Research Article

The Recursive Properties of the Error Term of the Fourth Power Mean of the Generalized Cubic Gauss Sums

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In this paper, we use the analytic methods and the properties of the classical Gauss sums to study the properties of the error term of the fourth power mean of the generalized cubic Gauss sums and give two recurrence formulae for it.

1. Introduction

For any integer $q \geq 2$ and any Dirichlet character $\chi \bmod q$, the definition of the classical Gauss sums $G(m, \chi; q)$ is

$$G(m, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma}{q}\right), \quad (1)$$

where m is an integer and $e(y) = e^{2\pi i y}$.

This sum and its properties are of great significance to the analytic number theory, and many number theory problems are closely related to them. Therefore, it is necessary to study the various properties of $G(m, \chi; q)$ and related sums. In this paper, we consider the generalized k -th Gauss sums $G(m, k, \chi; q)$, which is defined as follows:

$$G(m, k, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma^k}{q}\right), \quad (2)$$

where k is any positive integer and m is an integer with $(m, q) = 1$.

It is clear that this sum is a generalization of the classical Gauss sums $G(m, \chi; q)$. In fact, $G(m, 1, \chi; q) = G(m, \chi; q)$. Of course, the value of $G(m, k, \chi; q)$ is irregular as χ varies. However, some scholars have found that $G(m, k, \chi; q)$ has good value distribution properties in some problems of weighted mean value, even if we can get their exact calculation formulae for some $2k$ th power mean. In addition, there are some good upper bound estimates for $|G(m, k, \chi; q)|$. For example, for any integer n with $(n, q) = 1$, from the general result of Cochrane and Zheng [1], we can deduce

$$|G(m, k, \chi; q)| \leq 2^{\omega(q)} q^{(1/2)}, \quad (3)$$

where $\omega(q)$ denotes the number of distinct prime divisors of q . The case that q is a prime is due to Weil [2].

For $k = 2$, by the results of Zhang [3], let n be any integer with $(n, p) = 1$, and there are the following two identities:

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(n, 2, \chi; p)|^4 = \begin{cases} 3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}, & \text{if } p \equiv 1 \pmod{4}, \\ 3p^2 - 6p - 1, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (4)$$

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(n, 2, \chi; p)|^6 = 10p^3 - 25p^2 - 4p - 1, \quad \text{if } p \equiv 3 \pmod{4},$$

where $(*/p)$ denotes the Legendre symbol modulo p .

Zhang and Liu [4] have studied the sum

$$\sum_{\chi \bmod p} |G(n, 3, \chi; q)|^4 \quad (5)$$

and obtained the following calculation formula:

$$\sum_{\chi \bmod p} |G(1, 3, \chi; p)|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U, \quad (6)$$

where p is a prime with $3|p-1$ and $U = \sum_{a=1}^p e(a^3/p)$ is a real constant.

However, the value of U was not given in [4], and the form of U was not concise. Now, for any integer $1 \leq m \leq p-1$, we let

$$E(m, p) = \frac{1}{p-1} \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 - 5p^2 + 13p + 1. \quad (7)$$

In this paper, we use the analytic methods and the properties of the classical Gauss sums to study the

calculating problem of the n th power mean of $E(m, p)$ and give two recurrence formulae for it. That is, we shall prove the following main results.

Theorem 1. Let p be an odd prime with $3|p-1$. Then, for any positive integer n and integer m with $(m, p) = 1$, we have the third-order linear recurrence formula:

$$E^n(m, p) = U(p, d) \cdot E^{n-3}(m, p) + V(p, d) \cdot E^{n-2}(m, p), \quad (8)$$

where $U(p, d)$ and $V(p, d)$ are defined as

$$\begin{aligned} U(p, d) &= 64dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\ &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\ V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16), \end{aligned} \quad (9)$$

and $4p = d^2 + 27b^2$, where d is uniquely determined by $d \equiv 1 \pmod{3}$ (see [5]).

Theorem 2. Let p be an odd prime with $3|p-1$. Then, for any positive integer $n \geq 3$, we have the recurrence formula

$$\begin{aligned} \sum_{m=1}^{p-1} E^n(m, p) &= U(p, d) \sum_{m=1}^{p-1} E^{n-3}(m, p) \\ &\quad + V(p, d) \sum_{m=1}^{p-1} E^{n-2}(m, p), \end{aligned} \quad (10)$$

where the first three terms of $\sum_{m=1}^{p-1} E^n(m, p)$ are

$$\begin{aligned} \sum_{m=1}^{p-1} E(m, p) &= 0, \\ \sum_{m=1}^{p-1} E^2(m, p) &= 2p(p-1)(p-2d+4)^2, \\ \sum_{m=1}^{p-1} E^3(m, p) &= (p-1)U(p, d). \end{aligned} \quad (11)$$

Theorem 3. Let p be an odd prime with $3|p-1$. Then, for any positive integer $n \geq 1$, we have the third-order linear recurrence formula

$$\sum_{m=1}^{p-1} E^{-n}(m, p) = \frac{1}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-3)}(m, p) - \frac{V(p, d)}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-1)}(m, p), \quad (12)$$

where the initial values of $\sum_{m=1}^{p-1} E^{-n}(m, p)$ are

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-1}(m, p) &= \frac{2p(p-1)(p-2d+4)^2 - (p-1)V(p, d)}{U(p, d)}, \\ \sum_{m=1}^{p-1} E^{-2}(m, p) &= \frac{(p-1)V^2(p, d) - 2p(p-1)(p-2d+4)^2 V(p, d)}{U^2(p, d)}, \\ \sum_{m=1}^{p-1} E^{-3}(m, p) &= \frac{p-1}{U(p, d)} + \frac{2p(p-1)(p-2d+4)^2 V^2(p, d) - (p-1)V^3(p, d)}{U^3(p, d)}.\end{aligned}\tag{13}$$

Taking $n = 4$ in Theorem 3, we may immediately deduce the following corollary.

Corollary 1. Let p be an odd prime with $3|p-1$; then, we have the identity

$$\begin{aligned}\frac{1}{p-1} \sum_{m=1}^{p-1} \frac{1}{|E(m, p)|^4} \\ = \frac{2p(p-2d+4)^2 - 2V(p, d)}{U^2(p, d)} + \frac{V^4(p, d) - 2p(p-2d+4)^2 V^3(p, d)}{U^4(p, d)}.\end{aligned}\tag{14}$$

2. Several Lemmas

In this section, we give three lemmas which are necessary in the proofs of our theorems. In the process of proving our lemmas, we need some knowledge of the analytic number theory; all of which can be found in [6–8], so it is not necessary to repeat them here.

Lemma 1. Let p be an odd prime with $p \equiv 1 \pmod{3}$. Then, for any third-order character $\lambda \pmod{p}$, we have

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \tag{15}$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \pmod{3}$.

Proof. This result can be found in [9] or [10]. $\square$

Lemma 2. Let p be an odd prime with $3|p-1$. Then, for any cubic character $\lambda \pmod{p}$, we have the identities

$$\begin{aligned}E(m, p) &= \bar{\lambda}(m) \left(\sum_{a=1}^{p-1} \bar{\lambda}(a^3 - 1) \right)^2 \cdot \tau(\lambda) + \lambda(m) \left(\sum_{b=1}^{p-1} \lambda(b^3 - 1) \right)^2 \cdot \tau(\bar{\lambda}) \\ &= \bar{\lambda}(m) \tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2.\end{aligned}\tag{16}$$

Proof. For any integer $1 \leq a \leq p-1$, it is easy to show that

From the properties of the cubic character modulo p , we have

$$1 + \lambda(a) + \lambda^2(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue mod } p, \\ 0, & \text{otherwise.} \end{cases}\tag{17}$$

$$\begin{aligned}\lambda^2 &= \bar{\lambda}, \\ \lambda(-1) &= 1, \\ \overline{\tau(\bar{\lambda})} &= \tau(\bar{\lambda}).\end{aligned}\tag{18}$$

So, we have the identity

$$\begin{aligned}
 \sum_{a=1}^{p-1} \lambda(a^3 - 1) &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \bar{\lambda}(b) e\left(\frac{b(a^3 - 1)}{p}\right) \\
 &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{-b}{p}\right) \sum_{a=1}^{p-1} (1 + \lambda(a) + \bar{\lambda}(a)) e\left(\frac{ba}{p}\right) \\
 &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{-b}{p}\right) (-1 + \bar{\lambda}(b)\tau(\lambda) + \lambda(b)\tau(\bar{\lambda})) \\
 &= \frac{1}{\tau(\bar{\lambda})} \left(-\lambda(-1)\tau(\bar{\lambda}) + \tau(\lambda) \sum_{b=1}^{p-1} \lambda(b) e\left(\frac{-b}{p}\right) + \tau(\bar{\lambda}) \sum_{b=1}^{p-1} e\left(\frac{-b}{p}\right) \right) \\
 &= \frac{1}{\tau(\bar{\lambda})} (-\tau(\bar{\lambda}) + \tau^2(\lambda) - \tau(\bar{\lambda})) = \frac{\tau^3(\lambda)}{p} - 2.
 \end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned}
 E(m, p) &= \bar{\lambda}(m)\tau(\lambda) \left(\sum_{a=1}^{p-1} \bar{\lambda}(a^3 - 1) \right)^2 + \lambda(m)\tau(\bar{\lambda}) \left(\sum_{a=1}^{p-1} \lambda(a^3 - 1) \right)^2 \\
 &= \bar{\lambda}(m)\tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m)\tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2.
 \end{aligned} \tag{20}$$

This proves Lemma 2. $\square$

$$E^3(m, p) = U(p, d) + V(p, d) \cdot E(m, p), \tag{21}$$

where

Lemma 3. Let p be an odd prime with $3|p-1$. Then, for any cubic character λ modulo p , we have the identity

$$\begin{aligned}
 U(p, d) &= 64 dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\
 &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\
 V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16),
 \end{aligned} \tag{22}$$

and $E(m, p)$ is defined as the same as in Lemma 2.

Proof. From Lemma 2, we have

$$\begin{aligned}
 E^3(m, p) &= \left(\bar{\lambda}(m) \tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\
 &= \left(\tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \right)^3 + \left(\tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\
 &\quad + 3p \cdot \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \cdot E(m, p).
 \end{aligned} \tag{23}$$

Note that $\tau(\lambda)\tau(\bar{\lambda}) = p$, and from (17), we have

Therefore,

$$\begin{aligned}
 \tau^6(\lambda) + \tau^6(\bar{\lambda}) &= \left(\tau^3(\lambda) + \tau^3(\bar{\lambda}) \right)^2 - 2\tau^3(\lambda)\tau^3(\bar{\lambda}) \\
 &= d^2 p^2 - 2p^3.
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 &\left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \\
 &= \left(4 - \frac{4\tau^3(\bar{\lambda})}{p} + \frac{\tau^6(\bar{\lambda})}{p^2} \right) \cdot \left(4 - \frac{4\tau^3(\lambda)}{p} + \frac{\tau^6(\lambda)}{p^2} \right) \\
 &= 16 - \frac{16}{p} \left(\tau^3(\lambda) + \tau^3(\bar{\lambda}) \right) + \frac{4}{p^2} \left(\tau^6(\lambda) + \tau^6(\bar{\lambda}) \right) + \frac{16}{p^2} \tau^3(\lambda)\tau^3(\bar{\lambda}) \\
 &\quad - \frac{4}{p^3} \left(\tau^3(\lambda)\tau^6(\bar{\lambda}) + \tau^3(\bar{\lambda})\tau^6(\lambda) \right) + \frac{1}{p^4} \tau^6(\lambda)\tau^6(\bar{\lambda}) \\
 &= 16 - \frac{16}{p} \cdot dp + \frac{4}{p^2} \cdot (d^2 p^2 - 2p^3) + \frac{16}{p^2} \cdot p^3 - \frac{4}{p^3} \cdot p^3 \cdot dp + \frac{1}{p^4} \cdot p^6 \\
 &= p^2 + 8p - 4dp + 4d^2 - 16d + 16.
 \end{aligned} \tag{25}$$

In addition,

$$\begin{aligned}
 \left(\tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \right)^3 &= \tau^3(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^6 \\
 &= \tau^3(\lambda) \left(\frac{\tau^{18}(\bar{\lambda})}{p^6} + 36 \frac{\tau^{12}(\bar{\lambda})}{p^4} + 144 \frac{\tau^6(\bar{\lambda})}{p^2} + 64 - 12 \frac{\tau^{15}(\bar{\lambda})}{p^5} \right. \\
 &\quad \left. + 24 \frac{\tau^{12}(\bar{\lambda})}{p^4} - 16 \frac{\tau^9(\bar{\lambda})}{p^3} - 144 \frac{\tau^9(\bar{\lambda})}{p^3} + 96 \frac{\tau^6(\bar{\lambda})}{p^2} - 192 \frac{\tau^3(\bar{\lambda})}{p} \right) \\
 &= 64 \tau^3(\lambda) + 240 p \tau^3(\bar{\lambda}) - 160 \tau^6(\bar{\lambda}) + \frac{60}{p} \tau^9(\bar{\lambda}) \\
 &\quad - \frac{12}{p^2} \tau^{12}(\bar{\lambda}) + \frac{1}{p^3} \tau^{15}(\bar{\lambda}) - 192 p^2.
 \end{aligned} \tag{26}$$

Using the method similar to (24), we obtain

$$\begin{aligned}
 \tau^9(\lambda) + \tau^9(\bar{\lambda}) &= (\tau^6(\lambda) + \tau^6(\bar{\lambda}))(\tau^3(\lambda) + \tau^3(\bar{\lambda})) - p^3(\tau^3(\lambda) + \tau^3(\bar{\lambda})) \\
 &= d^3 p^3 - 3 d p^4, \\
 \tau^{12}(\lambda) + \tau^{12}(\bar{\lambda}) &= d^4 p^4 - 4 d^2 p^5 + 2 p^6, \\
 \tau^{15}(\lambda) + \tau^{15}(\bar{\lambda}) &= d^5 p^5 - 5 d^3 p^6 + 5 d p^7.
 \end{aligned} \tag{27}$$

Combining formulae (23)–(27), we have

$$\begin{aligned}
 E^3(m, p) &= 64(\tau^3(\lambda) + \tau^3(\bar{\lambda})) + 240 p(\tau^3(\bar{\lambda}) + \tau^3(\lambda)) - 160(\tau^6(\bar{\lambda}) + \tau^6(\lambda)) \\
 &\quad + \frac{60}{p}(\tau^9(\bar{\lambda}) + \tau^9(\lambda)) - \frac{12}{p^2}(\tau^{12}(\bar{\lambda}) + \tau^{12}(\lambda)) + \frac{1}{p^3}(\tau^{15}(\bar{\lambda}) + \tau^{15}(\lambda)) \\
 &\quad - 384 p^2 + 3 p \cdot (p^2 + 8 p - 4 d p + 4 d^2 - 16 d + 16) \cdot E(m, p) \\
 &= (5 d - 24) p^4 - (5 d^3 - 48 d^2 + 180 d - 320) p^3 \\
 &\quad + (d^5 - 12 d^4 + 60 d^3 - 160 d^2 + 240 d - 384) p^2 + 64 d p \\
 &\quad + 3 p \cdot (p^2 + 8 p - 4 d p + 4 d^2 - 16 d + 16) \cdot E(m, p) \\
 &= U(p, d) + V(p, d) \cdot E(m, p).
 \end{aligned} \tag{28}$$

This proves Lemma 3.

□

3. Proofs of the Theorems

Now, we shall complete the proofs of our main results. Firstly, we prove Theorem 1. Let p be an odd prime with

$3|p-1$, χ be any Dirichlet character modulo p , and λ be a cubic character modulo p . Then, from the properties of the classical Gauss sums and (17), we have

$$\begin{aligned}
 \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right|^4 \\
 &= \sum_{\chi \bmod p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi(abc\bar{d}) e\left(\frac{m(a^3 + b^3 - c^3 - d^3)}{p}\right) \\
 &= (p-1) \sum_{\substack{a=1 \\ ab \equiv c \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{m(a^3 + b^3 - c^3 - d^3)}{p}\right) \\
 &= (p-1) \sum_{\substack{a=1 \\ ab \equiv c \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 + b^3 - c^3 - 1)}{p}\right) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) \tag{29} \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} (1 + \lambda(d) + \bar{\lambda}(d)) e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) + (p-1)\bar{\lambda}(m)\tau(\lambda) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \bar{\lambda}((a^3 - 1)(b^3 - 1)) \\
 &\quad + (p-1)\lambda(m)\tau(\bar{\lambda}) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \lambda((a^3 - 1)(b^3 - 1)) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) + (p-1)E(m, p),
 \end{aligned}$$

where $E(m, p)$ is the same as in Lemma 2.

For any integer n , we have the trigonometric identity

$$\sum_{a=1}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p-1, & \text{if } (n, p) = p, \\ -1, & \text{if } (n, p) = 1. \end{cases} \tag{30}$$

From (30), we have the identity

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{m d(a^3 - 1)(b^3 - 1)}{p}\right) \\
 &= 2(p-1) \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 - (p-1) \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 - \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 \\
 &= 6(p-1)^2 - 9(p-1) - (p-4)^2 = 5p^2 - 13p - 1.
 \end{aligned} \tag{31}$$

Combining (29)–(31), we have

$$\begin{aligned}
 \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right|^4 \\
 &= (p-1)(5p^2 - 13p - 1) + (p-1)E(m, p).
 \end{aligned} \tag{32}$$

For any positive integer n , from Lemma 3, we have

where

$$\begin{aligned}
 E^n(m, p) &= E^3(m, p) \cdot E^{n-3}(m, p) \\
 &= [U(p, d) + V(p, d) \cdot E(m, p)] \cdot E^{n-3}(m, p) \\
 &= U(p, d) \cdot E^{n-3}(m, p) + V(p, d) \cdot E^{n-2}(m, p),
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 U(p, d) &= 64 dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\
 &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\
 V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16).
 \end{aligned} \tag{34}$$

This proves Theorem 1.

Now, we prove Theorem 2. From Theorem 1, we have

$$\sum_{m=1}^{p-1} E^n(m, p) = U(p, d) \sum_{m=1}^{p-1} E^{n-3}(m, p) + V(p, d) \sum_{m=1}^{p-1} E^{n-2}(m, p). \tag{35}$$

From Lemma 2, we have

$$\sum_{m=1}^{p-1} E(m, p) = \sum_{m=1}^{p-1} \left(\bar{\lambda}(m) \tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right) = 0, \quad (36)$$

$$\begin{aligned} \sum_{m=1}^{p-1} E^2(m, p) &= \sum_{m=1}^{p-1} \left(\bar{\lambda}(m) \tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^2 \\ &= \sum_{m=1}^{p-1} \lambda(m) \tau^2(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^4 + \sum_{m=1}^{p-1} \bar{\lambda}(m) \tau^2(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^4 \\ &\quad + \sum_{m=1}^{p-1} 2p \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \\ &= 2p(p-1)(p-2d+4)^2. \end{aligned} \quad (37)$$

From (36) and Lemma 3, we have

$$\begin{aligned} \sum_{m=1}^{p-1} E^3(m, p) &= \sum_{m=1}^{p-1} \left(\bar{\lambda}(m) \tau(\lambda) \left(\frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left(\frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\ &= \sum_{m=1}^{p-1} (U(p, d) + V(p, d)E(m, p)) = (p-1)U(p, d). \end{aligned} \quad (38)$$

Now, Theorem 2 follows from (35)–(38).

Finally, we prove Theorem 3. For any integer $n \geq 1$, from Lemma 3, we have

$$\sum_{m=1}^{p-1} E^{-n}(m, p) = \frac{1}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-3)}(m, p) - \frac{V(p, d)}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-1)}(m, p), \quad (39)$$

$$\begin{aligned} E^2(m, p) &= U(p, d) \cdot E^{-1}(m, p) + V(p, d), \\ E(m, p) &= U(p, d) \cdot E^{-2}(m, p) + V(p, d) \cdot E^{-1}(m, p), \\ E^0(m, p) &= U(p, d) \cdot E^{-3}(m, p) + V(p, d) \cdot E^{-2}(m, p), \\ E^{-1}(m, p) &= U(p, d) \cdot E^{-4}(m, p) + V(p, d) \cdot E^{-3}(m, p). \end{aligned} \quad (40)$$

Therefore, we have

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-1}(m, p) &= \sum_{m=1}^{p-1} \frac{E^2(m, p) - V(p, d)}{U(p, d)} \\ &= \frac{2p(p-1)(p-2d+4)^2 - (p-1)V(p, d)}{U(p, d)},\end{aligned}\quad (41)$$

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-2}(m, p) &= \sum_{m=1}^{p-1} \frac{E(m, p) - V(p, d) \cdot E^{-1}(m, p)}{U(p, d)} \\ &= \frac{(p-1)V^2(p, d) - 2p(p-1)(p-2d+4)^2V(p, d)}{U^2(p, d)},\end{aligned}\quad (42)$$

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-3}(m, p) &= \sum_{m=1}^{p-1} \frac{1 - V(p, d) \cdot E^{-2}(m, p)}{U(p, d)} \\ &= \frac{p-1}{U(p, d)} + \frac{2p(p-1)(p-2d+4)^2V^2(p, d) - (p-1)V^3(p, d)}{U^3(p, d)}.\end{aligned}\quad (43)$$

Combining (39)–(43), we may immediately complete the proof of Theorem 3.

4. Conclusion

The main results of this paper give two third-order linear recurrence formulae for the error term of the fourth power mean of the generalized cubic Gauss sums, and these results are the improvement and generalization of [4]. They are some new contributions in the relevant fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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References

- [1] T. Cochrane and Z. Zheng, "Pure and mixed exponential sums," *Acta Arithmetica*, vol. 91, no. 3, pp. 249–278, 1999.
- [2] A. Weil, "On some exponential sums," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 34, pp. 203–210, 1948.
- [3] W. P. Zhang, "Moments of generalized quadratic Gauss sums weighted by L-functions," *Journal of Number Theory*, vol. 92, pp. 304–314, 2002.
- [4] W. P. Zhang and H. N. Liu, "On the general Gauss sums and their fourth power mean," *Osaka Journal of Mathematics*, vol. 42, pp. 189–199, 2005.
- [5] S. Chowla, J. Cowles, and M. Cowles, "On the number of zeros of diagonal cubic forms," *Journal of Number Theory*, vol. 9, no. 4, pp. 502–506, 1977.
- [6] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, NY, USA, 1976.
- [7] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, NY, USA, 1982.
- [8] C. D. Pan and C. B. Pan, *Goldbach Conjecture*, Science Press, Beijing, China, 1992.
- [9] W. P. Zhang and J. Y. Hu, "The number of solutions of the diagonal cubic congruence equation mod p ," *Mathematical Reports*, vol. 20, pp. 70–76, 2018.
- [10] B. C. Berndt and R. J. Evans, "The determination of Gauss sums," *Bulletin of the American Mathematical Society*, vol. 5, no. 2, pp. 107–130, 1981.