We will assume the following:

(a) For each $\omega \in C[0,1]$, the mapping $\mathfrak{J}\colon [0,1]\times[0,1]\times\mathbb{R} \longrightarrow \chi_{cv}(\mathbb{R})$ is such that $\mathfrak{J}(\kappa,\tau,\omega(\tau))$ is lower semicontinuous in $[0,1]\times[0,1]$

(b) There exists $l\colon [0,1] \longrightarrow [0,+\infty)$ which is continuous such that

$$|\mathfrak{J}(\kappa,\tau,\omega) - \mathfrak{J}(\kappa,\tau,\varpi)|^p \leq l(\tau)|\omega(\tau) - \varpi(\tau)| \tag{56}$$

for all $\kappa, \tau \in [0,1]$, $\omega, \varpi \in C[0,1]$.

(c) There exists $k \in (0,1)$ so that

$$\left(\int_0^{\kappa} l(\tau)d\tau\right)^p \leq \frac{k}{2^{p-1}}. \tag{57}$$

Theorem 5. *Under the assumptions (a)–(c), the integral inclusion (54) has a solution in $C[0,1]$.*

Proof. Let $\mathscr{M} = C[0,1]$. Define $\mathcal{O}\colon \mathscr{M} \longrightarrow \mathscr{F}(\mathscr{M})$ by

$$[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)} = \left\{\varpi \in \mathscr{M}\colon \varpi(\kappa) \in \mathfrak{h}(\kappa) + \int_0^{\kappa} \mathfrak{J}(\kappa,\tau,\omega(\tau))d\tau\right\}, \tag{58}$$

for all $\kappa \in [0,1]$. Let $\omega \in \mathscr{M}$ be arbitrary, then there exists $\alpha_{\mathcal{O}}(\omega) \in (0,1]$. For $\mathfrak{J}_{\omega}(\kappa,\tau)\colon [0,1]\times[0,1] \longrightarrow \chi_{cv}(\mathbb{R})$, it follows from Michael's selection theorem that there exists $\mathfrak{j}_{\omega}(\kappa,\tau)\colon [0,1]\times[0,1] \longrightarrow \mathbb{R}$ such that $\mathfrak{j}_{\omega}(\kappa,\tau) \in \mathfrak{J}_{\omega}(\kappa,\tau)$ for each $\kappa, \tau \in [0,1]$. It follows that $\mathfrak{h}(\kappa) + \int_0^{\kappa} \mathfrak{j}_{\omega}(\kappa,\tau)d\tau \in [\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}$. Hence, $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)} \neq \varnothing$. It is a simple matter to show that $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}$ is closed, and so details are excluded (see also [17]). Moreover, since $\mathfrak{h}$ is continuous on $[0,1]$, and $\mathfrak{J}_\omega(\kappa,\tau)$ is continuous, their ranges are bounded. This means that $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}$ is bounded. Thus, $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)} \in CB(\mathcal{M})$. ■

Let $\omega_1, \omega_2 \in \mathcal{M}$, then there exists $\alpha_{\mathcal{O}}(\omega_1), \alpha_{\mathcal{O}}(\omega_1) \in (0,1]$ such that $[\mathcal{O}\omega_1]_{\alpha_{\mathcal{O}}(\omega_1)}, [\mathcal{O}\omega_2]_{\alpha_{\mathcal{O}}(\omega_2)} \in CB(\mathcal{M})$. Let $\varpi_1 \in [\mathcal{O}\omega_1]_{\alpha_{\mathcal{O}}(\omega_1)}$ be arbitrary such that

$$\varpi_1(\kappa) \in \mathfrak{h}(\kappa) + \int_0^\kappa \mathfrak{J}(\kappa,\tau,\omega_1(\tau))\mathrm{d}\tau, \tag{59}$$

for $\kappa \in [0,1]$ holds. This means that for all $\kappa,\tau \in [0,1]$, there exists $\mathfrak{j}_{\omega_1}(\kappa,\tau) \in \mathfrak{J}_{\omega_1}(\kappa,\tau) = \mathfrak{J}(\kappa,\tau,\omega_1(\tau))$ such that

$$\varpi_1(\kappa) = \mathfrak{h}(\kappa) + \int_0^\kappa \mathfrak{j}_{\omega_1}(\kappa,\tau)\mathrm{d}\tau, \tag{60}$$

for $\kappa \in [0,1]$. For all $\omega_1, \omega_2 \in \mathcal{M}$, it follows from (b) that

$$\left|\mathfrak{J}(\kappa,\tau,\omega_1) - \mathfrak{J}(\kappa,\tau,\omega_2)\right|^p \le l(\tau)\left|\omega_1(\tau) - \omega_2(\tau)\right|. \tag{61}$$

It means that there exists $z(\kappa,\tau) \in \mathfrak{J}_{\omega_2}(\kappa,\tau)$ such that

$$\left|\mathfrak{j}_{\omega_1}(\kappa,\tau) - z(\kappa,\tau)\right|^p \le l(\tau)\left|\omega_1(\tau) - \omega_2(\tau)\right|, \tag{62}$$

for all $\kappa,\tau \in [0,1]$.

Now, we can take the set-valued operator U defined by

$$U(\kappa,\tau) = \mathfrak{J}_{\omega_2}(\kappa,\tau) \cap \left\{u \in \mathbb{R}: \left|\mathfrak{j}_{\omega_1}(\kappa,\tau) - u\right| \le l(\tau)\left|\omega_1(\tau) - \omega_2(\tau)\right|\right\}. \tag{63}$$

Hence, by (a), U is lower semicontinuous. It follows that there exists a continuous operator $\mathfrak{j}_{\omega_2}(\kappa,\tau)\colon [0,1]\times[0,1] \longrightarrow \mathbb{R}$ such that $\mathfrak{j}_{\omega_2}(\kappa,\tau) \in U(\kappa,\tau)$ for $\kappa,\tau \in [0,1]$. Then, $\varpi_2(\kappa) = \mathfrak{h}(\kappa) + \int_0^\kappa \mathfrak{j}_{\omega_1}(\kappa,\tau)\mathrm{d}\tau$ satisfies that

$$\varpi_2(\kappa) \in \mathfrak{h}(\kappa) + \int_0^\kappa \mathfrak{J}(\kappa,\tau,\omega_2(\tau))\mathrm{d}\tau, \quad \kappa \in [0,1]. \tag{64}$$

That is $\varpi_2 \in [\mathcal{O}\omega_2]_{\alpha_{\mathcal{O}}(\omega_2)}$ and

$$\begin{aligned}\left|\varpi_1(\kappa) - \varpi_2(\kappa)\right|^p &\le \left(\int_0^\kappa \left|\mathfrak{j}_{\omega_1}(\kappa,\tau) - \mathfrak{j}_{\omega_2}(\kappa,\tau)\right|\mathrm{d}\tau\right)^p \\ &\le \left(\int_0^\kappa l(\tau)\left|\omega_1(\tau) - \omega_2(\tau)\right|\mathrm{d}\tau\right)^p \le \max\nolimits_{\tau\in[0,1]}^p \left|\omega(\tau) - \varpi(\tau)\right|^p\left(\int_0^\kappa l(\tau)\mathrm{d}\tau\right) \le \frac{k^2}{2^{p-1}} d_b(\omega_1,\omega_2),\end{aligned} \tag{65}$$

for all $\kappa,\tau \in [0,1]$. Thus, we obtain that

$$2^{p-1} d_b(\varpi_1,\varpi_2) \le k^2 d_b(\omega_1,\omega_2). \tag{66}$$

Interchanging the roles of ω_1 and ω_2, we obtain that

$$sH_b\left([\mathcal{O}\omega_1]_{\alpha_{\mathcal{O}}(\omega_1)}, [\mathcal{O}\omega_2]_{\alpha_{\mathcal{O}}(\omega_2)}\right) \le k^2 d_b(\omega_1,\omega_2). \tag{67}$$

This implies that

$$\sqrt{sH_b\left([\mathcal{O}\omega_1]_{\alpha_{\mathcal{O}}(\omega_1)}, [\mathcal{O}\omega_2]_{\alpha_{\mathcal{O}}(\omega_2)}\right)} \le k\sqrt{d_b(\omega_1,\omega_2)}. \tag{68}$$

Taking exponential, we have

$$e^{\sqrt{sH_b\left([\mathcal{O}\omega_1]_{\alpha_{\mathcal{O}}(\omega_1)}, [\mathcal{O}\omega_2]_{\alpha_{\mathcal{O}}(\omega_2)}\right)}} \le e^{k\sqrt{d_b(\omega_1,\omega_2)}}. \tag{69}$$

Taking the function $\Theta \in \Omega_s$ defined by $\Theta(\eta) = e^{\sqrt{\eta}}$ for $\eta > 0$, we get that the condition (8) is satisfied. Using the result 6, we conclude that (54) has a solution. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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