Research Article

Computation of the Double Metric Dimension in Convex Polytopes

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A source detection problem in complex networks has been studied widely. Source localization has much importance in order to model many real-world phenomena, for instance, spreading of a virus in a computer network, epidemics in human beings, and rumor spreading on the internet. A source localization problem is to identify a node in the network that gives the best description of the observed diffusion. For this purpose, we select a subset of nodes with least size such that the source can be uniquely located. This is equivalent to find the minimal doubly resolving set of a network. In this article, we have computed the double metric dimension of convex polytopes R_n and Q_n by describing their minimal doubly resolving sets.

1. Introduction and Preliminaries

Let G be a finite and connected graph with vertex set V_G and edge set E_G . The cardinality of V_G and E_G is called the order and the size of the graph G , respectively. For any two vertices b and c of graph G , the number of edges in a shortest path between the vertices b and c is called the distance between them and is represented as $d(b, c)$. A vertex $b \in V_G$ is said to resolve two vertices c and e if $d(b, c)$ is not equal to $d(b, e)$. Consider $P = \{p_i | 1 \leq i \leq k\} \subseteq V_G$ be an ordered set and v be a vertex of G ; then, the representation $r(v, P)$ of v with respect to P is the k -vector $(d(v, p_i))_{i=1}^k$. The set P is called a resolving set for graph G if any vertex of G has unique representation with respect to P . A resolving set having least number of vertices is called a metric basis of G . $\dim(G)$ represents cardinality of metric basis and is known as the metric dimension of G . For the notions that are used and not defined in this paper, we refer [1, 2].

In graph theory, the notion of minimal resolving sets were first defined in [3, 4] by Slater, Harary, and Melter independently. Slater used the metric dimension as location number of the graph. He used this concept by placing sonar/loran detecting devices with the least cardinality in such a way that the location of each vertex is unique with respect to its

distances from the devices in the network. The idea of trilateration in real two-dimensional plane can be generalized in terms of the metric dimension of graphs. For instance, in Global Positioning System (GPS), three satellites in orbit uniquely determine the location of an object on earth by using distances. Also, for the theoretical study of the metric dimension, there are many inspirational applications regarding robot navigation [5] and chemical structure [6]. Different coin weighing problems discussed in [7, 8] and complete analysis of Mastermind game found in [9] have strong connections with the metric dimension of hamming graphs. Many applications of this concept have been explored in different fields such as geographical routing protocols [10], combinatorial optimization [11], network discovery and verification [12], pharmaceutical chemistry [6], etc. [13].

It is a computationally difficult problem to determine the exact value of the metric dimension of arbitrary graphs. Therefore, some useful bounds have been found for different classes of graphs.

Chartrand et al. [6], in 2000, characterized all graphs with metric dimension $n - 1$, $n - 2$, and 1. Buczkowski et al. [14] and Tomescu and Javaid [15] determined the metric dimension of wheel graphs and Jahangir graphs, respectively. Ali et al. [16] studied the metric dimension of Mobius

ladders. They proved that Mobius ladders constitute a family of cubic graphs with constant metric dimension. Baca et al. [17] computed the metric dimension of regular bipartite graphs. Imran and Baig [18, 19] determined the metric dimension for some classes of convex polytopes. In 2015, Tomescu and Imran used R -sets for the computation of the metric dimension of necklace graphs [20]. Ahmad et al. computed the metric dimension for the families of kayak paddle graphs and chorded cycles (for details, see [21–23]).

It is an interesting challenge in complex networks to localize the epidemic source. For example, in order to find a mysterious source of virus spreading throughout the network, the only information needed to identify the mysterious source is the infection times of the subset of vertices known as sensors. These sensors may record their least time of being infected. Now, the main problem consists of determining how many sensors are needed to assure that the virus source is exactly located. The characteristic known as the double metric dimension is the answer of this problem (see [24]).

The recognition of the virus source might be straightforward if one can notice the whole procedure of the spreading of virus. Unfortunately, the whole procedure might be very expensive because of the cost of collecting information. If the starting time of epidemic spread is not known, in that case accurate recognition of infection source is possible if and only if the network sensor set is a doubly resolving set.

Cáceres et al. presented the doubly resolving sets by demonstrating its relation with the metric dimension of the Cartesian product of the graph G and also proved that the least cardinality of a doubly resolving set is an upper bound of the metric dimension of the graph under consideration (see [25]). For $a_1, a_2, u_1, u_2 \in V_G$, a_1 and a_2 are said to doubly resolve u_1 and u_2 if $d(u_1, a_1) - d(u_1, a_2) \neq d(u_2, a_1) - d(u_2, a_2)$. If any two vertices of G are doubly resolved by some two vertices in $D \subseteq V_G$, then D is termed doubly resolving set of G . In other words, there do not exist $u_1, u_2 \in V_G$ such that $r(u_1|D) - r(u_2|D) \equiv m \cdot u$ for some $m \in \mathbb{Z}$, and u is a vector of length $|D|$ with all entries equal to 1. Such D of least cardinality is known as minimal doubly resolving set and its cardinality is called the double metric dimension of G represented by $\psi(G)$. It should be noted that, for $a_1, a_2 \in D$ and $u_1, u_2 \in V_G$, $d(u_1, a_1) - d(u_2, a_1) \neq 0$ or $d(u_1, a_2) - d(u_2, a_2) \neq 0$ implies D to be also a resolving set. Consequently, for any graph G , we have $\dim(G) \leq \psi(G)$. The NP-hardness of $\dim(G)$ and $\psi(G)$ were proven by Khuller et al. in [5] and Kratica et al. in [26], respectively.

In [24], it was discussed that to identify the virus source in a star-like network is more difficult compared to that in a path-like network. A star graph with n vertices must have a doubly resolving set consisting of all the external vertices, which show the double metric dimension is $n - 1$, and for the path graph, it is 2. This also implies that the double metric dimension always depends upon the topology of network.

In [27], the metric and double metric dimensions were found to be equal in case for some families of the circulant graph. Prism graphs, hamming graphs, and some convex

polytopes have been discussed in the context of the double metric dimension in [28–30]. Ahmad et al. determined the minimal doubly resolving sets for some families of Harary graphs (see [31]). Chen et al. [32] provided the first explicit approximation lower and upper bounds for the minimum doubly resolving set problem. In [33, 34], Ahmad et al. have determined the metric and double metric dimensions for the line graphs of prism graphs, n -sunlet graphs, and kayak paddle graphs.

The convex polytopes in the context of metric dimension had been studied widely by many authors in last few years. The computation of the double metric dimension in convex polytopes is a challenging problem nowadays. In this paper, we compute the double metric dimensions of convex polytopes R_n and Q_n defined by Baca in [35]. Imran et al. [36] determined the metric dimensions of these convex polytopes given in the following theorems:

Theorem 1. *Let R_n be the graph of convex polytopes; then, for every $n \geq 5$, $\dim(R_n) = 3$.*

Theorem 2. *Let Q_n be the graph of convex polytopes; then, for every $n \geq 6$, $\dim(Q_n) = 3$.*

The above results are also useful in finding the lower bound of the double metric dimensions of convex polytopes R_n and Q_n .

The remaining portion of the article is structured as follows.

The double metric dimension for the convex polytopes R_n and Q_n is described in Sections 2 and 3, respectively. Finally, we conclude in Section 4 that the double metric dimension for the convex polytopes R_n and Q_n is constant.

2. The Double Metric Dimension for the Convex Polytope R_n

This section will particularly focus to find out the double metric dimension for convex polytope R_n . The graph of convex polytope R_n consists of three-sided, four-sided, and n -sided faces, as illustrated in Figure 1.

We label the vertices of the inner cycle by $\{e_j: \forall 0 \leq j \leq n - 1\}$, the vertices of the center cycle by $\{f_j: \forall 0 \leq j \leq n - 1\}$, and the vertices of the outer cycle by $\{g_j: \forall 0 \leq j \leq n - 1\}$, as demonstrated in Figure 1.

Theorem 1 implies that $\psi(R_n) \geq 3$, for $n \geq 5$. Furthermore, we will show that $\psi(R_n) = 3$, for $n \geq 5$. Now, to calculate distances for the convex polytope R_n , define $S_j(f_0) = \{f \in V_{R_n}: d(f_0, f) = j\}$ is the set of vertices in V_{R_n} at distance j from f_0 . For $\psi(R_n)$ with $n \geq 5$, we can find the sets $S_j(f_0)$ that are elaborated in Table 1. Obviously, it can be seen from Figure 1 that $S_j(f_0) = \emptyset$, when $j > ((n + 2)/2)$, for even n and $S_j(f_0) = \emptyset$, when $j > ((n + 1)/2)$, for odd n . Note that the sets $S_j(f_0)$ can clearly be used to find the distance between the two arbitrary vertices of V_{R_n} as follows.

The symmetry in Figure 1 shows that

FIGURE 1: Convex polytope R_n .TABLE 1: $S_j(f_0)$ for R_n .

n	j	$S_j(f_0)$
Even	1	$\{e_0, e_{n-1}, f_1, f_{n-1}, g_0\}$
	$2 \leq j \leq \lfloor n/2 \rfloor - 1$	$\{e_{j-1}, e_{n-j}, f_j, f_{n-j}, g_{j-1}, g_{n-j+1}\}$
	$n/2$	$\{e_{(n-2)/2}, e_{n/2}, f_{n/2}, g_{(n-2)/2}, g_{(n+2)/2}\}$
	$(n+2)/2$	$\{g_{n/2}\}$
Odd	$(n-1)/2$	$\{e_{(n-3)/2}, e_{(n+1)/2}, f_{(n-1)/2}, f_{(n+1)/2}, g_{(n-3)/2}, g_{(n+3)/2}\}$
	$(n+2)/2$	$\{e_{(n-1)/2}, g_{(n-1)/2}, g_{(n+1)/2}\}$

$$\begin{aligned}
 d(f_j, f_k) &= d(f_0, f_{|j-k|}), \text{ if } 0 \leq |j-k| \leq n-1, \\
 d(f_j, g_k) &= d(f_0, g_{|j-k|}), \text{ if } 0 \leq |j-k| \leq n-1, \\
 d(g_j, g_k) &= d(f_0, g_{|j-k|}) - 1, \text{ if } 0 \leq |j-k| \leq n-1.
 \end{aligned} \tag{1}$$

For odd values of n , we have

$$\begin{aligned}
 d(f_j, e_k) &= \begin{cases} d(f_0, e_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, e_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\
 d(e_j, g_k) &= \begin{cases} d(f_0, g_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j \geq k, \\ d(f_0, g_{|j-k|}), & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } k > j, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \end{cases} \\
 d(e_j, e_k) &= \begin{cases} d(f_0, e_{|j-k|}) - 1, & \text{if } 0 \leq |j-k| \leq \frac{n-1}{2}, \\ d(f_0, e_{|j-k|}), & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1. \end{cases}
 \end{aligned} \tag{2}$$

For even values of n , we have

$$\begin{aligned}
 d(f_j, e_k) &= \begin{cases} d(f_0, e_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j > k, \\ d(f_0, e_{|j-k|}), & \text{if } |j-k| = \frac{n}{2} \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, e_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\
 d(e_j, e_k) &= \begin{cases} d(f_0, e_{|j-k|}) - 1, & \text{if } 0 \leq |j-k| \leq \frac{n-2}{2}, \\ d(f_0, e_{|j-k|}), & \text{if } \frac{n}{2} \leq |j-k| \leq n-1, \end{cases} \\
 d(e_j, g_k) &= \begin{cases} d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n}{2} \text{ for } k > j, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j \geq k, \\ d(f_0, g_{|j-k|}), & \text{if } \frac{n}{2} \leq |j-k| \leq n-1 \text{ for } j > k. \end{cases}
 \end{aligned} \tag{3}$$

As an outcome, if the distance $d(f_0, f)$ is known for each $f \in V_{R_n}$, then one can recreate the distances between any two vertices of V_{R_n} .

Lemma 1. $\psi(R_n) = 3$ for $n \geq 5$, whenever n is even.

Proof. According to the set $D = \{e_0, e_{(n-2)/2}, g_1\}$, the representations of R_n is explained in Table 2.

Now, from Table 2, the first metric coordinate of the vector of $f_0 \in S_j(f_0)$ is equal to 1. Similarly, it can be observed that there are no two vertices $h_1, h_2 \in S_j(f_0)$, for every $j \in \{1, 2, \dots, ((n+2)/2)\}$ such as $r(h_1, D) - r(h_2, D) = 0$. Thus, there are no two vertices $h_1 \in S_j(f_0)$ and $h_2 \in S_k(f_0)$, for any $j, k \in \{1, 2, \dots, ((n+2)/2)\}$, $j \neq k$, such as $r(h_1, D) - r(h_2, D) = j - k$. Therefore, the set $D = \{e_0, e_{(n-2)/2}, g_1\}$ is the minimal doubly resolving set. Hence, Lemma 1 holds. $\square$

Lemma 2. $\psi(R_n) = 3$, for $n \geq 5$, whenever n is odd.

Proof. The minimal doubly resolving sets for R_5 and R_7 are

$$D = \begin{cases} \{e_0, e_2, g_4\}, & \text{if } n = 5, \\ \{e_0, e_3, g_2\}, & \text{if } n = 7. \end{cases} \tag{4}$$

Now, Table 3 demonstrates that the representations of R_n (where $n \geq 9$) in relation to the set $D = \{e_0, e_{(n-1)/2}, g_{(n+5)/2}\}$.

Now, from Table 3, the first metric coordinate of the vector of $f_0 \in S_j(f_0)$ is equal to 1. Similarly, it can be observed that there are no two vertices $h_1, h_2 \in S_j(f_0)$, for every $j \in \{1, 2, \dots, ((n+1)/2)\}$ such as $r(h_1, D) - r(h_2, D) = 0$. Thus, there are no two vertices $h_1 \in S_j(f_0)$ and $h_2 \in S_k(f_0)$, for any $j, k \in \{1, 2, \dots, ((n+1)/2)\}$, $j \neq k$, such as $r(h_1, D) - r(h_2, D) = j - k$. Therefore, the set $D = \{e_0, e_{(n-1)/2}, g_{(n+5)/2}\}$ is the minimal doubly resolving set. Hence, Lemma 2 holds. $\square$

It is displayed from the whole technique that $\psi(R_n) = 3$, for $n \geq 5$. The statement of the theorem by using Lemmas 1 and 2 is mentioned below.

Theorem 3. Let R_n be the convex polytope; then, $\psi(R_n) = 3$ for $n \geq 5$.

TABLE 2: Vectors of metric coordinates for R_n whenever n is even, where $n \geq 5$.

j	$S_j(f_0)$	$D = \{e_0, e_{(n-2)/2}, g_1\}$
0	f_0	$(1, (n/2), 2)$
	e_0	$(0, ((n-2)/2), 2)$
1	e_{n-1}	$(1, (n/2), 3)$
	f_1	$(1, ((n-2)/2), 1)$
	f_{n-1}	$(2, (n/2), 3)$
	g_0	$(2, ((n+2)/2), 1)$
$2 \leq j \leq ((n-2)/2)$	e_{j-1}	$(j-1, ((n-2j)/2), j)$
	e_{n-j}	$(j, ((n-2j+2)/2), ((n+2j-6)/2))$
	f_j	$(j, ((n-2j)/2), j)$
	f_{n-j}	$(j+1, ((n-2j+2)/2), ((n+2j-6)/2))$
	g_{j-1}	$(j, ((n-2j+4)/2), j-2)$
	g_{n-j+1}	$(j+1, ((n-2j+6)/2), j)$
	$e_{(n-2)/2}$	$((n-2)/2, 0, (n/2))$
	$e_{n/2}$	$((n/2), 1, ((n+2)/2))$
$n/2$	$f_{n/2}$	$((n/2), 1, (n/2))$
	$g_{(n-2)/2}$	$((n/2), 2, ((n-4)/2))$
	$g_{(n+2)/2}$	$((n+2)/2, 3, (n/2))$

TABLE 3: Vectors of metric coordinates for R_n whenever n is odd, where $n \geq 9$.

j	$S_j(f_0)$	$D = \{e_0, e_{(n-1)/2}, g_{(n+5)/2}\}$
0	f_0	$(1, ((n+1)/2), ((n-3)/2))$
	e_0	$(0, ((n-1)/2), ((n-1)/2))$
1	e_{n-1}	$(1, ((n-1)/2), ((n-3)/2))$
	f_1	$(1, ((n-1)/2), ((n-1)/2))$
	f_{n-1}	$(2, ((n-1)/2), ((n-5)/2))$
	g_0	$(2, ((n+3)/2), ((n-5)/2))$
$2 \leq j \leq ((n-3)/2)$	e_{j-1}	$= \begin{cases} (1, ((n-3)/2), ((n+1)/2)), & \text{if } j = 2; \\ (j-1, ((n-2j+1)/2), ((n-2j+9)/2)), & \text{if } j \neq 2, \end{cases}$
	e_{n-j}	$= \begin{cases} (((n-3)/2), 2, 2), & \text{if } j = ((n-3)/2); \\ (j, ((n-2j+1)/2), ((n-2j-1)/2)), & \text{if } j \neq ((n-3)/2), \end{cases}$
	f_j	$= \begin{cases} (2, ((n-3)/2), ((n+1)/2)), & \text{if } j = 2; \\ (j, ((n-2j+1)/2), ((n-2j+7)/2)), & \text{if } j \neq 2, \end{cases}$
	f_{n-j}	$= \begin{cases} (((n-1)/2), 2, 2), & \text{if } j = ((n-3)/2); \\ (j+1, ((n-2j+1)/2), ((n-2j-3)/2)), & \text{if } j \neq ((n-3)/2), \end{cases}$
	g_{j-1}	$= \begin{cases} (j, ((n-2j+5)/2), ((n+2j-7)/2)), & \text{if } j \leq 3; \\ (j, ((n-2j+5)/2), ((n-2j+7)/2)), & \text{if } j > 3, \end{cases}$
	g_{n-j+1}	$(j+1, ((n-2j+5)/2), ((n-2j-3)/2))$
	$e_{(n-3)/2}$	$((n-3)/2, 1, 5)$
	$e_{(n+1)/2}$	$((n-1)/2, 1, 3)$
$(n-1)/2$	$f_{(n-1)/2}$	$((n-1)/2, 1, 4)$
	$f_{(n+1)/2}$	$((n+1)/2, 1, 3)$
	$g_{(n-3)/2}$	$((n-1)/2, 3, 4)$
	$g_{(n+3)/2}$	$((n+1)/2, 3, 1)$
$(n+1)/2$	$e_{(n-1)/2}$	$((n-1)/2, 0, 4)$
	$g_{(n-1)/2}$	$((n+1)/2, 2, 3)$
	$g_{(n+1)/2}$	$((n+3)/2, 2, 2)$

3. The Double Metric Dimension for the Convex Polytope Q_n

This section will particularly focus to find out the double metric dimension for convex polytope Q_n . The graph of convex polytope Q_n consists of three-sided, four-sided, five-sided, and n -sided faces, as demonstrated in Figure 2.

We label the vertices of the inner cycle by $\{e_j: \forall 0 \leq j \leq n-1\}$, the vertices of the interior cycle by $\{f_j: \forall 0 \leq j \leq n-1\}$, the set of exterior vertices by $\{g_j: \forall 0 \leq j \leq n-1\}$, and the vertices of the outer cycle by $\{h_j: \forall 0 \leq j \leq n-1\}$, as demonstrated in Figure 2.

Theorem 2 implies that $\psi(Q_n) \geq 3$, for $n \geq 6$. Furthermore, we will show that $\psi(Q_n) = 3$, for $n \geq 6$.

FIGURE 2: Convex polytope Q_n .

Now, to calculate distances for the convex polytope Q_n , define $S_j(f_0) = \{f \in V_{Q_n} : d(f_0, f) = j\}$ is the set of vertices in V_{Q_n} at distance j from f_0 . For $\psi(Q_n)$ with $n \geq 6$, we can find the sets $S_j(f_0)$ that are elaborated in Table 4. Obviously, it can be seen from Figure 2 that $S_j(f_0) = \emptyset$, when $j > ((n+2)/2)$, for even n , and $S_j(f_0) = \emptyset$, when $j > ((n+3)/2)$, for odd n . Note that the sets $S_j(f_0)$ can clearly be used to find the distance between the two arbitrary vertices of V_{Q_n} as follows.

The symmetry in Figure 2 shows that

$$\begin{aligned} d(f_j, f_k) &= d(e_j, e_k) = d(h_j, h_k) \\ &= d(f_0, f_{|j-k|}), \text{ if } 0 \leq |j-k| \leq n-1, \\ d(f_j, e_k) &= d(f_0, e_{|j-k|}), \text{ if } 0 \leq |j-k| \leq n-1. \end{aligned} \quad (5)$$

For odd values of n , we have

$$\begin{aligned} d(f_j, g_k) &= \begin{cases} d(f_0, g_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\ d(g_j, g_k) &= \begin{cases} d(f_0, g_{|j-k|}) - 1, & \text{if } |j-k| = 0, \\ d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2}, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1, \end{cases} \\ d(e_j, g_k) &= \begin{cases} d(f_0, e_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq \frac{n-1}{2} \text{ for } k \geq j, \\ d(f_0, e_{|j-k|}), & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \end{cases} \end{aligned}$$

TABLE 4: $S_j(f_0)$ for Q_n .

n	j	$S_j(f_0)$
Even	1	$\{e_0, f_1, f_{n-1}, g_0, g_{n-1}\}$
	$2 \leq j \leq \lfloor n/2 \rfloor - 1$	$\{e_{j-1}, e_{n-j+1}, f_j, f_{n-j}, g_{j-1}, g_{n-j}, h_{j-2}, h_{n-j+1}\}$
	$n/2$	$\{e_{(n-2)/2}, e_{(n+2)/2}, f_{n/2}, g_{(n-2)/2}, g_{n/2}, h_{(n-4)/2}, h_{(n+2)/2}\}$
	$(n+2)/2$	$\{e_{n/2}, h_{(n-2)/2}, h_{n/2}\}$
Odd	$(n-1)/2$	$\{e_{(n-3)/2}, e_{(n+3)/2}, f_{(n-1)/2}, f_{(n+1)/2}, g_{(n-3)/2}, g_{(n+1)/2}, h_{(n-5)/2}, h_{(n+3)/2}\}$
	$(n+1)/2$	$\{e_{(n-1)/2}, e_{(n+1)/2}, g_{(n-1)/2}, h_{(n-3)/2}, h_{(n+1)/2}\}$
	$(n+3)/2$	$\{h_{(n-1)/2}\}$

$$\begin{aligned}
d(g_j, h_k) &= \begin{cases} d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } 0 \leq |j-k| \leq \frac{n-1}{2} \text{ for } k \geq j, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \end{cases} \\
d(f_j, h_k) &= \begin{cases} d(f_0, h_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 1, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, h_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\
d(e_j, h_k) &= \begin{cases} d(f_0, h_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-1}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 2, & \text{if } \frac{n+1}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j. \end{cases}
\end{aligned} \tag{6}$$

For even values of n , we have

$$d(g_j, g_k) = \begin{cases} d(f_0, g_{|j-k|}) - 1, & \text{if } |j-k| = 0, \\ d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2}, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n}{2} \leq |j-k| \leq n-1, \end{cases}$$

$$\begin{aligned}
d(f_j, g_k) &= \begin{cases} d(f_0, g_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } |j-k| = \frac{n}{2} \text{ for } j > k, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\
d(e_j, g_k) &= \begin{cases} d(f_0, e_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n}{2} \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, e_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq \frac{n-2}{2} \text{ for } k \geq j, \\ d(f_0, e_{|j-k|}), & \text{if } \frac{n}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \end{cases} \\
d(f_j, h_k) &= \begin{cases} d(f_0, h_{|j-k|}) - 1, & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}), & \text{if } |j-k| = \frac{n}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 1, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, h_{|j-k|}), & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j, \end{cases} \\
d(g_j, h_k) &= \begin{cases} d(f_0, g_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j > k, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, g_{|j-k|}), & \text{if } 0 \leq |j-k| \leq \frac{n-2}{2} \text{ for } k \geq j, \\ d(f_0, g_{|j-k|}) + 1, & \text{if } \frac{n}{2} \leq |j-k| \leq n-1 \text{ for } k > j, \end{cases} \\
d(e_j, h_k) &= \begin{cases} d(f_0, h_{|j-k|}), & \text{if } 1 \leq |j-k| \leq \frac{n-2}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 1, & \text{if } |j-k| = \frac{n}{2} \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 2, & \text{if } \frac{n+2}{2} \leq |j-k| \leq n-1 \text{ for } j > k, \\ d(f_0, h_{|j-k|}) + 1, & \text{if } 0 \leq |j-k| \leq n-1 \text{ for } k \geq j. \end{cases}
\end{aligned}$$

TABLE 5: Vectors of metric coordinates for Q_n whenever n is even, where $n \geq 6$.

j	$S_j(f_0)$	$D = \{e_0, e_{n/2}, h_0\}$
0	f_0	$(1, ((n+2)/2), 2)$
	e_0	$(0, (n/2), 3)$
	f_1	$(2, (n/2), 3)$
1	f_{n-1}	$(2, ((n-2)/2), 1)$
	g_0	$(2, ((n+2)/2), 1)$
	g_{n-1}	$(2, ((n+2)/2), 2)$
	e_{j-1}	$(j-1, ((n-2j+2)/2), j+1)$
	e_{n-j+1}	$(j-1, ((n-2j+2)/2), j+2)$
	f_j	$(j+1, ((n-2j+2)/2), j+1)$
$2 \leq j \leq ((n-2)/2)$	f_{n-j}	$(j+1, ((n-2j+2)/2), j+2)$
	g_{j-1}	$(j+1, ((n-2j+4)/2), j)$
	g_{n-j}	$(j+1, ((n-2j+4)/2), j+1)$
	h_{j-2}	$(j+1, ((n-2j+8)/2), j-2)$
	h_{n-j+1}	$(j+1, ((n-2j+8)/2), j-1)$
	$e_{(n-2)/2}$	$((n-2)/2, 1, ((n+2)/2))$
	$e_{(n+2)/2}$	$((n-2)/2, 1, ((n+4)/2))$
	$f_{n/2}$	$((n+2)/2, 1, ((n+2)/2))$
$n/2$	$g_{(n-2)/2}$	$((n+2)/2, 2, (n/2))$
	$g_{n/2}$	$((n+2)/2, 2, ((n+2)/2))$
	$h_{(n-4)/2}$	$((n+2)/2, 4, ((n-4)/2))$
	$h_{(n+2)/2}$	$((n+2)/2, 4, ((n-2)/2))$
	$e_{n/2}$	$((n/2), 0, ((n+4)/2))$
$(n+2)/2$	$h_{(n-2)/2}$	$((n+4)/2, 3, ((n-2)/2))$
	$h_{n/2}$	$((n+4)/2, 3, (n/2))$

TABLE 6: Vectors of metric coordinates for Q_n whenever n is odd, where $n \geq 13$.

j	$S_j(f_0)$	$D = \{e_0, e_{(n-1)/2}, h_{(n+5)/2}\}$
0	f_0	$(1, ((n+1)/2), ((n-3)/2))$
	e_0	$(0, ((n-1)/2), ((n-1)/2))$
	f_1	$(2, ((n-1)/2), ((n-1)/2))$
1	f_{n-1}	$(2, ((n+1)/2), ((n-5)/2))$
	g_0	$(2, ((n+1)/2), ((n-3)/2))$
	g_{n-1}	$(2, ((n+3)/2), ((n-5)/2))$
	e_{j-1}	$= \begin{cases} (j-1, ((n-2j+1)/2), ((n+2j-3)/2)), & \text{if } j \leq 4; \\ (j-1, ((n-2j+1)/2), ((n-2j+13)/2)), & \text{if } j > 4. \end{cases}$
	e_{n-j+1}	$= \begin{cases} (j-1, ((n-2j+3)/2), ((n-2j+1)/2)), & \text{if } j \neq ((n-3)/2); \\ ((n-5)/2, 3, 3), & \text{if } j = ((n-3)/2), \end{cases}$
	f_j	$= \begin{cases} (j+1, ((n-2j+1)/2), ((n+2j-3)/2)), & \text{if } j \leq 3; \\ (j+1, ((n-2j+1)/2), ((n-2j+9)/2)), & \text{if } j > 3, \end{cases}$
$2 \leq j \leq ((n-3)/2)$	f_{n-j}	$= \begin{cases} (j+1, ((n-2j+3)/2), ((n-2j-3)/2)), & \text{if } j \leq ((n-7)/2); \\ (j+1, ((n-2j+3)/2), ((2j-n+9)/2)), & \text{if } j > ((n-7)/2), \end{cases}$
	g_{j-1}	$= \begin{cases} (j+1, ((n-2j+3)/2), ((n+2j-5)/2)), & \text{if } j \leq 3; \\ (j+1, ((n-2j+3)/2), ((n-2j+9)/2)), & \text{if } j > 3, \end{cases}$
	g_{n-j}	$= \begin{cases} (j+1, ((n-2j+5)/2), ((n-2j-3)/2)), & \text{if } j \neq ((n-3)/2); \\ ((n-1)/2, 4, 2), & \text{if } j = ((n-3)/2), \end{cases}$
	h_{j-2}	$= \begin{cases} (j+1, ((n-2j+7)/2), ((n+2j-9)/2)), & \text{if } j \leq 4; \\ (j+1, ((n-2j+7)/2), ((n-2j+9)/2)), & \text{if } j > 4, \end{cases}$
	h_{n-j+1}	$(j+1, ((n-2j+9)/2), ((n-2j-3)/2))$

TABLE 6: Continued.

j	$S_j(f_0)$	$D = \{e_0, e_{(n-1)/2}, h_{(n+5)/2}\}$
$(n-1)/2$	$e_{(n-3)/2}$	$((n-3)/2, 1, 7)$
	$e_{(n+3)/2}$	$((n-3)/2, 2, 4)$
	$f_{(n-1)/2}$	$((n+1)/2, 1, 5)$
	$f_{(n+1)/2}$	$((n+1)/2, 2, 4)$
	$g_{(n-3)/2}$	$((n+1)/2, 2, 5)$
	$g_{(n+1)/2}$	$((n+1)/2, 3, 3)$
	$h_{(n-5)/2}$	$((n+1)/2, 4, 5)$
	$h_{(n+3)/2}$	$((n+1)/2, 5, 1)$
$(n+1)/2$	$e_{(n-1)/2}$	$((n-1)/2, 0, 6)$
	$e_{(n+1)/2}$	$((n-1)/2, 1, 5)$
	$g_{(n-1)/2}$	$((n+3)/2, 2, 4)$
	$h_{(n-3)/2}$	$((n+3)/2, 3, 4)$
$(n+3)/2$	$h_{(n+1)/2}$	$((n+3)/2, 4, 2)$
	$h_{(n-1)/2}$	$((n+5)/2, 3, 3)$

As an outcome, if the distance $d(f_0, f)$ for every $f \in V_{Q_n}$ is known, then one can recreate the distances between any two vertices of V_{Q_n} .

Lemma 3. $\psi(Q_n) = 3$, for $n \geq 6$, whenever n is even.

Proof. The representations of Q_n according to the set $D = \{e_0, e_{n/2}, h_0\}$ are explained in Table 5.

Now, from Table 5, the first metric coordinate of the vector of $f_0 \in S_j(f_0)$ is equal to 1. Similarly, it can be checked that there are no two vertices $p_1, p_2 \in S_j(f_0)$, for every $j \in \{1, 2, \dots, ((n+2)/2)\}$ such as $r(p_1, D) - r(p_2, D) = 0$. Thus, there are no two vertices $p_1 \in S_j(f_0)$ and $p_2 \in S_k(f_0)$, for any $j, k \in \{1, 2, \dots, ((n+2)/2)\}$, $j \neq k$, such as $r(p_1, D) - r(p_2, D) = j - k$. Therefore, the set $D = \{e_0, e_{n/2}, h_0\}$ is the minimal doubly resolving set. Thus, Lemma 3 holds. $\square$

Lemma 4. $\psi(Q_n) = 3$, for $n \geq 6$, whenever n is odd.

Proof. The minimal doubly resolving sets for Q_7 , Q_9 , and Q_{11} are

$$D = \begin{cases} \{e_0, e_3, h_1\}, & \text{if } n = 7, \\ \{e_0, e_4, h_7\}, & \text{if } n = 9, \\ \{e_0, e_5, h_2\}, & \text{if } n = 11. \end{cases} \quad (8)$$

The representations of convex polytope Q_n (where $n \geq 13$) according to the set $D = \{e_0, e_{(n-1)/2}, h_{(n+5)/2}\}$ are explained in Table 6.

Now, from Table 6, the first metric coordinate of the vector of $f_0 \in S_j(f_0)$ is equal to 1. Similarly, it can be checked that there are no two vertices $p_1, p_2 \in S_j(f_0)$, for every $j \in \{1, 2, \dots, ((n+3)/2)\}$ such as $r(p_1, D) - r(p_2, D) = 0$. Thus, there are no two vertices $p_1 \in S_j(f_0)$ and $p_2 \in S_k(f_0)$, for any $j, k \in \{1, 2, \dots, ((n+3)/2)\}$, $j \neq k$, such as $r(p_1, D) - r(p_2, D) = j - k$. Therefore, the set $D = \{e_0, e_{(n-1)/2}, h_{(n+5)/2}\}$ is the minimal doubly resolving set. Hence, Lemma 4 holds. $\square$

It is displayed from the whole technique that $\psi(Q_n) = 3$, for $n \geq 6$. The statement of the theorem by using Lemmas 3 and 4 is mentioned below.

Theorem 4. Let Q_n be the convex polytope, then $\psi(Q_n) = 3$, for $n \geq 6$.

4. Conclusion

Despite of the fact that determining the minimal resolving sets of general graphs is computationally tough, the metric dimension has been gaining all the attention due to its applications in the different fields such as computer networking, navigation of robots, sonar technology, and optimization problems. The doubly resolving sets are a reasonable tool to successfully diagnose the source of infection within a network. The metric and double metric dimensions are NP-hard in general case.

The focus of article was the computation of the double metric dimension regarding convex polytopes R_n and Q_n . We conclude that the double metric dimension of these classes of plane graphs is finite and independent of the number of vertices in these graphs. Just three vertices suitably picked up are sufficient to doubly resolve all the vertices of these classes of plane graphs.

Data Availability

All data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

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