Research Article

Hermite–Hadamard-Type Inequalities for the Generalized Geometrically Strongly Modified h -Convex Functions

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Convexity theory becomes a hot area of research due to its applications in pure and applied mathematics, especially in optimization theory. The aim of this paper is to introduce a broader class of convex functions by unifying geometrically strong convex function with h convex functions. This new class of functions is called as generalized geometrically strongly modified h -convex functions. We established Hermite–Hadamard-type inequalities for the generalized geometrically strongly modified h -convex functions. Our results can be considered as generalization and extension of literature.

1. Introduction

The modern analysis involves the applications of convexity, and no mathematician, working in applied mathematics, especially, in nonlinear programming and optimization theory, can ignore the significant role of convex sets and convex functions. It can be viewed as one of the most natural and simple notation in mathematics. The convexity of sets and the convexity of functions have been the object of many studies during the last century. Early contributions to convex analysis were made by Hölder [1] and Minkowski [2]. Convexity also plays a fundamental role in mathematical economics and statistics [3–5], and the importance of convex functions is well known in statistics, management sciences, and engineering. Since the classical convexity is not enough to attain certain goals in applied mathematics, the classical convexity has been generalized in many ways, see [6, 7].

The theory of inequalities produced important contributions in convex analysis, for example, the Hermite–Hadamard, Schur, and Fejér inequalities are very important and produce many applications [8, 9]. The

Hermite–Hadamard inequalities give us an estimate of the mean value of a convex function and are defined as follows.

The Hermite–Hadamard inequality for a convex function $\phi: L = [r, s] \subset \mathbb{R} \rightarrow \mathbb{R}$ is

$$\phi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r} \int_r^s \phi(x) dx \leq \frac{\phi(r) + \phi(s)}{2}. \quad (1)$$

There are many generalizations of the Hermite–Hadamard inequality, see [10–12]. Different notions of convex functions and different variants of fractional intergrades have been used in literature to establish Hermite–Hadamard inequalities, see, for example, Hermite–Hadamard-type inequalities for harmonically convex functions [6], Hermite–Hadamard inequality for Caputo-Fabrizio fractional integrals [13], Hermite–Hadamard-type inequalities for h -convexity [14], and Hermite–Hadamard inequalities for s -convex and s -concave functions [15]. For more details about inequality theory, we refer the readers to [16–20] and references therein.

The other interesting inequalities are Chebyshev-type inequalities [15, 21, 22], Trapezium-type inequalities [23], generalized fractional Hilfer integral inequalities [24], etc.

[25]. Also, many different extensions and versions appeared in a number of papers, see, for example, [26–30].

In this paper, we introduced a broad class of convex functions and established Hermite–Hadamard-type inequalities in the setting of newly introduced class of functions.

The paper is organized as follows. In Section 2, we will give some basic definitions and basic algebraic properties for this class of functions. Sections 3 and 4 are devoted to Hermite–Hadamard and some other inequalities for the generalized geometrically strongly modified h -convex functions.

2. Definitions and Basic Results

We start with some basic definitions related to our work. Throughout this paper, J is an interval in $\mathbb{R}$ and $\eta: N \times N \rightarrow M \subset \mathbb{R}$ is a bi-function.

Definition 1 (convex function, see [31]). Let $\phi: J \rightarrow \mathbb{R}$ be an extended real-valued function defined on a convex set $J \subset \mathbb{R}^n$. Then, the function ϕ is convex on J if

$$\phi(\alpha r + (1 - \alpha)s) \leq \alpha\phi(r) + (1 - \alpha)\phi(s) \quad (2)$$

holds, for all $r, s \in J$ and $\alpha \in (0, 1)$.

Definition 2 (h -convex function, see [14]). Let $\phi, h: J \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then, ϕ is h -convex function, if

$$\phi(\alpha r + (1 - \alpha)s) \leq h(\alpha)\phi(r) + h(1 - \alpha)\phi(s) \quad (3)$$

holds, for all $r, s \in J$ and $\alpha \in [0, 1]$.

Definition 3 (modified h -convex function, see [32]). Let $\phi, h: J \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then, ϕ is called modified h -convex function if

$$\phi(\alpha r + (1 - \alpha)s) \leq h(\alpha)\phi(r) + (1 - h(\alpha))\phi(s) \quad (4)$$

holds, for all $r, s \in J$ and $\alpha \in [0, 1]$.

Definition 4 (generalized convex function, see [33]). A function $\phi: J = [r, s] \rightarrow \mathbb{R}$ is said to be generalized convex function with respect to a bi-function $\eta(.,.): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if

$$\phi(\alpha r + (1 - \alpha)s) \leq \phi(s) + \alpha\eta(\phi(r), \phi(s)), \quad \forall r, s \in J, \alpha \in [0, 1], \quad (5)$$

holds.

Remark 1. If we take $\eta(r, s) = r - s$ in Definition 1, then the generalized convex function reduces to a convex function.

Definition 5 (generalized modified h -convex function). Let $\phi, h: J \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then, ϕ is the generalized modified h -convex function if

$$\phi(\alpha r + (1 - \alpha)s) \leq \phi(s) + h(\alpha)\eta(\phi(r), \phi(s)) \quad (6)$$

holds, for all $r, s \in J$ and $\alpha \in [0, 1]$.

Definition 6 (generalized strongly modified h -convex function, see [34]). Let $\phi, h: J \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then, ϕ is the generalized strongly modified h -convex function if

$$\phi(\alpha r + (1 - \alpha)s) \leq \phi(s) + h(\alpha)\eta(\phi(r), \phi(s)) - \mu\alpha(1 - \alpha)(r - s)^2 \quad (7)$$

holds, for all $r, s \in J$ and $\alpha \in [0, 1]$.

Definition 7 (additive). A function η is said to be additive if

$$\eta(r_1, s_1) + \eta(r_2, s_2) = \eta(r_1 + r_2, s_1 + s_2), \quad \forall r_1, r_2, s_1, s_2 \in \mathbb{R}. \quad (8)$$

Definition 8 (nonnegatively homogeneous). A function η is said to be nonnegatively homogeneous if

$$\eta(\lambda r_1, \lambda r_2) = \lambda\eta(r_1, r_2), \quad \forall r_1, r_2 \in \mathbb{R}. \quad (9)$$

Definition 9 (supermultiplicative function, see [35]). A function $\phi: J \subset \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be supermultiplicative function if

$$\phi(r, s) \geq \phi(r)\phi(s), \quad \forall r, s \in J, \alpha \in [0, 1]. \quad (10)$$

Definition 10 (logarithmic mean). Let $r, s \in \mathbb{R}$ such that $r, s \neq 0$ and $|r| \neq |s|$; then, logarithmic mean for real numbers is defined as

$$L(r, s) = \frac{r - s}{\ln|r| - \ln|s|}. \quad (11)$$

Definition 11 (geometrically convex function, see [14]). A function $\phi: J \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ is said to be a geometrically convex function or geometric arithmetic convexity with respect to a bi-function $\eta(.,.): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if

$$\phi(r^\alpha s^{1-\alpha}) \leq \alpha\phi(r) + (1 - \alpha)\phi(s), \quad \forall r, s \in [0, 1]. \quad (12)$$

Definition 12 (generalized geometrically convex function, see [24]). A function $\phi: J \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ is said to be generalized geometrically convex function with respect to a bi-function $\eta(.,.): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if

$$\phi(r^\alpha s^{1-\alpha}) \leq (1 - \alpha)\phi(s) + \alpha(\phi(s) + \eta(\phi(r), \phi(s))), \quad \forall r, s \in [0, 1]. \quad (13)$$

Definition 13 (generalized geometrically strongly modified h -convex functions). A function $\phi: J \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ is said to be generalized geometrically strongly modified h -convex function with respect to a bi-function $\eta(.,.): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1 - h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s))) - \mu\alpha(1 - \alpha)(r - s)^2. \end{aligned} \quad (14)$$

Lemma 1. For $r, s \in [0, \infty)$ and $m, \alpha \in (0, 1]$, if $r < s$ and $s \geq 1$, then

$$r^\alpha s^{m(1-\alpha)} \leq \alpha r + (1-\alpha)s. \quad (15)$$

Remark 2. The abovementioned definitions can be related with each other as follows:

- (1) For $\eta(r, s) = r - s$, the generalized convex function reduces to the convex function
- (2) For $h(\alpha) = \alpha$, the generalized modified h -convex function reduces to the generalized convex function
- (3) For $h(\alpha) = \alpha$ and $\eta(r, s) = r - s$, the generalized modified h -convex function reduces to the convex function
- (4) For $\eta(r, s) = r - s$, the generalized modified h -convex function reduces to the modified h -convex function
- (5) For $\mu = 0, \eta(r, s) = r - s$ and $h(\alpha) = \alpha$, the generalized geometrically strongly modified h -convex functions reduces to the convex function
- (6) For $h(\alpha) = \alpha$ and $\mu = 0$, the generalized geometrically strongly modified h -convex functions reduces to the generalized geometrically convex function

The following proposition is about the addition of two functions.

Proposition 1. Let η be additive $\mu^* \geq 0$ and $2\mu = \mu^*$; then, the sum of two generalized geometrically strongly modified h -convex functions is also generalized geometrically strongly modified h -convex functions.

proof. Let ϕ and ψ be two generalized geometrically strongly modified h -convex functions; then, by definition, we have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s))) - \mu\alpha(1-\alpha)(r-s)^2, \\ \psi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\psi(s) + h(\alpha)(\psi(s) \\ &\quad + \eta(\psi(r), \psi(s))) - \mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (16)$$

So,

$$\begin{aligned} &\phi(r^\alpha s^{1-\alpha}) + \psi(r^\alpha s^{1-\alpha}) \\ &\leq [1-h(\alpha)](\phi(s) + \psi(s)) + h(\alpha)[\phi(s) + \eta(\phi(r), \phi(s)) + \psi(s) \\ &\quad + \eta(\psi(r), \psi(s))] - 2\mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (17)$$

Now, by additive property of η and taking $2\mu = \mu^*$, we obtain

$$\begin{aligned} &\phi(r^\alpha s^{1-\alpha}) + \psi(r^\alpha s^{1-\alpha}) \\ &\leq [1-h(\alpha)](\phi(s) + \psi(s)) + h(\alpha)[(\phi(s) + \psi(s)) \\ &\quad + \eta(\phi(r) + \psi(r), \phi(s) + \psi(s))] - \mu^*\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (18)$$

This completes the proof.

The next result is about the scalar multiplication of a function. $\square$

Proposition 2. Let ϕ be generalized geometrically strongly modified h -convex function with $\lambda \geq 0$ and η is nonnegatively homogeneous. Then, $\lambda\phi$ is also a generalized geometrically strongly modified h -convex function.

Proof. Since ϕ is the generalized geometrically strongly modified h -convex function, so, by definition, we have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s))) - \mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (19)$$

So,

$$\begin{aligned} \lambda\phi(r^\alpha s^{1-\alpha}) &\leq \lambda[(1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) + \eta(\phi(r), \phi(s))) \\ &\quad - \mu\alpha(1-\alpha)(r-s)^2] \leq (1-h(\alpha))\lambda\phi(s) + \lambda h(\alpha)(\phi(s) + \eta(\phi(r), \phi(s))) \\ &\quad - \lambda\mu\alpha(1-\alpha)(r-s)^2 \leq (1-h(\alpha))\lambda\phi(s) + h(\alpha)(\lambda\phi(s) + \lambda\eta(\phi(r), \phi(s))) \\ &\quad - \lambda\mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (20)$$

As η is nonnegatively homogeneous and by taking $\lambda\mu = \mu^*$, we obtain

$$\begin{aligned} \lambda\phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\lambda\phi(s) + h(\alpha)(\lambda\phi(s) + \eta(\lambda\phi(r), \lambda\phi(s))) \\ &\quad - \mu^*\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (21)$$

This completes the proof. $\square$

Proposition 3. Let ϕ and ψ be two generalized geometrically strongly modified h -convex functions, where η is additive and nonnegatively homogeneous. Then, $r\phi + s\psi$ is also the generalized geometrically strongly modified h -convex function $\forall r, s \in \mathbb{R}$.

Proposition 4. Let ψ be a supermultiplicative function and ϕ be a generalized geometrically strongly modified h -convex function. If $\psi(r) - \psi(s) = r - s$, then $\phi \circ \psi$ is also a generalized geometrically strongly modified h -convex function.

Proof. Let ψ be a supermultiplicative function and ϕ is a generalized geometrically strongly modified h -convex function. Also, since $\psi(r) - \psi(s) = r - s$, so $\forall r, s \in J$ and $\alpha \in [0, 1]$, and we obtain

$$\begin{aligned} (\phi \circ \psi)(r^\alpha s^{1-\alpha}) &\geq \phi(\psi(r)\psi(s^{1-\alpha})) \\ &\leq (1-h(\alpha))(\phi \circ \psi)(s) + h(\alpha)((\phi \circ \psi)(s)) \\ &\quad + \eta((\phi \circ \psi)(r), (\phi \circ \psi)(s)) - \mu\alpha(1-\alpha)(\psi(r) - \psi(s))^2 \\ &= (1-h(\alpha))(\phi \circ \psi)(s) + h(\alpha)((\phi \circ \psi)(s)) \\ &\quad + \eta((\phi \circ \psi)(r), (\phi \circ \psi)(s)) - \mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (22)$$

This shows that $\phi \circ \psi$ is a generalized geometrically strongly modified h -convex function. $\square$

Proposition 5. Let us consider $\phi_i: J \subset \mathbb{R} \longrightarrow \mathbb{R}$ such that ϕ_i are generalized geometrically strongly modified h -convex functions, where $\sum_{i=1}^m \lambda_i = 1$ and η is additive and nonnegatively homogeneous. Then, the linear combination $\psi: \mathbb{R} \longrightarrow \mathbb{R}$ is also a generalized geometrically strongly modified h -convex function.

Proof. Letting $f(x) = \sum_{i=1}^m \lambda_i \phi_i(x)$ and setting $r = (r^\alpha s^{1-\alpha})$, we obtain

$$\begin{aligned} f(r^\alpha s^{1-\alpha}) &= \sum_{i=1}^m \lambda_i \phi_i(r^\alpha s^{1-\alpha}) \\ &\leq (1-h(\alpha)) \sum_{i=1}^m \lambda_i \phi_i(s) + h(\alpha) \sum_{i=1}^m \lambda_i \phi_i(s) \\ &\quad + \eta(\phi_i(r), \phi_i(s)) - \mu\alpha(1-\alpha)(r-s)^2 \\ &= \sum_{i=1}^m \lambda_i \phi_i(s) + h(\alpha) \eta \left(\sum_{i=1}^m \lambda_i \phi_i(r), \sum_{i=1}^m \lambda_i \phi_i(s) \right) \\ &\quad - \mu\alpha(1-\alpha)(r-s)^2 \\ &= f(s) + h(\alpha) \eta(f(r), f(s)) - \mu\alpha(1-\alpha)(r-s)^2. \end{aligned} \quad (23)$$

This completes the proof. $\square$

Proposition 6. If ϕ is a generalized geometrically strongly modified h -convex function, then ϕ is also a generalized geometrically modified h -convex function.

Proof. We have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s))) - \mu\alpha(1-\alpha)(r-s)^2 \\ &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) + \eta(\phi(r), \phi(s))), \end{aligned} \quad (24)$$

$\forall r, s \in J \subset \mathbb{R}$. This completes the proof. $\square$

Corollary 1. If ϕ is the generalized geometrically strongly convex function and $\alpha \leq h(\alpha)$, then ϕ is also a generalized geometrically strongly modified h -convex function.

3. Hermite–Hadamard-Type Inequality

In this section, we derive some new Hermite–Hadamard-type inequalities for generalized geometrically strongly modified h -convex functions. Throughout this section, consider $J = [r, s]$.

Theorem 1. Let $\phi, \psi: J = [r, s] \longrightarrow (0, \infty)$ be two generalized geometrically strongly modified h -convex functions and $r, s \in J$ with $r < s$; then, the following inequality holds:

$$\begin{aligned} &\frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} [\phi(x)\psi(x)] dx \\ &\leq \phi(s)\psi(s) \int_0^1 (1-h(\alpha))^2 d\alpha + M_1(r, s) \int_0^1 h(\alpha)(1-h(\alpha)) d\alpha \\ &\quad + M_2(r, s) \int_0^1 (h(\alpha))^2 d\alpha - \mu(r-s)^2 \eta(\phi(r) \\ &\quad + \psi(r), \phi(s) + \psi(s)) \int_0^1 \alpha(1-\alpha)h(\alpha) d\alpha \\ &\quad - \frac{1}{6} \mu(r-s)^2 \left[(\phi(s) + \psi(s)) - \frac{1}{5} \mu(r-s)^2 \right], \end{aligned} \quad (25)$$

where

$$M_1(r, s) = [\phi(s)[\psi(s) + \eta(\psi(r), \psi(s))] + \psi(s)[\phi(s) + \eta(\phi(r), \phi(s))] \text{ and } M_2(r, s) = [[\phi(s) + \eta(\phi(r), \phi(s))] [\psi(s) + \eta(\psi(r), \psi(s))]].$$

Proof. Since ϕ, ψ are generalized geometrically strongly modified h -convex functions, then $\forall r, s \in J$ and $\alpha \in [0, 1]$, and we have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s)) - \mu\alpha(1-\alpha)(r-s)^2, \\ \psi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\psi(s) + h(\alpha)(\psi(s) \\ &\quad + \eta(\psi(r), \psi(s)) - \mu\alpha(1-\alpha)(r-s)^2, \end{aligned} \quad (26)$$

which implies that

$$\begin{aligned}
& \phi(r^\alpha s^{1-\alpha})\psi(r^\alpha s^{1-\alpha}) \\
& \leq (1-h(\alpha))^2\phi(s)\psi(s) + h(\alpha)(1-h(\alpha))[\phi(s)[\psi(s) + \eta(\psi(r), \psi(s))] \\
& \quad + \psi(s)[\phi(s) + \eta(\phi(r), \phi(s))] + (h(\alpha))^2[[\phi(s) + \eta(\phi(r), \phi(s))] \\
& \quad \times [\psi(s) + \eta(\psi(r), \psi(s))]] - \mu\alpha(1-\alpha)(r-s)^2[(\phi(s) + \psi(s))(1-h(\alpha)) \\
& \quad + h(\alpha)[\phi(s) + \eta(\phi(r), \phi(s)) + \psi(s) + \eta(\psi(r), \psi(s))]] + \mu^2\alpha^2(1-\alpha)^2(r-s)^4.
\end{aligned} \tag{27}$$

Integrate both sides over α on $[0, 1]$ and by the additive property of η , i.e., $\eta(r_1, s_1) + \eta(r_2, s_2) = \eta(r_1 + r_2, s_1 + s_2)$, and we have

$$\begin{aligned}
\int_0^1 \phi(r^\alpha s^{1-\alpha})\psi(r^\alpha s^{1-\alpha})d\alpha &= \frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} [\phi(x)\psi(x)]dx \\
&\leq \phi(s)\psi(s) \int_0^1 (1-h(\alpha))^2 d\alpha + [\phi(s)[\psi(s) + \eta(\psi(r), \psi(s))] \\
&\quad + \psi(s)[\phi(s) + \eta(\phi(r), \phi(s))] \int_0^1 h(\alpha)(1-h(\alpha))d\alpha + [[\phi(s) \\
&\quad + \eta(\phi(r), \phi(s))][\psi(s) + \eta(\psi(r), \psi(s))] \int_0^1 (h(\alpha))^2 d\alpha - \mu(r-s)^2 \\
&\quad \times \int_0^1 \alpha(1-\alpha)[(1-h(\alpha))(\phi(s) + \psi(s)) + h(\alpha)[\phi(s) + \psi(s) \\
&\quad + \eta(\phi(r) + \psi(r), \phi(s) + \psi(s))]d\alpha + \mu^2(r-s)^4 \int_0^1 \alpha^2(1-\alpha)^2 d\alpha \\
&\leq \phi(s)\psi(s) \int_0^1 (1-h(\alpha))^2 d\alpha + [\phi(s)[\psi(s) + \eta(\psi(r), \psi(s))] \\
&\quad + \psi(s)[\phi(s) + \eta(\phi(r), \phi(s))] \int_0^1 h(\alpha)(1-h(\alpha))d\alpha + [[\phi(s) \\
&\quad + \eta(\phi(r), \phi(s))][\psi(s) + \eta(\psi(r), \psi(s))] \\
&\quad \int_0^1 (h(\alpha))^2 d\alpha - \mu(r-s)^2 \eta(\phi(r) + \psi(r), \phi(s) + \psi(s)) \int_0^1 \alpha(1-\alpha)h(\alpha)d\alpha \\
&\quad - \frac{1}{6}\mu(r-s)^2 \left[(\phi(s) + \psi(s)) - \frac{1}{5}\mu(r-s)^2 \right] \\
&\leq \phi(s)\psi(s) \int_0^1 (1-h(\alpha))^2 d\alpha + M_1(r, s) \int_0^1 h(\alpha)(1-h(\alpha))d\alpha \\
&\quad + M_2(r, s) \int_0^1 (h(\alpha))^2 d\alpha - \mu(r-s)^2 \eta(\phi(r) + \psi(r), \phi(s) + \psi(s)) \\
&\quad \times \int_0^1 \alpha(1-\alpha)h(\alpha)d\alpha - \frac{1}{6}\mu(r-s)^2 \left[(\phi(s) + \psi(s)) - \frac{1}{5}\mu(r-s)^2 \right].
\end{aligned} \tag{28}$$

This completes the proof. $\square$

Theorem 2. Let $\phi, \psi: J = [r, s] \longrightarrow (0, \infty)$ be generalized geometrically strongly modified h -convex functions and $r, s \in J$ with $r < s$; then, the following inequality holds:

$$\begin{aligned} & \frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \psi\left(\frac{rs}{x}\right) \right] dx \\ & \leq \phi(s) \psi(r) \int_0^1 (1 - h(\alpha))^2 d\alpha + M_1(r, s) \int_0^1 h(\alpha) (1 - h(\alpha)) d\alpha + M_2(r, s) \int_0^1 (h(\alpha))^2 d\alpha \\ & \quad - \mu(r - s)^2 \eta(\phi(r) + \psi(s), \phi(s) + \psi(r)) \int_0^1 \alpha(1 - \alpha) h(\alpha) d\alpha \\ & \quad - \frac{1}{6} \mu(r - s)^2 \left[(\phi(s) + \psi(r)) + \frac{1}{5} \mu(r - s)^2 \right], \end{aligned} \quad (29)$$

where $M_1(r, s) = [\phi(s)[\psi(r) + \eta(\psi(b), \psi(r))] + \psi(r)[\phi(s) + \eta(\phi(r), \phi(s))]$ and $M_2(r, s) = [[\phi(s) + \eta(\phi(r), \phi(s))][\psi(r) + \eta(\psi(s), \psi(r))]]$.

Proof. Since ϕ, ψ are generalized geometrically strongly modified h -convex functions, so $\forall r, s \in J$ and $\alpha \in [0, 1]$, and we have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) & \leq (1 - h(\alpha))\phi(s) + h(\alpha)(\phi(r) \\ & \quad + \eta(\phi(r), \phi(s)) - \mu\alpha(1 - \alpha)(r - s)^2), \\ \psi(r^{1-\alpha} s^\alpha) & \leq (1 - h(\alpha))\psi(r) + h(\alpha)(\psi(s) \\ & \quad + \eta(\psi(s), \psi(r)) - \mu\alpha(1 - \alpha)(r - s)^2), \end{aligned} \quad (30)$$

which implies that

$$\begin{aligned} & \phi(r^\alpha s^{1-\alpha}) \psi(r^{1-\alpha} s^\alpha) \\ & \leq \phi(s) \psi(r) (1 - h(\alpha))^2 + [\phi(s)[\psi(r) + \eta(\psi(s), \psi(r))] + \psi(r) \\ & \quad \times [\phi(s) + \eta(\phi(r), \phi(s))]] h(\alpha) (1 - h(\alpha)) + [[\phi(s) + \eta(\phi(r), \phi(s))]] \\ & \quad \times [\psi(r) + \eta(\psi(s), \psi(r))]] (h(\alpha))^2 - \mu(r - s)^2 [\phi(s) + \psi(r)] \alpha(1 - \alpha) \\ & \quad \times (1 - h(\alpha)) - \mu(r - s)^2 [[\phi(s) + \eta(\phi(r), \phi(s))] \\ & \quad + \psi(r) + \eta(\psi(s), \psi(r))]] \\ & \quad \times \alpha(1 - \alpha) (h(\alpha)) + \mu^2 (r - s)^4 \alpha^2 (1 - \alpha)^2. \end{aligned} \quad (31)$$

Integrate both sides of (31) over α on $[0, 1]$ and by the additive property of η , that is, $\eta(r_1, s_1) + \eta(r_2, s_2) = \eta(r_1 + r_2, s_1 + s_2)$, and we have

$$\begin{aligned} & \int_0^1 \phi(r^\alpha s^{1-\alpha}) \psi(r^{1-\alpha} s^\alpha) d\alpha = \frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \psi\left(\frac{rs}{x}\right) \right] dx \\ & \leq \phi(s) \psi(r) \int_0^1 (1 - h(\alpha))^2 d\alpha + [\phi(s)[\psi(r) + \eta(\psi(s), \psi(r))] \\ & \quad + \psi(r)[\phi(s) + \eta(\phi(r), \phi(s))]] \int_0^1 h(\alpha) (1 - h(\alpha)) d\alpha \\ & \quad + [[\phi(s) + \eta(\phi(r), \phi(s))]] [\psi(r) + \eta(\psi(s), \psi(r))] \int_0^1 (h(\alpha))^2 d\alpha \\ & \quad - \mu(r - s)^2 \times \int_0^1 \alpha(1 - \alpha) [(1 - h(\alpha))(\phi(s) + \psi(r)) + h(\alpha)(\phi(r) + \psi(s))] \\ & \quad + \eta(\phi(r) + \psi(s), \phi(s) + \psi(r))] d\alpha + \mu^2 (a - b)^4 \int_0^1 \alpha^2 (1 - \alpha)^2 d\alpha \end{aligned}$$

$$\begin{aligned}
&\leq \phi(s)\psi(r) \int_0^1 (1-h(\alpha))^2 d\alpha + [\phi(s)[\psi(r) + \eta(\psi(s), \psi(r))] \\
&\quad + \psi(r)[\phi(s) + \eta(\phi(r), \phi(s))]] \int_0^1 h(\alpha)(1-h(\alpha)) d\alpha + [[\phi(s) \\
&\quad + \eta(\phi(r), \psi(s))][\psi(r) + \eta(\psi(s), \psi(r))]] \int_0^1 (h(\alpha))^2 d\alpha \\
&\quad - \mu(r-s)^2 \eta(\phi(r) + \psi(s), \phi(s) + \psi(r)) \int_0^1 \alpha(1-\alpha)h(\alpha) d\alpha \\
&\quad - \frac{1}{6} \mu(r-s)^2 \left[(\phi(s) + \psi(r)) - \frac{1}{5} \mu(r-s)^2 \right] \\
&\leq \phi(s)\psi(r) \int_0^1 (1-h(\alpha))^2 d\alpha + M_1(r, s) \int_0^1 h(\alpha)(1-h(\alpha)) dt + M_2(r, s) \int_0^1 (h(\alpha))^2 d\alpha \\
&\quad - \mu(r-s)^2 \eta(\phi(r) + \psi(s), \phi(s) + \psi(r)) \times \int_0^1 \alpha(1-\alpha)h(\alpha) d\alpha \\
&\quad - \frac{1}{6} \mu(r-s)^2 \left[(\phi(s) + \psi(r)) + \frac{1}{5} \mu(r-s)^2 \right].
\end{aligned} \tag{32}$$

This completes the proof. $\square$

Corollary 2. Taking $\phi = \psi$ and $\eta(r, s) = r - s$ in (25), we obtain

Remark 3. Taking $h(\alpha) = \alpha$ and $\mu = 0$ in (25) and (31), we obtained Hermite–Hadamard-type inequalities for generalized geometrically convex functions [24].

$$\begin{aligned}
&\frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \phi\left(\frac{rs}{x}\right) \right] dx \\
&\leq \phi(s)\phi(r) \int_0^1 (1-h(\alpha))^2 d\alpha + [\phi^2(s) + \phi^2(r)] \int_0^1 h(\alpha)(1-h(\alpha)) d\alpha \\
&\quad + [\phi(r)\phi(s)] \int_0^1 (h(\alpha))^2 d\alpha - \frac{1}{6} \mu(r-s)^2 \left[(\phi(s) + \phi(r)) - \frac{1}{5} \mu(r-s)^2 \right].
\end{aligned} \tag{33}$$

Theorem 3. Let $\phi, \psi: [r, s] \longrightarrow (0, \infty)$ be generalized geometrically strongly modified h -convex functions and $r, s \in J$ with $r < s$; then, the following inequality holds:

$$\begin{aligned}
&\frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \psi\left(\frac{rs}{x}\right) \right] dx \\
&\leq \frac{1}{2} \left\{ [\phi^2(s) + \psi^2(r)] \int_0^1 (1-h(\alpha))^2 d\alpha \right. \\
&\quad + [[\psi(r) + \eta(\psi(s), \psi(r))]^2 + [\phi(s) + \eta(\phi(r), \phi(s))]^2] \\
&\quad \times \int_0^1 (h(\alpha))^2 d\alpha + [[\phi(s)][\phi(s) + \eta(\phi(r), \phi(s))] + [\psi(r)][\psi(r) \\
&\quad + \eta(\psi(s), \psi(r))]] \int_0^1 2h(\alpha)(1-h(\alpha)) d\alpha + [[\phi(s) + \eta(\phi(r), \phi(s))] \\
&\quad + [\psi(r) + \eta(\psi(s), \psi(r))]] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(h(\alpha)) d\alpha \\
&\quad \left. + [\phi(s) + \psi(r)] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(1-h(\alpha)) d\alpha + \frac{\mu^2(r-s)^4}{15} \right\}.
\end{aligned} \tag{34}$$

Proof. Since $\phi, \psi: [r, s] \longrightarrow (0, \infty)$ are generalized geometrically strongly modified h -convex functions, so $\forall r, s \in J$ and $\alpha \in [0, 1]$, and we have

$$\begin{aligned} \phi(r^\alpha s^{1-\alpha}) &\leq (1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) \\ &\quad + \eta(\phi(r), \phi(s)) - \mu\alpha(1-\alpha)(r-s)^2), \\ \psi(r^{1-\alpha} s^\alpha) &\leq (1-h(\alpha))\psi(r) + h(\alpha)(\psi(r) \\ &\quad + \eta(\psi(s), \psi(r)) - \mu\alpha(1-\alpha)(r-s)^2), \end{aligned} \quad (35)$$

$$\begin{aligned} &\frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \psi\left(\frac{rs}{x}\right) \right] dx \\ &= \int_0^1 \phi(r^\alpha s^{1-\alpha}) \psi(r^{1-\alpha} s^\alpha) d\alpha \\ &\leq \frac{1}{2} \int_0^1 \left[\phi(r^\alpha s^{1-\alpha}) \right]^2 + \left[\psi(r^{1-\alpha} s^\alpha) \right]^2 d\alpha \\ &\leq \frac{1}{2} \left\{ \int_0^1 [(1-h(\alpha))\phi(s) + h(\alpha)(\phi(s) + \eta(\phi(r), \phi(s)) \right. \\ &\quad \left. - \mu\alpha(1-\alpha)(r-s)^2)]^2 + [(1-h(\alpha))\psi(r) + h(\alpha)(\psi(r) \right. \\ &\quad \left. + \eta(\psi(s), \psi(r)) - \mu\alpha(1-\alpha)(r-s)^2)]^2 d\alpha \right\} \\ &= \frac{1}{2} \left\{ [\phi^2(s) + \psi^2(r)] \int_0^1 (1-h(\alpha))^2 d\alpha \right. \\ &\quad + [(\psi(r) + \eta(\psi(s), \psi(r)))^2 + (\phi(s) + \eta(\phi(r), \phi(s)))^2] \int_0^1 (h(\alpha))^2 d\alpha \\ &\quad + [(\phi(s))(\phi(s) + \eta(\phi(r), \phi(s))) + (\psi(r))(\psi(r) + \eta(\psi(s), \psi(r)))] \times \\ &\quad \int_0^1 2h(\alpha)(1-h(\alpha)) d\alpha + [(\phi(s) + \eta(\phi(r), \phi(s))) \\ &\quad + (\psi(r) + \eta(\psi(s), \psi(r)))] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(h(\alpha)) d\alpha \\ &\quad + [\phi(s) + \psi(r)] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(1-h(\alpha)) d\alpha \\ &\quad \left. + 2\mu^2(r-s)^4 \int_0^1 \alpha^2(1-\alpha)^2 d\alpha \right\} \\ &\leq \frac{1}{2} \left\{ [\phi^2(s) + \psi^2(r)] \int_0^1 (1-h(\alpha))^2 d\alpha + [(\psi(r) + \eta(\psi(s), \psi(r)))^2 \right. \\ &\quad + (\phi(s) + \eta(\phi(r), \phi(s)))^2] \int_0^1 (h(\alpha))^2 d\alpha + [(\phi(s))(\phi(s) + \eta(\phi(r), \phi(s))) \\ &\quad + (\psi(r))(\psi(r) + \eta(\psi(s), \psi(r)))] \int_0^1 2h(\alpha)(1-h(\alpha)) d\alpha \\ &\quad + [(\phi(s) + \eta(\phi(r), \phi(s))) + (\psi(r) + \eta(\psi(s), \psi(r)))] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(h(\alpha)) d\alpha \\ &\quad \left. + [\phi(s) + \psi(r)] 2\mu(r-s)^2 \int_0^1 \alpha(1-\alpha)(1-h(\alpha)) d\alpha + \frac{\mu^2(r-s)^4}{15} \right\}. \end{aligned} \quad (36)$$

This completes the proof. $\square$

Remark 4. Taking $h(\alpha) = \alpha$ and $\mu = 0$ in (34) we obtained Hermite–Hadamard-type inequality for the generalized geometrically convex functions [24].

Corollary 3. Taking $g_1 = g_2$ and $\eta(x, y) = x - y$ in Theorem 3, we obtain

$$\begin{aligned} & \frac{1}{\ln s - \ln r} \int_r^s \frac{1}{x} \left[\phi(x) \phi\left(\frac{rs}{x}\right) \right] dx \\ & \leq \frac{1}{2} \left\{ \left[\phi^2(s) + \phi^2(r) \right] \left[\int_0^1 (1 - 2(h(\alpha))^2 + 2(h(\alpha))^2 d\alpha \right] + [\phi(s)\phi(r) \right. \\ & \quad \left. + \phi(r)\phi(s)] \int_0^1 2h(\alpha)(1 - h(\alpha)) d\alpha + 2\mu(r - s)^2 [\phi(r) + \phi(s)] \right. \\ & \quad \left. \times \left[\int_0^1 \alpha(1 - \alpha)h(\alpha) d\alpha + \int_0^1 \alpha(1 - \alpha)(1 - h(\alpha)) d\alpha \right] + \frac{\mu^2(r - s)^4}{15} \right\}. \end{aligned} \quad (37)$$

Lemma 2. Let $\phi: J \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in J$ with $r < s$ and $\phi'' \in L_1[r, s]$. Then, we have

$$\begin{aligned} & \left| \frac{\phi(r) + \phi(s)}{2} - \frac{1}{s - r} \int_r^s \phi(x) dx \right| \\ & = \frac{(s - r)^2}{2} \int_0^1 \alpha(1 - \alpha) \phi''(\alpha r + (1 - \alpha)s) d\alpha. \end{aligned} \quad (38)$$

Theorem 4. Let $\phi: J \subset \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a twice differentiable function on J° , $r, s \in J$ with $r < s$ and $\phi'' \in L_1[r, s]$. If $|\phi''|^q$ is a generalized geometrically strongly modified h -convex function, monotonically decreasing on $[r, s]$ for $q \geq 1$ and $\alpha \in [0, 1]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\phi(r) + \phi(s)}{2} - \frac{1}{s - r} \int_r^s \phi(x) dx \right| \\ & \leq \frac{(s - r)^2}{2} \left(\frac{1}{6} \right)^{1 - (1/q)} \left(\frac{1}{6} |\phi''(b)|^q + \int_0^1 \alpha(1 - \alpha) |h(\alpha)\eta(\phi''(r), \phi''(s))|^q d\alpha \right. \\ & \quad \left. - \mu(r - s)^2 \int_0^1 \alpha(1 - \alpha) |\alpha(1 - \alpha)|^q d\alpha \right)^{(1/q)}. \end{aligned} \quad (39)$$

Proof. From Lemma 2 and using the well-known power-mean inequality, we have

$$\begin{aligned} & \left| \frac{\phi(r) + \phi(s)}{2} - \frac{1}{s - r} \int_r^s \phi(x) dx \right| \\ & = \frac{(s - r)^2}{2} \int_0^1 \alpha(1 - \alpha) |\phi''(\alpha r + (1 - \alpha)s)| d\alpha \\ & \leq \frac{(s - r)^2}{2} \left(\int_0^1 \alpha(1 - \alpha) \right)^{1 - (1/q)} \left(\int_0^1 \alpha(1 - \alpha) |\phi''(\alpha r + (1 - \alpha)s)|^q d\alpha \right)^{(1/q)} \\ & = \frac{(s - r)^2}{2} \left(\frac{1}{6} \right)^{1 - (1/q)} \left(\int_0^1 \alpha(1 - \alpha) |\phi''(\alpha r + (1 - \alpha)s)|^q d\alpha \right)^{(1/q)}. \end{aligned} \quad (40)$$

Since $|\phi''|^q$ is generalized geometrically strongly modified h -convex function and monotonically decreasing on $[r, s]$, we obtain

$$\begin{aligned} & r^\alpha s^{1 - \alpha} \leq \alpha r + (1 - \alpha)s, \\ & |\phi''(\alpha r + (1 - \alpha)s)|^q \leq |\phi''(r^\alpha s^{1 - \alpha})|^q. \end{aligned} \quad (41)$$

Therefore, we have

$$\begin{aligned}
 & \left| \frac{\phi(r) + \phi(s)}{2} - \frac{1}{s-r} \int_r^s \phi(x) dx \right| \\
 & \leq \frac{(s-r)^2}{2} \left(\frac{1}{6} \right)^{1-(1/q)} \left(\int_0^1 \alpha(1-\alpha) |\phi''(r^\alpha s^{1-\alpha})|^q d\alpha \right)^{(1/q)} \\
 & \leq \frac{(s-r)^2}{2} \left(\frac{1}{6} \right)^{1-(1/q)} \left(\int_0^1 \alpha(1-\alpha) |(1-h(\alpha))\phi''(s) \right. \\
 & \quad \left. + h(\alpha)\phi''(s) + \eta(\phi''(r), \phi''(s)) - \mu(r-s)^2 \alpha(1-\alpha)|^q d\alpha \right)^{(1/q)} \\
 & \leq \frac{(s-r)^2}{2} \left(\frac{1}{6} \right)^{1-(1/q)} \left(\frac{1}{6} |\phi''(s)|^q \right. \\
 & \quad \left. + \int_0^1 \alpha(1-\alpha) |h(\alpha)\eta(\phi''(r), \phi''(s))|^q d\alpha \right. \\
 & \quad \left. - \mu(r-s)^2 \int_0^1 \alpha(1-\alpha) |\alpha(1-\alpha)|^q d\alpha \right)^{(1/q)}. \tag{42}
 \end{aligned}$$

This completes the proof. $\square$

Remark 5. Taking $h(\alpha) = \alpha$, $\mu = 0$, and $\eta(r, s) = r - s$ in (39), we obtained Hermite–Hadamard inequality for the geometric arithmetic convex function.

4. Some Other Inequalities

Lemma 3. Let $\phi: J \subset \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in J$ with $r < s$ and $\phi' \in L[r, s]$. Then, we have

$$\begin{aligned}
 & s\phi(s) - r\phi(r) - \int_r^s \phi(x) dx \\
 & = \frac{(\ln s - \ln r)}{2} \left[\int_0^1 s^{1+\alpha} r^{1-\alpha} \phi'(s^{(1+\alpha/2)} r^{(1-\alpha/2)}) d\alpha \right. \\
 & \quad \left. + \int_0^1 s^{1+\alpha} r^{1-\alpha} \phi'(s^{(1-\alpha/2)} r^{(1+\alpha/2)}) d\alpha \right]. \tag{43}
 \end{aligned}$$

Theorem 5. Let $\phi: J \subset \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in J$ with $r < s$ and $\phi' \in L[r, s]$. If $|\phi'|$ is generalized geometrically strongly modified h -convex function for $q \geq 1$, then the following inequality holds:

$$\begin{aligned}
 & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x) dx \right| \\
 & \leq \frac{(s-r)^{1-(1/q)}}{2} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(\frac{s}{r} \right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2} \right) \right) d\alpha \right. \right. \\
 & \quad \left. \left. + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{s}{r} \right)^\alpha h\left(\frac{1-\alpha}{2} \right) d\alpha \right]^{(1/q)} \right. \\
 & \quad \left. + r \left[|\phi'(s)|^q \int_0^1 \left(\frac{r}{s} \right)^\alpha \left(1 - h\left(\frac{1+\alpha}{2} \right) \right) d\alpha \right. \right. \\
 & \quad \left. \left. + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{r}{s} \right)^\alpha h\left(\frac{1+\alpha}{2} \right) d\alpha \right]^{(1/q)} \right. \\
 & \quad \left. - \left[\frac{\mu(r-s)^2}{4} L(r, s) \left(\frac{s^2 + r^2}{rs} \right) \right]^{(1/q)} \right\}. \tag{44}
 \end{aligned}$$

Proof. Using Lemma 3 and Holder's inequality, we have

$$\begin{aligned}
 & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x) dx \right| \\
 & = \frac{rs(\ln s - \ln r)}{2} \left[\int_0^1 \left(\frac{s}{r} \right)^\alpha |\phi'(s^{1+\alpha/2} r^{1-\alpha/2})| d\alpha \right. \\
 & \quad \left. + \int_0^1 \left(\frac{r}{s} \right)^\alpha |\phi'(s^{1-\alpha/2} r^{1+\alpha/2})| d\alpha \right] \\
 & \leq \frac{rs(\ln s - \ln r)}{2} \left\{ \left[\int_0^1 \left(\frac{s}{r} \right)^\alpha d\alpha \right]^{1-(1/q)} \right. \\
 & \quad \cdot \left[\int_0^1 \left(\frac{s}{r} \right)^\alpha |\phi'(s^{1+\alpha/2} r^{1-\alpha/2})| d\alpha \right]^{(1/q)} \\
 & \quad \left. + \left[\int_0^1 \left(\frac{r}{s} \right)^\alpha d\alpha \right]^{1-(1/q)} \left[\int_0^1 \left(\frac{r}{s} \right)^\alpha |\phi'(s^{1-\alpha/2} r^{1+\alpha/2})| d\alpha \right]^{(1/q)} \right\}. \tag{45}
 \end{aligned}$$

Now, consider

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{s}{r}\right)^\alpha \left|\phi'(s^{1+\alpha/2} r^{1-\alpha/2})\right|^q d\alpha \\
&\leq \int_0^1 \left(\frac{s}{r}\right)^\alpha \left[|\phi'(s)|^q \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q h\left(\frac{1-\alpha}{2}\right) \right. \\
&\quad \left. - \mu(r-s)^2 \left(\frac{1-\alpha}{2}\right) \left(\frac{1+\alpha}{2}\right)\right] d\alpha \\
&= |\phi'(s)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \\
&\quad \int_0^1 \left(\frac{s}{r}\right)^\alpha h\left(\frac{1-\alpha}{2}\right) d\alpha - \mu(r-s)^2 \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(\frac{1-\alpha}{2}\right) \left(\frac{1+\alpha}{2}\right) d\alpha \\
&= |\phi'(s)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \\
&\quad \times \int_0^1 \left(\frac{s}{r}\right)^\alpha h\left(\frac{1-\alpha}{2}\right) d\alpha - \mu(r-s)^2 \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(\frac{1-\alpha}{2}\right) \left(\frac{1+\alpha}{2}\right) d\alpha.
\end{aligned} \tag{46}$$

Similarly, I_2 becomes

$$\begin{aligned}
&\int_0^1 \left(\frac{r}{s}\right)^\alpha \left|\phi'(s^{(1-\alpha)/2} r^{(1+\alpha)/2})\right|^q d\alpha \\
&\leq |\phi'(s)|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \\
&\quad \times \int_0^1 \left(\frac{r}{s}\right)^\alpha h\left(\frac{1+\alpha}{2}\right) d\alpha - \mu(r-s)^2 \int_0^1 \left(\frac{r}{s}\right)^\alpha \left(\frac{1+\alpha}{2}\right) \left(\frac{1-\alpha}{2}\right) d\alpha.
\end{aligned} \tag{47}$$

Also,

$$\left[\int_0^1 \left(\frac{s}{r}\right)^\alpha d\alpha \right]^{1-(1/q)} = \left(\frac{s-r}{r(\ln s - \ln r)} \right)^{1-(1/q)}. \tag{48}$$

Combining (45)–(48), we have

$$\begin{aligned}
&\left| s\phi(s) - r\phi(r) - \int_r^s \phi(x) dx \right| \\
&\leq \frac{(s-r)^{1-(1/q)}}{2} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha \right. \right. \\
&\quad + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha h\left(\frac{1-\alpha}{2}\right) d\alpha \\
&\quad - \frac{s\mu(r-s)^2}{4} \left(\frac{s-r}{r(\ln s - \ln r)} \right)^{(1/q)} + r \left[|\phi'(s)|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha \right. \\
&\quad + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha h\left(\frac{1+\alpha}{2}\right) d\alpha \\
&\quad \left. \left. - \frac{r\mu(r-s)^2}{4} \left(\frac{r-s}{s(\ln r - \ln s)} \right)^{(1/q)} \right] \right\} \\
&\leq \frac{(s-r)^{1-(1/q)}}{2} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha \right. \right. \\
&\quad + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha h\left(\frac{1-\alpha}{2}\right) d\alpha \left. \right]^{1/q} \\
&\quad + r \left[|\phi'(s)|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha h\left(\frac{1+\alpha}{2}\right) d\alpha \right]^{1/q} \\
&\quad \left. - \left[\frac{\mu(r-s)^2}{4} L(r, s) \left(\frac{s^2 + r^2}{rs} \right) \right]^{1/q} \right\}.
\end{aligned} \tag{49}$$

This completes the proof.

If we take $\eta(r, s) = r - s$, then (45) reduces to the following result. $\square$

Corollary 4. Let $\phi: J \subset \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in J$ with $r < s$ and $\phi' \in L[r, s]$. If $|\phi'|$ is a generalized geometrically strongly modified h -convex function for $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x)dx \right| \\ & \leq \frac{(s-r)^{1-(1/q)}}{2} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha \right. \right. \\ & \quad \left. \left. + |\phi'(r)|^q \int_0^1 \left(\frac{s}{r}\right)^\alpha h\left(\frac{1-\alpha}{2}\right) d\alpha \right]^{1/q} \right. \\ & \quad \left. + r \left[|\phi'(s)|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha \right. \right. \\ & \quad \left. \left. + |\phi'(r)|^q \int_0^1 \left(\frac{r}{s}\right)^\alpha h\left(\frac{1+\alpha}{2}\right) d\alpha \right]^{1/q} \right. \\ & \quad \left. - \left[\frac{\mu(r-s)^2}{4} L(r, s) \left(\frac{s^2 + r^2}{rs} \right) \right]^{1/q} \right\}. \end{aligned} \quad (50)$$

Theorem 6. Let $\phi: J \subset \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in J$ with $r < s$ and $\phi' \in L[r, s]$. If $|\phi'|$ is generalized geometrically strongly modified h -convex function for $q > 1$, then the following inequality holds:

$$\begin{aligned} & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x)dx \right| \\ & \leq \frac{\ln s - \ln r}{2} \left[L(r^{q/(q-1)}, s^{q/(q-1)}) \right]^{1-(1/q)} \\ & \quad \cdot \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha \right. \right. \\ & \quad \left. \left. + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 h\left(\frac{1-\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{1/q} \right. \\ & \quad \left. + r \left[|\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \right. \right. \\ & \quad \left. \left. \times \int_0^1 h\left(\frac{1+\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{1/q} \right\}. \end{aligned} \quad (51)$$

Proof. Using Lemma 3 and Holder's inequality, we have

$$\begin{aligned} & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x)dx \right| \\ & = \frac{rs(\ln s - \ln r)}{2} \left[\int_0^1 \left(\frac{s}{r}\right)^\alpha |\phi'(s^{(1+\alpha)/2} r^{(1-\alpha)/2})| d\alpha \right. \\ & \quad \left. + \int_0^1 \left(\frac{r}{s}\right)^\alpha |\phi'(s^{(1-\alpha)/2} r^{(1+\alpha)/2})| d\alpha \right] \\ & \leq \frac{rs(\ln s - \ln r)}{2} \left\{ \left[\int_0^1 \left(\frac{s}{r}\right)^{q\alpha/(q-1)} d\alpha \right]^{1-(1/q)} \right. \\ & \quad \cdot \left[\int_0^1 |\phi'(s^{(1+\alpha)/2} r^{(1-\alpha)/2})|^q d\alpha \right]^{1/q} \\ & \quad \left. + \left[\int_0^1 \left(\frac{r}{s}\right)^{q\alpha/(q-1)} d\alpha \right]^{1-(1/q)} \right. \\ & \quad \cdot \left[\int_0^1 |\phi'(s^{(1-\alpha)/2} r^{(1+\alpha)/2})|^q d\alpha \right]^{1/q} \Big\}. \end{aligned} \quad (52)$$

Now, consider

$$\begin{aligned} I_1 & = \int_0^1 |\phi'(s^{((1+\alpha)/2)} r^{(1-\alpha)/2})|^q d\alpha \\ & \leq \int_0^1 \left[|\phi'(s)|^q \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \right. \\ & \quad \left. \times h\left(\frac{1-\alpha}{2}\right) - \mu(r-s)^2 \left(\frac{1-\alpha}{2}\right) \left(\frac{1+\alpha}{2}\right) \right] d\alpha \\ & = |\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1-\alpha}{2}\right)\right) d\alpha + |\phi'(s) \\ & \quad + \eta(\phi'(r), \phi'(s))|^q \int_0^1 h\left(\frac{1-\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6}. \end{aligned} \quad (53)$$

Similarly, I_2 becomes

$$\begin{aligned} I_2 & = \int_0^1 |\phi'(s^{(1-\alpha)/2} r^{(1+\alpha)/2})|^q d\alpha \\ & = |\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1+\alpha}{2}\right)\right) d\alpha + |\phi'(s) \\ & \quad + \eta(\phi'(r), \phi'(s))|^q \int_0^1 h\left(\frac{1+\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6}. \end{aligned} \quad (54)$$

Also,

$$\begin{aligned} & \left[\int_0^1 \left(\frac{s}{r}\right)^{(q\alpha/(q-1))} d\alpha \right]^{1-(1/q)} \\ & = \left(\frac{s^{(q/(q-1))} - r^{(q/(q-1))}}{(q/q-1)r^{(q/(q-1))}(\ln s - \ln r)} \right)^{1-(1/q)}. \end{aligned} \quad (55)$$

Combining (52)–(55), we have

$$\begin{aligned}
 & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x)dx \right| \\
 & \leq \frac{\ln s - \ln r}{2} \left[L(r^{(q/q-1)}, s^{(q/q-1)}) \right]^{1-(1/q)} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1-\alpha}{2}\right) \right) d\alpha \right. \right. \\
 & \quad \left. \left. + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 h\left(\frac{1-\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{1/q} \right. \\
 & \quad \left. + r \left[|\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1+\alpha}{2}\right) \right) d\alpha + |\phi'(s) + \eta(\phi'(r), \phi'(s))|^q \int_0^1 h\left(\frac{1+\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{1/q} \right\}.
 \end{aligned} \tag{56}$$

This completes the proof. $\square$

Remark 6. Taking $h(\alpha) = \alpha$ and $\mu = 0$ in (45) and (47), we obtained fractional integral inequalities for generalized geometrically convex function, see [24].

If $\eta(r, s) = r - s$, then (47) reduces to the following result:

Corollary 8. Let $\phi: J \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on the interior J° of J , where $r, s \in I$ with $r < s$ and $\phi' \in L[r, s]$. If $|\phi'|$ is generalized geometrically strongly modified h -convex function for $q > 1$, then the following inequality holds:

$$\begin{aligned}
 & \left| s\phi(s) - r\phi(r) - \int_r^s \phi(x)dx \right| \\
 & \leq \frac{\ln s - \ln r}{2} \left[L(r^{(q/q-1)}, s^{(q/q-1)}) \right]^{1-(1/q)} \left\{ s \left[|\phi'(s)|^q \int_0^1 \left(1 - h\left(\frac{1-\alpha}{2}\right) \right) d\alpha \right. \right. \\
 & \quad \left. \left. + |\phi'(r)|^q \int_0^1 h\left(\frac{1-\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{1/q} \right. \\
 & \quad \left. + r \left[|\phi'(s)|^q \times \int_0^1 \left(1 - h\left(\frac{1+\alpha}{2}\right) \right) d\alpha + |\phi'(s)|^q \int_0^1 h\left(\frac{1+\alpha}{2}\right) d\alpha - \frac{\mu(r-s)^2}{6} \right]^{(1/q)} \right\}.
 \end{aligned} \tag{57}$$

5. Conclusions

The theory of the convex function is applicable in almost every field of mathematics specially in approximation theory. The inequality theory got application in diverse areas of pure and applied mathematics. In this paper, we introduced a broader class of convex functions by unifying geometrically strong convex function with η_h convex function and developed the Hermite–Hadamard-type and fractional integral inequalities for this class of function.

Data Availability

All data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Xishan Yu analyzed the results and designed the whole paper, Muhammad Shoaib Saleem proved the results, Shumaila Waheed wrote the first version of the paper, and Ilyas Khan wrote the final version of the paper and arranged the funding for this research.

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