Research Article

Study on Twisted Product Almost Gradient Yamabe Solitons

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In this paper, we first study gradient Yamabe solitons on the twisted product spaces. Then, we classify and characterize the warped product and twisted product spaces with almost gradient Yamabe solitons. We also study the construction of almost gradient Yamabe solitons in the Riemannian product spaces.

1. Introduction

The notion of Yamabe flow was introduced by Hamilton [1] in 1989, which is defined on a Riemannian manifold (M, g) as $(\partial g / \partial t) = -rg$, where g is the Riemannian metric on M and r is the scalar curvature of M . The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. A Riemannian manifold (M, g) is called a Yamabe soliton if there exist a smooth vector field X and constant ρ such that

$$(r - \rho)g = \frac{1}{2}\mathcal{L}_X g, \quad (1)$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X . The Yamabe soliton is called shrinking if $\rho > 0$, steady if $\rho = 0$, and expanding if $\rho < 0$. When $X = \nabla h$ for some function h on M , we say that M is a gradient Yamabe soliton with a potential function h . In this case, equation (1) becomes

$$(r - \rho)g = \text{Hess}(h), \quad (2)$$

[2–7]. In (2), if ρ is a function on M , then M is called an almost gradient Yamabe soliton with (h, ρ) [8]. The Yamabe soliton (resp., gradient Yamabe soliton) is said to be trivial if X is killing (resp., h is a constant).

The warped product or twisted product metric $\tilde{g}$ on the product manifold of two Riemannian manifolds (B, g) and $(F, \bar{g})$ is given by $\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & f^2 \bar{g} \end{pmatrix}$, where f is a positive function with $f: B \rightarrow \mathbb{R}^+$ for a warped product metric and $f: B \times F \rightarrow \mathbb{R}^+$ for a twisted product metric.

It was known [6] that the metric of any compact Yamabe soliton is a metric of constant scalar curvature when the dimension of the manifold $n \geq 3$. In 2011, Cao and two coauthors [3] studied classification theorems for complete nontrivial locally conformally flat gradient Yamabe soliton. In [9], they found sufficient conditions on the soliton vector field under which the metric of a Yamabe soliton is a Yamabe metric, that is, a metric of constant scalar curvature. Moreover, in [8], we can see the various examples of compact and noncompact almost gradient Yamabe soliton. The present authors [4] studied gradient Yamabe soliton in the warped product manifolds and admittance of gradient Yamabe solitons and geometric structures for some model spaces. In 2019, Karaca [9] obtained a classification theorem regarding a gradient Yamabe soliton on multiplying warped product space with splitting potential function. With respect to the result, we obtain similar classification theorems for the warped product or the twisted product space with almost gradient Yamabe soliton. Especially considering the

condition of the conformal flatness, the classification and characterizations of the space were greatly helped. In addition to these, there are many works on solitons on the twisted product spaces [10–14] and Yamabe solitons [2, 5–9].

From this point of view, the purpose of this paper is to get a more generalized classification theorem of the results which is already published on the warped and twisted product space for a gradient Yamabe soliton and almost gradient Yamabe soliton. This paper is organized as follows. In Section 2, we discuss gradient Yamabe soliton in the twisted product space. Sections 3–5 are devoted to studying

almost gradient Yamabe soliton in the Riemannian, warped, and twisted product spaces.

2. Gradient Yamabe Solitons in the Twisted Product Spaces

In this section, we consider the case that the twisted product space $M = B \times_f F$ of n -dimensional Riemannian manifold (B, g) and p -dimensional Riemannian manifold $(F, \bar{g})$ is a gradient Yamabe soliton with $(h, \bar{\rho})$. Let ∇ and $\bar{\nabla}$ are Riemannian connections in B and F , respectively. Then, we have

$$\begin{aligned} (\tilde{r} - \bar{\rho})g_{ab} &= \nabla_b h_a, \\ \partial_b h_x &= \frac{f_b h_x}{f}, \\ (\tilde{r} - \bar{\rho})f^2 \bar{g}_{yx} &= \bar{\nabla}_y h_x + f f^c h_c \bar{g}_{yx} - \frac{1}{f} (f_y h_x + f_x h_y - f^z h_z \bar{g}_{xy}), \\ \tilde{r} &= r + \frac{\bar{r}}{f^2} - \frac{2p\Delta f}{f} - \frac{2(p-1)(\bar{\Delta} f)}{f^3} - \frac{p(p-1)\|f_e\|^2}{f^2} - \frac{(p-1)(p-4)\|f_x\|^2}{f^4}, \end{aligned} \quad (3)$$

where $\{u_a: a = 1, 2, \dots, n\}$ and $\{u_x: x = n+1, 2, \dots, n+p\}$ are local coordinate systems in B and F , respectively. Moreover, we have put $h_a = (\partial h / \partial u_a)$, $f_c = (\partial f / \partial u_c)$, and $f_x = (\partial f / \partial u_x)$.

Assume that the potential function h is decomposed by $h = k + l$ for some functions k and l on B and F , respectively. Then, from equation (3), we see that $(\tilde{r} - \bar{\rho})g_{ab} = \nabla_b k_a$, so $\tilde{r} - \bar{\rho}$ becomes a function on B because $\bar{\rho}$ is a constant. Moreover, we obtain

$$\tilde{r} - \bar{\rho} = r - \left[-\frac{\bar{r}}{f^2} + \frac{2p\Delta f}{f} + \frac{2(p-1)(\bar{\Delta} f)}{f^3} + \frac{p(p-1)\|f_e\|^2}{f^2} + \frac{(p-1)(p-4)\|f_x\|^2}{f^4} + \bar{\rho} \right], \quad (4)$$

and since $\tilde{r} - \bar{\rho}$ and r are functions on B , the quantity $-(\bar{r}/f^2) + (2p\Delta f/f) + (2(p-1)(\bar{\Delta} f)/f^3) + (p(p-1)\|f_e\|^2/f^2) + ((p-1)(p-4)\|f_x\|^2/f^4) + \bar{\rho}$ is a function on B . Hence, by using the first equation of (3), we get

$$\nabla_b k_a = \left(r - \left[-\frac{\bar{r}}{f^2} + \frac{2p\Delta f}{f} + \frac{2(p-1)(\bar{\Delta} f)}{f^3} + \frac{p(p-1)\|f_e\|^2}{f^2} + \frac{(p-1)(p-4)\|f_x\|^2}{f^4} + \bar{\rho} \right] \right) g_{ab} = (r - \rho)g_{ab}, \quad (5)$$

where we have put $\rho = -(\bar{r}/f^2) + (2p\Delta f/f) + (2(p-1)(\bar{\Delta} f)/f^3) + (p(p-1)\|f_e\|^2/f^2) + ((p-1)(p-4)\|f_x\|^2/f^4) + \bar{\rho}$; that is, B becomes an almost gradient Yamabe soliton. Thus, we have the following theorem.

Theorem 1. *If the twisted product manifold $M = B \times_f F$ is gradient Yamabe soliton with $(h, \bar{\rho})$ and $h = k + l$, for some functions k and l on B and F , respectively, then the base space*

B becomes an almost gradient Yamabe soliton with (k, ρ) and $\tilde{r}$ depends only on B .

In [15], the authors proved the following theorem.

Theorem 2. *If the twisted product manifold $M = B \times_f F$ is conformally flat and $p \neq 1$ and $n \neq 1$, then M is the warped product space $B \times_{f^*} F^*$ of B and F^* .*

In [4], the present authors proved the following theorem.

Theorem 3. *If the product manifold $M = B \times F$ is a gradient Yamabe soliton, then B, F , and M become trivial gradient Yamabe soliton. This means that there is a nontrivial gradient Yamabe soliton in the Riemannian product manifold.*

$$\begin{aligned}(\tilde{r} - \tilde{\rho})g_{ab} &= \nabla_b k_a, \\ (f^*)_b \tilde{f} l_x &= 0, \\ (\tilde{r} - \tilde{\rho})f^2 \tilde{g}_{yx} &= \tilde{\nabla}_y l_x + f f^c k_c \tilde{g}_{yx} - \frac{1}{f} (f_y l_x + f_x l_y - f^z l_z \tilde{g}_{xy}),\end{aligned}\tag{6}$$

where $f = f^* \tilde{f}$ and $l_x = (\partial/\partial u_x)$. From the first and second equations of (6), we see that $\tilde{r} - \tilde{\rho}$ is a function only on B and f^* is constant or $\tilde{f} l_x = 0$. Since $f = f^* \tilde{f}$ is a positive function, we see that f^* is constant or l is constant. Let us consider the first case, that is, f^* is constant. Then, $f = f^* \tilde{f}$ depends only on F and that $\tilde{g}$ becomes a Riemannian product metric. Hence, B, F , and M become trivial gradient Yamabe soliton due to Theorem 3. Moreover, from the third equation of (6), we get

$$(\tilde{r} - \tilde{\rho}) = \frac{1}{f^2} \left(\tilde{\Delta} l + \frac{p-2}{f} \tilde{g}^{xy} \tilde{f}_x l_y \right), \tag{7}$$

and that $(\tilde{r} - \tilde{\rho})$ depends only on F because the quantities of right-hand side of (7) depend only on F . Therefore, $(\tilde{r} - \tilde{\rho})$ becomes a constant and that Δk becomes a constant due to the first equation of (6). On the other hand, if l is constant, then the potential function h becomes a function on B so that B becomes an almost gradient Yamabe soliton by Theorem 1 and the first and third equations of (6). Thus, we have the following theorem.

Theorem 4. *Let the twisted product manifold $M = B \times_f F$ be a gradient Yamabe soliton and conformally flat. If $h = k + l$ for some functions k and l on B and F , respectively, and $p \neq 1$ and $n \neq 1$, then $\tilde{r} - \tilde{\rho}$ depends only on B , and M is one of the following two cases:*

- (1) M is a Riemannian product of B and F , and that B, F , and M become trivial gradient Yamabe soliton, and $(\tilde{r} - \tilde{\rho})$ and Δk become constant
- (2) The potential function h depends only on B , and B becomes an almost gradient Yamabe soliton

In 1965, Tashiro [16] proved that the following theorem.

Theorem 5. *Let M be an n -dimensional complete Riemannian manifold of dimension $n \geq 2$ and suppose it admits a special concircular field ρ satisfying the equation*

$$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b)g_{\mu\lambda}, \tag{8}$$

for constants k and b . Then, M becomes either direct product $V \times I$ of an $(n-1)$ -dimensional complete Riemannian

manifold V with a straight line I when $k = b = 0$ or a Euclidean space when $k = 0$ but $b \neq 0$.

Hence, if we combine Theorems 4 and 5, then we can state the following theorem.

Theorem 6. *Let the twisted product manifold $M = B \times_f F$ be a gradient Yamabe soliton and conformally flat. If $h = k + l$ and $p \neq 1$ and $n \neq 1$, then we have two cases as follows:*

- (i) M becomes a Riemannian product space $M = V \times I$ of an $(n-1)$ -dimension complete Riemannian manifold V with a straight line I when $\tilde{r} - \tilde{\rho} = 0$ or M becomes a Euclidean space when $\tilde{r} - \tilde{\rho} \neq 0$. In any case, M, B , and F become trivial gradient Yamabe soliton.
- (ii) The potential function h depends only on B , and B becomes an almost gradient Yamabe soliton.

3. Almost Gradient Yamabe Solitons in the Riemannian Product Spaces

Tokura and other coauthors [7] introduced various examples of an almost gradient Yamabe soliton in R^3 . In this section, we consider the relation between the structure of an almost gradient Yamabe soliton with $(h, \tilde{\rho})$ in the Riemannian product manifold $(B \times F, \tilde{g})$ of (B, g) and $(F, \bar{g})$ and the structure of an almost gradient Yamabe soliton in B and F . If the potential function h is expressed by $h = k + l$ for some functions k and l on B and F , respectively, then we have

$$\begin{aligned}(\tilde{r} - \tilde{\rho})g_{ab} &= \nabla_b k_a, \\ \partial_b l_x &= 0, \\ (\tilde{r} - \tilde{\rho})\tilde{g}_{yx} &= \tilde{\nabla}_y l_x, \tilde{r} = r + \tilde{r}.\end{aligned}\tag{9}$$

Theorem 7. *Let the Riemannian product manifold $M = B \times F$ be an almost gradient Yamabe soliton with $(h, \tilde{\rho})$ with $h = k + l$ for some functions k and l in B and F , respectively. Then, $\tilde{r} - \tilde{\rho}$ is a constant on M , B becomes an almost gradient Yamabe soliton with $(k, \rho = \tilde{\rho} - \tilde{r})$, and F becomes an almost gradient Yamabe soliton with $(l, \bar{\rho} = \tilde{\rho} - r)$ and $\rho + \bar{\rho} = \tilde{\rho} + r$. In this case, $r - \rho$ and $\tilde{r} - \bar{\rho}$ become constants on B and F , respectively.*

Proof. From the first and third equations of equation (9), we can see that $\tilde{r} - \tilde{\rho}$ is a quantity on B and F , respectively. Hence, $\tilde{r} - \tilde{\rho}$ becomes constant. Since $M = B \times F$ is an almost gradient Yamabe soliton with $(h = k + l, \tilde{\rho})$ for some function k and l on B and F , respectively, we get $(r - (\tilde{\rho} - \tilde{r}))g_{ab} = \nabla_b k_a$ and $(\tilde{r} - (\tilde{\rho} - r))g_{yx} = \tilde{\nabla}_y l_x$ from equation (9). If we put ρ and $\bar{\rho}$ by $\rho = \tilde{\rho} - \tilde{r}$ and $\bar{\rho} = \tilde{\rho} - r$, then $\rho = r - (\nabla_b k_a)g^{ba}$ and $\bar{\rho} = \tilde{r} - (\tilde{\nabla}_y l_x)g^{yx}$. Therefore, ρ and $\bar{\rho}$ become functions on B and F , respectively. Hence, we can see that B and F become an almost gradient Yamabe soliton with (ρ, k) and $(\bar{\rho}, l)$, respectively. Moreover, we obtain $\rho + \tilde{r} = \tilde{\rho} = \bar{\rho} + r$. Hence, we obtain $\tilde{r} - \bar{\rho} = \tilde{r} - \tilde{\rho} = r - \rho$ and that $r - \rho$ and $\tilde{r} - \bar{\rho}$ are also constants.

For the converse case of Theorem 7, if we assume that B and F are almost gradient Yamabe solitons with (ρ, k) and $(\bar{\rho}, l)$, respectively, and $\rho + \tilde{r} = \bar{\rho} + r$, then we get $(r - \rho)g_{ba} = \nabla_b k_a$ and $(\tilde{r} - \bar{\rho})g_{yx} = \tilde{\nabla}_y l_x$. If we take $h = k + l$ and put $\rho + \tilde{r} = \bar{\rho} + r = \tilde{\rho}$, then we can derive $(\tilde{r} - \bar{\rho})\tilde{g}_{ba} = \tilde{\nabla}_b \tilde{\nabla}_a h$ and $(\tilde{r} - \bar{\rho})\tilde{g}_{yx} = \tilde{\nabla}_y \tilde{\nabla}_x h$; that is, $B \times F$ becomes an almost gradient soliton with $(h, \tilde{\rho})$, where $\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix}$. Thus, we have the following theorem. $\square$

Theorem 8. Let B be an almost Yamabe soliton with (ρ, k) and F be an almost Yamabe soliton with $(\bar{\rho}, l)$. If $\rho + \tilde{r} = \bar{\rho} + r$, then $B \times F$ becomes an almost Yamabe soliton with $((1/2)(\rho + \bar{\rho} + r + \tilde{r}), k + l)$.

If we combine Theorems 7 and 8, we can state the following theorem.

Theorem 9. The Riemannian product space $M = B \times F$ with $(h = k + l, \tilde{\rho})$ is almost gradient Yamabe soliton for some functions k and l on B and F , respectively, if and only if B and F are almost gradient Yamabe soliton with (ρ, k) and $(\bar{\rho}, l)$, respectively, and $\rho + \tilde{r} = \bar{\rho} = \tilde{\rho} + r$.

If we combine Theorems 5 and 7, then we can state the following theorem.

Theorem 10. If the complete Riemannian product manifold $M = B \times F$ is an almost gradient Yamabe soliton with $(h = k + l, \tilde{\rho})$, then $\tilde{r} - \tilde{\rho}$ is a constant on M and that M becomes one of the following:

- (a) If $\tilde{r} - \tilde{\rho} = 0$, then $M = V \times I$ of an $(n - 1)$ -dimensional complete Riemannian manifold V with a straight line I
- (b) If $\tilde{r} - \tilde{\rho} \neq 0$, then a Euclidean space

Proof. We see that $\tilde{r} - \tilde{\rho}$ is a constant by Theorem 7. Then, from the first equation of (7), $\tilde{\nabla} \tilde{\nabla} h = \beta \tilde{g}$ for some constant $\beta = \tilde{r} - \tilde{\rho}$. Hence, we can prove Theorem 10 by using Theorem 1.

From Theorems 7 and 10, we see that if the constant $\tilde{r} - \tilde{\rho} = 0$, then the product manifold M becomes $M = V \times I_1$ of an $(n - 1)$ -dimension complete Riemannian manifold V with the first equation of (9), I_1 . Moreover, V becomes an almost gradient Yamabe soliton with $(k, \rho = \tilde{\rho})$ and $\nabla \nabla k = 0$

because $\tilde{r} - \tilde{\rho} = r - \rho$ is constant and $\tilde{r} = 0$. Hence, the $(n - 1)$ -dimension Riemannian manifold V becomes $V = W^{n-2} \times I_2$ of an $(n - 2)$ -dimensional Riemannian manifold W with I_2 from Theorem 7, that is, $M = (W^{n-2} \times I_2) \times I_1$. Hence, M becomes $M = W^{n-2} \times I^2$, where we put $I^2 = I_2 \times I_1$. Hence these facts, Theorems 5 and 7 give the following theorem $\square$

Theorem 11. Let the complete Riemannian product manifold M be an almost gradient Yamabe soliton with $(h = k + l, \tilde{\rho})$. Then, $\tilde{r} - \tilde{\rho}$ becomes a constant. If the constant $\tilde{r} - \tilde{\rho} = 0$, then M becomes $M = W \times I^2$ of an $(n - 2)$ -dimensional complete Riemannian manifold W with a 2-dimensional Euclidean space I^2 .

4. Almost Gradient Yamabe Solitons in the Warped Product Spaces

In this section, we study that the warped product space of $M = B \times_f F$ is an almost gradient Yamabe soliton with $(h, \tilde{\rho})$. Then, we have

$$\tilde{\nabla}_b h_a = \nabla_b h_a, \tilde{\nabla}_b h_x = \partial_b h_x - \frac{f_b}{f} h_x, \tilde{\nabla}_y h_x = \bar{\nabla}_y h_x + f f^c h_c \bar{g}_{yx}, \quad (10)$$

and that

$$(\tilde{r} - \tilde{\rho})g_{ab} = \nabla_b h_a, \quad \partial_b h_x = \frac{f_b}{f} h_x, \quad (11)$$

$$(\tilde{r} - \tilde{\rho})f^2 \bar{g}_{yx} = \bar{\nabla}_y h_x + f f^c h_c \bar{g}_{yx}.$$

Theorem 12. If the warped product space $M = B \times_f F$ is an almost gradient Yamabe soliton with $(h, \tilde{\rho})$, then we have the following:

- (a) If $h_a = 0$ for all a , then $\tilde{r} = \tilde{\rho}$, $\bar{\nabla}_x h_y = 0$, and M is either a Riemannian product of B and F or the potential function h is a constant
- (b) If $h_x = 0$ for all x , then B is an almost gradient Yamabe soliton

Proof.

- (a) If $h_a = 0$, then h becomes a function on F , $\tilde{r} = \tilde{\rho}$, and $\bar{\nabla}_y h_x = 0$ from the first and third equations of (11). Moreover, we get $f_b h_x = 0$ from the second equation of (11); that is, M is either a Riemannian product of B and F or the potential function h is a constant.
- (b) From the first equation of (11), we see that $\tilde{r} - \tilde{\rho}$ becomes a function on B . Moreover, we have $\nabla_b h_a = (\tilde{r} - \tilde{\rho})g_{ab} = (r - ((\tilde{r}/f^2) + (2p\Delta f/f) + (p(p - 1)\|f_e\|^2/f^2) + \tilde{\rho}))g_{ab}$ from the fourth equation of (3) and the first equation of (11). If we put $\rho = -(\tilde{r}/f^2) + (2p\Delta f/f) + (p(p - 1)\|f_e\|^2/f^2) + \tilde{\rho}$,

then we obtain $\rho = r - \tilde{r} - \tilde{\rho} = r - (\Delta h/n)$ and that ρ is a function on B . Hence, B becomes an almost gradient Yamabe soliton.

Consider the case of $h = k + l$ for some functions k and l on B and F , respectively. Then, from equations (3) and (11), we obtain

$$\begin{aligned} (\tilde{r} - \tilde{\rho})g_{ab} &= \nabla_b k_a, \\ f_b l_x &= 0, \\ (\tilde{r} - \tilde{\rho})f^2 \bar{g}_{yx} &= \bar{\nabla}_y l_x + f f^c k_c \bar{g}_{yx}, \quad \tilde{r} = r + \frac{\bar{r}}{f^2} - \frac{2p\Delta f}{f} - \frac{p(p-1)\|f_e\|^2}{f^2}. \end{aligned} \quad (12)$$

Hence, we can see that the function f or l is constant from the second equation of (12). If the function f is constant, then M becomes a product space; henceforth, we can apply Theorem 9.

On the other hand, if the function l is a constant, then $l_x = 0$ for all x . Hence, if we apply this fact to Theorem 12 (b), then we have. $\square$

Theorem 13. *If the warped product space $M = B \times_f F$ is an almost gradient Yamabe soliton with $(h = k + l, \tilde{\rho})$ for some functions k and l in B and F , respectively, then we have two cases as follows:*

- (i) M is either $M = W \times I^2$ of an $(n-2)$ -dimensional complete Riemannian manifold W with a 2-dimensional Euclidean space I^2 if the constant $\tilde{r} - \tilde{\rho} = 0$ or a Euclidean space if the constant $\tilde{r} - \tilde{\rho} \neq 0$
- (ii) B is an almost gradient Yamabe soliton with $(k, \rho = -(\bar{r}/f^2) + (2p\Delta f/f) + (p(p-1)\|f_e\|^2/f^2) + \tilde{\rho})$

5. Almost Gradient Yamabe Solitons in the Twisted Product Spaces

Let the twisted product space $M = B \times_f F$ be an almost gradient Yamabe soliton with $(h, \tilde{\rho})$. Then, we have

$$\begin{aligned} (\tilde{r} - \tilde{\rho})g_{ab} &= \nabla_b h_a, \\ \partial_b h_x &= \frac{f_b h_x}{f}, \\ (\tilde{r} - \tilde{\rho})f^2 \bar{g}_{yx} &= \bar{\nabla}_y h_x + f f^c h_c \bar{g}_{yx} - \frac{1}{f}(f_y h_x + f_x h_y - f^z h_z \bar{g}_{xy}), \\ \tilde{r} &= r + \frac{\bar{r}}{f^2} - \frac{2p\Delta f}{f} - \frac{2(p-1)(\bar{\Delta} f)}{f^3} - \frac{p(p-1)\|f_e\|^2}{f^2} - \frac{(p-1)(p-4)\|f_x\|^2}{f^4}. \end{aligned} \quad (13)$$

Assume that the potential function h is decomposed by $h = k + l$ for some functions k and l on B and F , respectively. Then, from the first equation of (13), we see that

$(\tilde{r} - \tilde{\rho})g_{ab} = \nabla_b k_a$; that is, $\tilde{r} - \tilde{\rho}$ is a function on B . Moreover, we obtain

$$\tilde{r} - \tilde{\rho} = r - \left[-\frac{\bar{r}}{f^2} + \frac{2p\Delta f}{f} + \frac{2(p-1)(\bar{\Delta} f)}{f^3} + \frac{p(p-1)\|f_e\|^2}{f^2} + \frac{(p-1)(p-4)\|f_x\|^2}{f^4} + \tilde{\rho} \right], \quad (14)$$

and since $\tilde{r} - \tilde{\rho}$ and r are functions on B , the quantity $-(\bar{r}/f^2) + (2p\Delta f/f) + (2(p-1)(\bar{\Delta} f)/f^3) + (p(p-1)$

$\|f_e\|^2/f^2) + ((p-1)(p-4)\|f_x\|^2/f^4) + \tilde{\rho}$ is a function on B . Hence, by use of the first equation of (11), we get

$$\nabla_b k_a = \left(r - \left[-\frac{\bar{r}}{f^2} + \frac{2p\Delta f}{f} + \frac{2(p-1)(\bar{\Delta} f)}{f^3} + \frac{p(p-1)\|f_e\|^2}{f^2} + \frac{(p-1)(p-4)\|f_x\|^2}{f^4} + \tilde{\rho} \right] \right) g_{ab}, \quad (15)$$

and that B becomes an almost gradient Yamabe soliton. Thus, we have the following theorem.

Theorem 14. *If the twisted product manifold $M = B \times_f F$ is an almost gradient Yamabe soliton with $(h, \bar{\rho})$ and $h = k + l$, then the base space B becomes an almost gradient Yamabe soliton and $\bar{\tau} - \bar{\rho}$ is a function on B .*

If the conformally flat twisted product space $M = B \times_f F$ is an almost gradient Yamabe soliton with $(h, \bar{\rho})$ and $h = k + l$ for some functions k and l on B and F , respectively, then we get

$$(\bar{\tau} - \bar{\rho})g_{ab} = \nabla_b k_a,$$

$$(f^*)_b \bar{f} l_x = 0,$$

$$(\bar{\tau} - \bar{\rho})f^2 \bar{g}_{yx} = \bar{\nabla}_y l_x + f f^c k_c \bar{g}_{yx} - \frac{1}{f} (f_y l_x + f_x l_y - f^z l_z \bar{g}_{xy}), \quad (16)$$

and $f = f^* \bar{f}$, where f^* and $\bar{f}$ are functions on B and F , respectively, which come from the conformally flatness and the Proof of Theorem 2. From the first and second equations of (16), we see that $\bar{\tau} - \bar{\rho}$ is a function only on B , and f^* is constant or $\bar{f} l_x = 0$, respectively. Since $f = f^* \bar{f}$ is a positive function, we see that f^* is a constant or l is a constant.

Let us consider the first case, that is, f^* is a constant. Then, $f = f^* \bar{f}$ depends only on F and that $\bar{g}$ becomes a Riemannian product metric. Hence, B , F , and M become trivial gradient Yamabe solitons due to Theorem 3. Moreover, from the third equation of (16), we get

$$(\bar{\tau} - \bar{\rho}) = \frac{1}{f^2} \left(\bar{\Delta} l + \frac{p-2}{f} \bar{g}^{xy} \bar{f}_x l_y \right), \quad (17)$$

and that $(\bar{\tau} - \bar{\rho})$ depends only on F because the quantities of right-hand side of (17) depend only on F . Therefore, $(\bar{\tau} - \bar{\rho})$ becomes a constant and that Δk becomes a constant due to the first equation of (16). Moreover, we see that $\text{Hess } k = \alpha g$ where we have put $\alpha = \bar{\tau} - \bar{\rho}$. Then, we can apply this fact to Theorem 5, and we see that M becomes either direct product $V \times I$ of an $(n-1)$ -dimensional complete Riemannian manifold V with a straight line I when $\alpha = 0$ or a Euclidean space when $\alpha \neq 0$.

On the other hand, if l is constant, then the potential function h becomes a function on B and B becomes an almost gradient Yamabe soliton by Theorem 14. Thus, we have the following theorem.

Theorem 15. *Let the twisted product manifold $M = B \times_f F$ be an almost gradient Yamabe soliton with $(h, \bar{\rho})$ and conformally flat. If $h = k + l$ and $p \neq 1$ and $n \neq 1$, then $(\bar{\tau} - \bar{\rho})$ depends only on B , and M is one of the following two cases:*

- (1) M is either direct product $V \times I$ of an $(n-1)$ -dimensional complete Riemannian manifold V with a straight line I when $\alpha = 0$ or a Euclidean space when $\alpha \neq 0$ moreover $(\bar{\tau} - \bar{\rho})$ and Δk become constants
- (2) The potential function h depends only on B , and B becomes an almost gradient Yamabe soliton.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts interest.

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