Research Article

An Inertial Iterative Algorithm with Strong Convergence for Solving Modified Split Feasibility Problem in Banach Spaces

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In this paper, we propose an iterative scheme for a special split feasibility problem with the maximal monotone operator and fixed-point problem in Banach spaces. The algorithm implements Halpern's iteration with an inertial technique for the problem. Under some mild assumption of the monotonicity of the related mapping, we establish the strong convergence of the sequence generated by the algorithm which does not require the spectral radius of $A^T A$. Finally, the numerical example is presented to demonstrate the efficiency of the algorithm.

1. Introduction

The split feasibility problem (shortly **SFP**) introduced by Censor and Elfving [1] in 1994 can be defined as follows: find a point x^* satisfying

$$\begin{aligned} x^* &\in C, \\ Ax^* &\in Q, \end{aligned} \quad (1)$$

where C and Q are nonempty closed-convex subsets of real Hilbert spaces H^1 and H^2 , respectively, and $A: H^1 \rightarrow H^2$ is a bounded linear operator. The SFP has broad application in modelling real-world problems such as the inverse problem in signal processing, radiotherapy, and data compression (for example, see [2–4]). Various algorithms have been invented by several authors for solving the SFP and related optimization problems (for example, see [5–10]). The CQ algorithm is one of the most popular solvers for SFP which was first proposed by Byrne [11], taking an initial point arbitrarily and defining the iterative step as

$$x^{k+1} = P_C(x^k - \mu A^T(I - P_Q)Ax^k), \quad \forall k \geq 1, \quad (2)$$

where $\mu \in (0, (2/\rho(A^T A)))$, $\rho(A^T A)$ is the spectral radius of $A^T A$, and P_C and P_Q denote the metric projections of H^1 onto C and H^2 onto Q , respectively, that is, $P_C(x) = \operatorname{argmin}_{y \in C} \|x - y\|$ over all $x \in C$. It was proved that the sequence $\{x_n\}$ generated by (2) converges weakly to a solution of the SFP provided the step size $\mu \in (0, (2/\rho(A^T A)))$. As an extension of this CQ algorithm, several iterative algorithms have been invented for solving SFP in Hilbert spaces and Banach spaces (for example, see [12–17]).

Let H be a real Hilbert space, F be a strictly convex, reflexive smooth Banach space, J_F denotes the duality mapping on F , and C and Q be nonempty closed-convex subsets of H and F , respectively. The following Halpern's type iteration algorithm was proposed by Alsulami and Takahashi [13] in 2015. Let $\{t^k\}$ be a sequence in H such that $t^k \rightarrow t \in H$ and $x^1, t^1 \in H$:

$$\begin{cases} v^k = \lambda_k t^k + (1 - \lambda_k)P_C(x^k - rA^*J_F(I - P_Q)Ax^k), \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k)v^k, \quad k \geq 1, \end{cases} \quad (3)$$

where $0 < r < \infty$ and $\{\alpha_k\} \subset (0, 1)$. It was proved that the sequence $\{x^k\}$ defined by (3) converges strongly to a point $\omega_0 \in C \cap A^{-1}(Q)$; for some $\omega_0 = P_{C \cap A^{-1}Q} t^1$, $\forall a, b \in \mathbb{R}$ if $0 < r\|A\|^2 < 2$, $\lim_{k \rightarrow \infty} \lambda_k = 0$, $\sum_{k=1}^{\infty} \lambda_k = \infty$, and $0 < a \leq \alpha_k \leq b < 1$.

However, the previous algorithms only use the current point to get the next iteration which can lead slow convergence. Inertial technique, as an accelerated method, was first proposed by Polyak [18] to speed up the convergence rate of smooth convex minimization. Subsequently, F. Alvarez in [19] combined with a proximal method to solve the problem of finding the zero of a maximal monotone operator. The main idea of this method is to make use of two previous iterates in order to update the next iterate. Due to the fact that the presence of the inertial term in an algorithm speeds up the convergence rate, inertial type algorithms have been widely studied by authors [5, 20, 21].

In this paper, we study the following modified SFP in the real Banach space:

Find $x \in F(T) \cap C$ such that

$$Ax \in B^{-1}(0), \quad (4)$$

where C is a nonempty, closed-convex subset of E_1 , $T: C \rightarrow C$ is a Bregman weak relatively nonexpansive mapping, and $B: E_2 \rightarrow 2^{E_2^*}$ is a maximal monotone operator. E_1 and E_2 are p -uniformly convex and uniformly smooth real Banach spaces, E_1^* and E_2^* be the duals of E_1 and E_2 , respectively, $A: E_1 \rightarrow E_2$ be a bounded linear operator, and $A^*: E_2^* \rightarrow E_1^*$ be the adjoint of A . We shall denote the value of the functional $x^* \in E^*$ at $x \in E$ by $\langle x^*, x \rangle$. Obviously, the modified SFP (4) is more general than (1).

Motivated by the above results, in this paper, we present an inertial algorithm for solving (4) in p -uniformly convex and uniformly smooth Banach spaces which have strong convergence. Our algorithm is designed to employ previous iterations x^k and x^{k-1} to obtain the next iterative point; all the implementation process does not compute the spectral radius of $A^T A$, which improves the feasibility of the algorithm.

The paper is organized as follows. Section 2 reviews some preliminaries. Section 3 gives the inertial iterative algorithm and its convergence analysis. Section 4 gives a numerical experiment. Some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries' results which will be useful for our convergence analysis in this paper. We denote the strong and weak convergence of the sequence $\{x_n\}$ to a point x by $x^k \rightarrow x$ and $x^k \rightharpoonup x$, respectively.

Let E be a real Banach space and $1 < q \leq 2 \leq p < \infty$ and $(1/p) + (1/q) = 1$. Define the modulus of smoothness of E as

$$\begin{aligned} \rho_E(\tau) &= \sup \left\{ \frac{\|x - y\| + \|x + y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}, \\ &= \sup \left\{ \frac{\|x - \tau y\| + \|x + \tau y\|}{2} - 1 : \|x\| = 1 = \|y\| \right\}, \end{aligned} \quad (5)$$

where E is uniformly smooth if and only if $\lim_{\tau \rightarrow 0^+} (\rho_E(\tau)/\tau) = 0$ and E is said to be q -uniformly smooth if there exists a constant $D_q > 0$ such that $\rho_E(\tau) \leq D_q \tau^q$.

Define the modulus of convexity of E as

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1; \varepsilon = \|x - y\| \right\}, \quad (6)$$

where E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ and E is p -uniformly convex if there is a constant $C_p > 0$ such that $\delta_E(\varepsilon) \geq C_p \varepsilon^p$ for all $\varepsilon \in (0, 2]$. Every uniformly convex Banach space is strictly convex and reflexive. It is known that if E is p -uniformly convex and uniformly smooth, then its dual E^* is q -uniformly smooth and uniformly convex.

Definition 1 (see [22]). Let $p > 1$. Define the generalized duality mapping $J_E^p: E \rightarrow 2^{E^*}$ as

$$J_E^p = \{x^* \in E^* : \langle x^*, x \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}. \quad (7)$$

It is known that when E is uniformly smooth, then J_E^p is norm-to-norm uniformly continuous on bounded subsets of E , and E is smooth if and only if J_E^p is single valued. J_E^p is said to be weak-to-weak continuous if

$$x^k \rightharpoonup x \Rightarrow \langle J_E^p x^k, y \rangle \rightarrow \langle J_E^p x, y \rangle, \quad \text{for any } y \in E. \quad (8)$$

Lemma 1 (see [23]). Let x and $y \in E$. If E is a q -uniformly smooth Banach space, then there exists a $D_q > 0$ such that

$$\|x - y\|^q \leq \|x\|^q - q \langle J_E^q(x), y \rangle + D_q \|y\|^q. \quad (9)$$

Definition 2 (see [24]). A function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be

- (i) proper if its effective domain $f = \{x \in E : f(x) < +\infty\}$ is nonempty
- (ii) convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$, $\forall \lambda \in (0, 1)$ and x and $y \in D(f)$
- (iii) lower semicontinuous at $x_0 \in D(f)$ if $f(x_0) \leq \lim_{x \rightarrow x_0} \inf f(x)$

Definition 3. Let $f: E \longrightarrow R$ be a differentiable and convex function. The Bregman distance denoted as $\Delta_f: \text{dom} f \times \text{dom} f \longrightarrow [0, +\infty)$ is defined as

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x \rangle, \quad x, y \in E, \quad (10)$$

where $f'(x)$ is the value of the gradient f at x .

It is worthy to note that the duality mapping J_E^p is actually the gradient of the function $f_p(x) = (1/p)\|x\|^p$, for $2 \leq p < \infty$. Hence, if $f = f_p$ in (12), the Bregman distance with respect to f_p now becomes

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle J_E^p x, y - x \rangle, \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y \rangle, \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) \langle -J_E^p x - J_E^p y, x \rangle. \end{aligned} \quad (11)$$

It is generally known that the Bregman distance is not a metric as a result of absence of symmetry, but it possesses some distance-like properties which are stated as follows:

$$\begin{aligned} \Delta_p(x, y) &= \Delta_p(x, z) + \Delta_p(z, y) + \langle J_E^p x - J_E^p z, z - y \rangle, \\ \Delta_p(x, y) + \Delta_p(y, x) &= \langle J_E^p x - J_E^p y, x - y \rangle. \end{aligned} \quad (12)$$

The relationship between the metric and Bregman distance in the p -uniformly convex space is as follows:

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle J_E^p x - J_E^p y, x - y \rangle, \quad (13)$$

where $\tau > 0$ is a fixed number.

Let C be a nonempty closed-convex subset of E . The Bregman projection is defined as

$$\Pi_C x = \arg \min_{y \in C} \Delta_{f_p}(y, x), \quad x \in E. \quad (14)$$

And, the metric projection can be defined similarly as

$$P_C(x) = \arg \min_{y \in C} \|y - x\|, \quad x \in E. \quad (15)$$

The Bregman projection is the unique minimizer of the Bregman distance and can be characterized by a variational inequality [25]:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x \rangle \leq 0, \quad \forall z \in C, \quad (16)$$

from which we have

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x \rangle \leq 0, \quad \forall z \in C. \quad (17)$$

The metric projection which is also the unique minimizer of the norm distance can be characterized by the following variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x \rangle \leq 0, \quad \forall z \in C. \quad (18)$$

We define the functional $V_p: E \times E \longrightarrow [0, \infty]$ associated with $f_p(x) = (1/p)\|x\|^p$ by

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle \bar{x}, x \rangle + \frac{1}{q}\|\bar{x}\|^q, \quad x \in E, \bar{x} \in E^*, \quad (19)$$

where $V_p(x, \bar{x}) \geq 0$. It then follows that

$$V_p(x, \bar{x}) = \Delta_p(x, J_{E^*}^q(\bar{x})), \quad \forall x \in E, \bar{x} \in E^*. \quad (20)$$

Chuasuk et al. [26] proved the following inequality:

$$V_p(x, \bar{x}) + \bar{y}, \langle J_{E^*}^q(\bar{x}) - x \rangle \leq V_p(x, \bar{x} + \bar{y}), \quad \forall x \in E, \bar{x}, \bar{y} \in E^*. \quad (21)$$

Furthermore, V_p is convex in the second variable, and thus, for all $z \in E$, $\{x_i\}_{i=1}^N$, $\{t_i\}_{i=1}^N \subset (0, 1)$, and $\sum_{i=1}^N t_i = 1$, we have (see [23])

$$\begin{aligned} \Delta_p\left(z, J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) &= V_p\left(z, \left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) \\ &\leq \sum_{i=1}^N t_i \Delta_p(z, x_i). \end{aligned} \quad (22)$$

Let C be a nonempty, closed, and convex subset of a smooth Banach space E , and let $T: C \longrightarrow C$. A point $x^* \in C$ is called an asymptotic fixed point of T if a sequence $\{x_n\}_{n \in \mathbb{N}}$ exists in C and converges weakly to x^* such that $\lim_{n \rightarrow \infty} \|x^k - T(x^k)\| = 0$. We denote the set of all asymptotic fixed points of T by $\tilde{F}(T)$. Moreover, a point $x^* \in C$ is said to be a strong asymptotic fixed point of T if there exists a sequence $\{x^k\}_{k \in \mathbb{N}}$ in C which converges strongly to x^* such that $\lim_{k \rightarrow \infty} \|x^k - T(x^k)\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\bar{F}(T)$. It follows from the definitions that $F(T) \subset \bar{F}(T) \subset \tilde{F}(T)$.

Definition 4 (see [27]). Let T be a mapping such that $T: C \longrightarrow E$. T is said to be

- (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each x and $y \in C$
- (ii) quasi-nonexpansive if $\|Tx - y^*\| \leq \|x - y^*\|$ such that $F(T) \neq \emptyset$, $\forall x \in C$, and $y^* \in F(T)$

Definition 5 (see [28]). Let $T: C \longrightarrow E$ be a mapping. T is said to be

- (1) Bregman nonexpansive if
$$\Delta_p(Tx, Ty) \leq \Delta_p(x, y), \quad \forall x \text{ and } y \in C. \quad (23)$$
- (2) Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and
$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C \text{ and } y^* \in F(T). \quad (24)$$

- (3) Bregman weak relatively nonexpansive if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$, and

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C \text{ and } y^* \in F(T). \quad (25)$$

- (4) Bregman relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$, and

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C \text{ and } y^* \in F(T). \quad (26)$$

From the definitions, it is evident that the class of Bregman quasi-nonexpansive maps contains the class of Bregman weak relatively nonexpansive maps. The class of Bregman weak relatively nonexpansive maps contains the class of Bregman relatively nonexpansive maps.

Let E be a smooth, strictly convex, and reflexive Banach space and $A: E \rightarrow 2^{E^*}$ be a maximal monotone operator. We define a mapping $Q_r^A: E \rightarrow D(A)$ by (see [25])

$$Q_r^A(x) = \left(I + r(J_E^p)^{-1} A \right)^{-1}(x), \quad \text{for all } x \in E \text{ and } r > 0. \quad (27)$$

This mapping is known as metric resolvent of A . Obviously, for all $r > 0$, we have

$$0 \in J_E^p(Q_r^A(x) - x) + rAQ_r^A(x), \quad (28)$$

and $F(Q_r^A) = A^{-1}(0)$. Furthermore, for all x and $y \in E$ and by the monotonicity of A , we can show that

$$\langle J_E^p(x - Q_r^A(x)) - J_E^p(y - Q_r^A(y)), Q_r^A(x) - Q_r^A(y) \rangle \geq 0. \quad (29)$$

From (29), we have for all x and $y \in E$,

$$\frac{J_E^p(x - Q_r^A(x))}{r} \in AQ_r^A(x), \quad (30)$$

$$\frac{J_E^p(y - Q_r^A(y))}{r} \in AQ_r^A(y). \quad (31)$$

Since A is monotone, we can obtain (30) from (31) and (32). This implies that for all $x \in E$, $t \in A^{-1}(0)$, and whenever $A^{-1}(0) \neq \emptyset$, we have

$$\langle J_E^p(x - Q_r^A(x)), Q_r^A(x) - t \rangle \geq 0. \quad (32)$$

Lemma 2 (see [29]). *Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E , $x_0 \in C$ and $x \in E$. Then, the following assertions are equivalent:*

- (1) $x_0 = \Pi_C(x)$
- (2) $\langle J_E^p(x_0) - J_E^p(x), z - x_0 \rangle \geq 0$, $\forall z \in C$

Furthermore, for $\forall y \in C$, we have

$$\Delta_p(\Pi_C(x), y) + \Delta_p(x, \Pi_C(x)) \leq \Delta_p(x, y). \quad (33)$$

Lemma 3 (see [30]). *Let E be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in E . Then, $\lim_{n \rightarrow \infty} \Delta_p(x_n, y_n) = 0$ if and only if*

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (34)$$

Lemma 4 (see [30]). *Let $q \geq 1$ and $r > 0$ be two fixed real numbers, then a Banach space E is uniformly convex if and only if there exists a continuous, strictly, increasing, and convex function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $g(0) = 0$ such that for all x and $y \in B_r$ and $0 \leq \alpha \leq 1$,*

$$\| \alpha x + (1 - \alpha)y \|^q \leq \alpha \|x\|^q + (1 - \alpha) \|y\|^q - W_q(\alpha) g(\|x - y\|), \quad (35)$$

where $W_q := \alpha^q(1 - \alpha) + \alpha(1 - \alpha)^q$ and $B_r := \{x \in E: \|x\| \leq r\}$.

3. Inertial Iteration Algorithm and Its Strong Convergence

In this section, we present our inertial iterative algorithm for solving the modified SFP (4) in Banach spaces. We also prove its strong convergence under some suitable conditions.

3.1. Inertial Iteration Algorithm. Now, we give our inertial iterative algorithm.

Algorithm 3.1. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) = \emptyset$. Let $\{\alpha_k\} \subset (0, 1)$, $0 < \liminf_{k \rightarrow \infty} \alpha_k \leq \limsup_{k \rightarrow \infty} \alpha_k < 1$, x^0 and $x^1 \in C = H_1$, and $\{\theta_k\} \subset (0, 1)$ be a real sequence, and $r_k > 0$. Assuming x^{k-1} and x^k have been constructed, we calculate the next iterate x^{k+1} via the following formulas:

$$w^k = J_{E_1}^q \left[J_{E_1}^p(x^k) + \theta_k (J_{E_1}^p(x^k) - J_{E_1}^p(x^{k-1})) \right], \quad (36a)$$

$$v^k = \Pi_C J_{E_1}^* \left[J_{E_1}^p(w^k) - \gamma_k \mu_k A^* J_{E_2}^p(I - Q_{r_k}^B) A w^k \right], \quad (36b)$$

$$u^k = J_{E_1}^q \left[\alpha_k J_{E_1}^p(v^k) + (1 - \alpha_k) J_{E_1}^p(T v^k) \right], \quad (36c)$$

$$C_k = \left\{ u \in E_1: \langle J_{E_1}^p(w^k) - J_{E_1}^p(u^k), u \rangle \leq \frac{1}{q} (\|w^k\|^p - \|u^k\|^p) \right\}, \quad (36d)$$

$$H_k = \{ u \in E_1: \langle x^k - u, J_{E_1}^p(x^1) - J_{E_1}^p(x^k) \rangle \geq 0 \}, \quad (36e)$$

$$x^{k+1} = \Pi_{C_k \cap H_k} x^1, \quad \forall k \in \mathbb{N}, \quad (36f)$$

where

$$\mu_k = \left(\frac{q \| (I - Q_{r_k}^B) A w^k \|^p}{D_q \| A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k \|^q} \right)^{(1/(q-1))}, \quad \gamma_k^{q-1} \in (0, 1). \quad (37)$$

We can see that during the iteration, it does not require to compute the spectral radius of ATA.

3.2. Convergence. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Now, we prove the following lemmas which will be used to establish the strong convergence.

Lemma 5. $\{x^k\}$ generated by (36a)–(36f) is well-defined.

Proof. Obviously, $C_k \cap H_k$ is a nonempty closed and convex set for $\forall k \geq 1$. Now, we show that $\Gamma \subset C_k \cap H_k$. Let $x^* \in \Gamma$. By (22) and Bregman weak relatively nonexpansive of T , we get

$$\begin{aligned} \Delta_p(x^*, u^k) &= \Delta_p\left(x^*, J_{E_1}^q \left[\alpha_k J_{E_1}^p v^k + (1 - \alpha_k) J_{E_1}^p T v^k \right]\right), \\ &\leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, T v^k), \\ &\leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_n) \Delta_p(x^*, v^k), \\ &= \Delta_p(x^*, v^k). \end{aligned} \quad (38)$$

From (36b), Lemma 1, and the definition of Bregman projection, we have

$$\begin{aligned} \Delta_p(x^*, v^k) &= \Delta_p\left(x^*, \Pi_C J_{E_1}^q \left[J_{E_1}^p(w^k) - \gamma_k \mu_k A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k \right]\right), \\ &\leq \Delta_p\left(x^*, J_{E_1}^q \left[J_{E_1}^p(w^k) - \gamma_k \mu_k A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k \right]\right), \\ &= \frac{\|x^*\|^p}{p} - \langle J_{E_1}^p w^k, x^* \rangle + \langle \gamma_k \mu_k A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k, x^* \rangle \\ &\quad + \frac{1}{q} \|J_{E_1}^p w^k - \gamma_k \mu_k A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k\|^q, \\ &\leq \frac{\|x^*\|^p}{p} - \langle J_{E_1}^p w^k, x^* \rangle + \gamma_k \mu_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A x^* \rangle + \frac{1}{q} \|J_{E_1}^p w^k\|^q \\ &\quad - \gamma_k \mu_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A w^k \rangle + \frac{D_q \mu_k^q}{a} \|A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k\|^q, \\ &= \frac{\|x^*\|^p}{p} - \langle J_{E_1}^p w^k, x^* \rangle + \frac{1}{q} \|J_{E_1}^p w^k\|^q + \gamma_k \mu_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A x^* - A w^k \rangle \\ &\quad + \frac{D_q \gamma_k^q \mu_k^q}{q} \|A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k\|^q, \\ &= \Delta_p(x^*, w^k) + \mu_k \gamma_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A x^* - Q_{r_k}^B A w^k + Q_{r_k}^B A w^k - A w^k \rangle \\ &\quad + \frac{D_q \gamma_k^q \mu_k^q}{q} \|A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k\|^q, \\ &= \Delta_p(x^*, w^k) + \gamma_k \mu_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A x^* - Q_{r_k}^B A w^k \rangle \\ &\quad - \gamma_k \mu_k \langle J_{E_2}^p (I - Q_{r_k}^B) A w^k, A w^k - Q_{r_k}^B A w^k \rangle \\ &\quad + \frac{D_q \gamma_k^q \mu_k^q}{-\alpha} \|A^* J_{E_2}^p (I - Q_{r_k}^B) A w^k\|^q, \end{aligned} \quad (39)$$

where the second inequality is from Lemma 2.

Furthermore, from (32), we have

$$\begin{aligned} \Delta_p(x^*, v^k) &\leq \Delta_p(x^*, w^k) \\ &\quad - \gamma_k \mu_k \langle J_{E_2}^p(I - Q_{r_k}^B)Aw^k, (I - Q_{r_k}^B)Aw^k \rangle \\ &\quad + \frac{D_q \gamma_k^q \mu_k^q}{q} \|A^* J_{E_2}^p(I - Q_{r_k}^B)Aw^k\|^q. \end{aligned} \quad (40)$$

By the definitions of μ_k , we have

$$\begin{aligned} \Delta_p(x^*, v^k) &\leq \Delta_p(x^*, w^k) \\ &\quad - \gamma_k \mu_k \left\{ \|(I - Q_{r_k}^B)Aw^k\|^p \right. \\ &\quad \left. - \frac{D_q \gamma_k^{q-1} \mu_k^{q-1}}{q} \|A^* J_{E_2}^p(I - Q_{r_k}^B)Aw^k\|^q \right\}, \\ &= \Delta_p(x^*, w^k) - \gamma_k \mu_k (1 - \gamma_k^{q-1}) \|(I - Q_{r_k}^B)Aw^k\|^p. \end{aligned} \quad (41)$$

Since $\gamma_k^{q-1} \in (0, 1)$, we have

$$\Delta_p(x^*, v^k) \leq \Delta_p(x^*, w^k). \quad (42)$$

Then,

$$\Delta_p(x^*, u^k) \leq \Delta_p(x^*, w^k), \quad (43)$$

which implies $\langle J_{E_1}^p(w^k) - J_{E_1}^p(u^k), u \rangle \leq (1/q)(\|w^k\|^p - \|u^k\|^p)$. So $\Gamma \subset C_k$, for all $k \in \mathbb{N}$. Since $x^{k+1} = \Pi_{C_k \cap H_k}(x^1)$, then $\langle J_{E_1}^p x^1 - J_{E_1}^p x^{k+1}, v - x^{k+1} \rangle \leq 0$, $\forall v \in C_k \cap H_k \subset C$. Ofcourse, for $x^* \in \Gamma$, we also have $\langle J_{E_1}^p x^1 - J_{E_1}^p x^{k+1}, x^* - x^{k+1} \rangle \leq 0$. It implies that $\Gamma \subset H_k$ for all $k \in \mathbb{N}$. So, we obtain that $\Gamma \subset C_k \cap H_k$ for all $k \in \mathbb{N}$. Therefore, $C_k \cap H_k$ is nonempty, and thus, $x^{k+1} = \Pi_{C_k \cap H_k}(x^1)$ is well-defined. $\square$

Lemma 6. Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. Then,

- (1) $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$
- (2) $\lim_{k \rightarrow \infty} \|x^k - w^k\| = 0$
- (3) $\lim_{k \rightarrow \infty} \|Tv^k - v^k\| = 0$
- (4) $\lim_{k \rightarrow \infty} \|x^k - v^k\| = 0$
- (5) $\lim_{k \rightarrow \infty} \|(I - Q_{r_k}^B)Aw^k\| = 0$

Proof

- (1) Let $z \in \Gamma$. Since $\Gamma \subset C_k \cap H_k$, $\forall k \geq 1$, and $x_{k+1} = \Pi_{C_k \cap H_k}(x^1)$, we have

$$\Delta_p(x^{k+1}, x^1) \leq \Delta_p(z, x^1), \quad \forall k \geq 1. \quad (44)$$

Thus, $\{\Delta_p(x^{k+1}, x^1)\}$ is bounded.

We observe that $x^{k+1} \in H_k$, and by (17), we have

$$\langle J_{E_1}^p(x^k) - J_{E_1}^p(x^1), x^k - x^{k+1} \rangle \leq 0. \quad (45)$$

Also, by (18), we have

$$\Delta_p(x^{k+1}, x^k) \leq \Delta_p(x^{k+1}, x^1) - \Delta_p(x^k, x^1), \quad \forall k \geq 1, \quad (46)$$

that is, $\Delta_p(x^k, x^1) \leq \Delta_p(x^{k+1}, x^1) - \Delta_p(x^{k+1}, x^k)$.

Thus, $\Delta_p(x^k, x^1) \leq \Delta_p(x^{k+1}, x^1)$; therefore, $\{\Delta_p(x^k, x^1)\}$ is a bounded monotone nondecreasing sequence. Hence, $\lim_{k \rightarrow \infty} \{\Delta_p(x^k, x^1)\}$ exists. From (46), we have $\lim_{n \rightarrow \infty} \{\Delta_p(x^{k+1}, x^k)\} = 0$. Thus, using Lemma 2.8, we get

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \quad (47)$$

- (2) By the uniform continuity of $J_{E_1}^p$ on bounded subsets of E_1 , from (47) we have

$$\lim_{k \rightarrow \infty} \|J_{E_1}^p(x^{k+1}) - J_{E_1}^p(x^k)\| = \lim_{k \rightarrow \infty} \|J_{E_1}^p(x^k) - J_{E_1}^p(x^{k-1})\| = 0. \quad (48)$$

From (36a), we obtain

$$w^k = J_{E_1}^q(J_{E_1}^p(x^k) + \theta_k(J_{E_1}^p(x^k) - J_{E_1}^p(x^{k-1}))). \quad (49)$$

Then,

$$J_{E_1}^p(w^k) = J_{E_1}^p(x^k) + \theta_k(J_{E_1}^p(x^k) - J_{E_1}^p(x^{k-1})), \quad (50)$$

which gives

$$\|J_{E_1}^p(w^k) - J_{E_1}^p(x^k)\| = |\theta_k| \|J_{E_1}^p(x^k) - J_{E_1}^p(x^{k-1})\|. \quad (51)$$

Therefore,

$$\lim_{k \rightarrow \infty} \|J_{E_1}^p(w^k) - J_{E_1}^p(x^k)\| = 0. \quad (52)$$

Since $J_{E_1}^p$ is also uniformly continuous on bounded subsets of E_1^* , we have

$$\lim_{k \rightarrow \infty} \|x^k - w^k\| = 0. \quad (53)$$

- (3) From (47) and (53), we obtain

$$\begin{aligned} \|x^{k+1} - w^k\| &= \|x^{k+1} - x^k + x^k - w^k\|, \\ &\leq \|x^{k+1} - x^k\| + \|x^k - w^k\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned} \quad (54)$$

Note that, from the construction of C_k , we have that

$$\Delta_p(x^{k+1}, u^k) \leq \Delta_p(x^{k+1}, w^k) \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (55)$$

Thus, $\Delta_p(x^{k+1}, u^k) \longrightarrow 0$ as $k \longrightarrow \infty$; also from Lemma 3, we have

$$\lim_{k \rightarrow \infty} \|x^{k+1} - u^k\| = 0. \quad (56)$$

Similarly, $\|x^k - u^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\|$; it then follows that

$$\lim_{n \rightarrow \infty} \|x^k - u^k\| = 0. \quad (57)$$

Using Lemma 4, from (36c) we have

$$\lim_{k \rightarrow \infty} \|w^k - u^k\| = 0. \quad (58)$$

It follows from (53) and (57) that

$$\begin{aligned} \Delta_p(x^*, u^k) &= \Delta_p(x^*, J_{E_1}^q [\alpha_k J_{E_1}^p(v^k) + (1 - \alpha_k) J_{E_1}^p(Tv^k)]), \\ &= V_p(x^*, \alpha_k J_{E_1}^p(v^k) + (1 - \alpha_k) J_{E_1}^p(Tv^k)), \\ &= \frac{1}{p} \|x^*\|^p - \alpha_k \langle J_{E_1}^p(v^k), x^* \rangle - (1 - \alpha_k) \langle J_{E_1}^p(Tv^k), x^* \rangle + \frac{1}{q} \|\alpha_k J_{E_1}^p v^k + (1 - \alpha_k) J_{E_1}^p T v^k\|^q, \\ &\leq \frac{1}{p} \|x^*\|^p - \alpha_k \langle J_{E_1}^p(v^k), x^* \rangle - (1 - \alpha_k) \langle J_{E_1}^p(Tv^k), x^* \rangle + \frac{1}{q} \alpha_k \|v^k\|^p + \frac{(1 - \alpha_k)}{q} \|Tv^k\|^p \\ &\quad - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|), \\ &= \alpha_k \frac{1}{p} \|x^*\|^p + (1 - \alpha_k) \frac{1}{p} \|x^*\|^p - \alpha_k \langle J_{E_1}^p(v^k), x^* \rangle - (1 - \alpha_k) \langle J_{E_1}^p(Tv^k), x^* \rangle \\ &\quad + \frac{1}{q} \alpha_k \|v^k\|^p + \frac{(1 - \alpha_k)}{q} \|Tv^k\|^p - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|), \\ &= \alpha_k \left\{ \frac{1}{p} \|x^*\|^p - \langle J_{E_1}^p(v^k), x^* \rangle + \frac{1}{q} \|v^k\|^p \right\} + (1 - \alpha_k) \left\{ \frac{1}{p} \|x^*\|^p - \langle J_{E_1}^p(Tv^k), x^* \rangle + \frac{1}{q} \|Tv^k\|^p \right\} \\ &\quad - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|), \\ &= \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, T(v^k)) - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|). \end{aligned} \quad (59)$$

Since T is Bregman weak relatively nonexpansive, we have

$$\begin{aligned} \Delta_p(x^*, u^k) &\leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, v^k) - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|), \\ &= \Delta_p(x^*, v^k) - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p v^k - J_{E_1}^p(Tv^k)\|), \\ &\leq \Delta_p(x^*, w^k) - \frac{W_q(\alpha_k)}{q} g(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|). \end{aligned} \quad (60)$$

Hence, from (12) and (13), we get

$$\begin{aligned}
 \frac{W_q(\alpha_k)}{q} g\left(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|\right) &\leq \Delta_p(x^*, w^k) - \Delta_p(x^*, u^k), \\
 &= \Delta_p(u^k, w^k) + \langle J_{E_1}^p(x^*) - J_{E_1}^p(u^k), u^k - w^k \rangle, \\
 &\leq \langle J_{E_1}^p(u^k) - J_{E_1}^p(w^k), u^k - w^k \rangle + \langle J_{E_1}^p(x^*) - J_{E_1}^p(u^k), u^k - w^k \rangle, \\
 &= \langle J_{E_1}^p(x^*) - J_{E_1}^p(w^k), u^k - w^k \rangle.
 \end{aligned} \tag{61}$$

From (58), we have

$$\lim_{k \rightarrow \infty} \frac{W_q(\alpha_k)}{q} g\left(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|\right) = 0, \tag{62}$$

which implies

$$\lim_{k \rightarrow \infty} g\left(\|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\|\right) = 0. \tag{63}$$

By the property of mapping g , we obtain

$$\lim_{k \rightarrow \infty} \|J_{E_1}^p(v^k) - J_{E_1}^p(Tv^k)\| = 0. \tag{64}$$

Since $J_{E_1}^p$ is uniformly continuous on bounded subsets of E_1^* , we have

$$\lim_{k \rightarrow \infty} \|T(v^k) - v^k\| = 0. \tag{65}$$

(4) From Algorithm 3.1, we have that

$$J_{E_1}^p(u^k) - J_{E_1}^p(v^k) = (1 - \alpha_k)(J_{E_1}^p(Tv^k) - J_{E_1}^p(v^k)). \tag{66}$$

Since $0 < \liminf_{k \rightarrow \infty} \alpha_k \leq \limsup_{k \rightarrow \infty} \alpha_k < 1$ and from (65), we have

$$\lim_{k \rightarrow \infty} \|J_{E_1}^p u^k - J_{E_1}^p v^k\| = 0. \tag{67}$$

Hence,

$$\lim_{k \rightarrow \infty} \|u^k - v^k\| = 0. \tag{68}$$

It is easy to obtain

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \|w^k - v^k\| &= 0, \\
 \lim_{k \rightarrow \infty} \|v^k - x^k\| &= 0.
 \end{aligned} \tag{69}$$

(5) From (41), we have

$$\gamma_k \mu_k (1 - \gamma_k^{q-1}) \| (I - Q_{r_k}^B) A w^k \|^p \leq \Delta_p(x^*, w^k) - \Delta_p(x^*, v^k). \tag{70}$$

From (13) and (16), we get

$$\begin{aligned}
 \gamma_k \mu_k (1 - \gamma_k^{q-1}) \| (I - Q_{r_k}^B) A w^k \|^p &\leq \Delta_p(v^k, w^k) + \langle J_{E_1}^p x^* - J_{E_1}^p v^k, v^k - w^k \rangle, \\
 &\leq \langle J_{E_1}^p v^k - J_{E_1}^p w^k, v^k - w^k \rangle + \langle J_{E_1}^p x^* - J_{E_1}^p v^k, v^k - w^k \rangle, \\
 &= \langle J_{E_1}^p x^* - J_{E_1}^p w^k, v^k - w^k \rangle.
 \end{aligned} \tag{71}$$

It follows from (69) that

$$\lim_{k \rightarrow \infty} \| (I - Q_{r_k}^B) A w^k \|^p = 0. \tag{72}$$

Then,

$$\lim_{k \rightarrow \infty} \| (I - Q_{r_k}^B) A w^k \| = 0. \tag{73}$$

Now, we present the following strong convergence theorem for Algorithm 3.1. $\square$

Theorem 1. Suppose $\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_1$.

Proof. We have known in Lemma 6 (1) that $\lim_{k \rightarrow \infty} \Delta_p(x^k, x^1)$ exists. Now, we show that $x^k \rightarrow \bar{x} \in \Gamma$. Let m and $k \in \mathbb{N}$, then from Lemma 2, we have

$$\begin{aligned}
 \Delta_p(x^m, x^k) &= \Delta_p(x^m, \Pi_{C_{k-1} \cap H_{k-1}} x^1) \\
 &\leq \Delta_p(x^m, x^1) - \Delta_p(x^k, x^1) \rightarrow 0.
 \end{aligned} \tag{74}$$

Therefore, by Lemma 3, we get that $\|x^m - x^k\| \rightarrow 0$ as m and $k \rightarrow \infty$. Thus, $\{x^k\}$ is a Cauchy sequence in C . Since C is closed and convex, it implies that there exists $\bar{x} \in C$ such that $x^k \rightarrow \bar{x}$ as $k \rightarrow \infty$. Since $\|x^k - v^k\| \rightarrow 0$, $\|Tv^k - v^k\| \rightarrow 0$, and T is a Bregman weak relatively nonexpansive mapping, then $\bar{x} \in F(T)$. More so, since $\|x^k - w^k\| \rightarrow 0$, then $w^k \rightarrow \bar{x}$, and by the linearity of A , we have $Aw^k \rightarrow A\bar{x}$. Also from (73), $Q_{r_k}^B Aw^k \rightarrow A\bar{x}$. Since $Q_{r_k}^B$ is a resolvent metric of B for $r_k > 0$, then for all $k \in \mathbb{N}$, we have

$$\frac{J_{E_2}^P(Aw^k - Q_{r_k}^B(Aw^k))}{r_k} \in BQ_{r_k}^B(Aw^k). \quad (75)$$

So for all $(s, s^*) \in B$, we have

$$0 \leq \langle s - Q_{r_k}^B(Aw^k), s^* - \frac{J_{E_2}^P(Aw^k - Q_{r_k}^B(Aw^k))}{r_k} \rangle. \quad (76)$$

It follows from (73) that for all $(s, s^*) \in B$, we have

$$0 \leq \langle s^* - 0, s - A\bar{x} \rangle. \quad (77)$$

Since B is maximal monotone, then it implies that $A\bar{x} \in B^{-1}(0)$; hence, $\bar{x} \in A^{-1}B^{-1}(0)$. Therefore, $\bar{x} \in \Gamma$.

Finally, we show that $\bar{x} = \Pi_\Gamma x^1$. Suppose there exists $\bar{y} \in \Gamma$ such that $\bar{y} = \Pi_\Gamma x^1$. Then,

$$\Delta_p(\bar{y}, x^1) \leq \Delta_p(\bar{x}, x^1). \quad (78)$$

We have shown in Lemma 5 that $\Gamma \subset C_k$ and $\forall k \geq 1$, then $\Delta_p(x^k, x^1) \leq \Delta_p(\bar{x}, x^1)$. By the lower semicontinuity of the norm, we have

$$\begin{aligned} \Delta_p(\bar{x}, x^1) &= \frac{\|\bar{x}\|^p}{q} - \langle J_{E_1}^p \bar{x}, x^1 \rangle + \frac{\|x^1\|^p}{p}, \\ &\leq \liminf_{k \rightarrow \infty} \left\{ \frac{\|x^k\|^p}{q} - \langle J_{E_1}^p x^k, x^1 \rangle + \frac{\|x^1\|^p}{p} \right\}, \\ &= \liminf_{k \rightarrow \infty} \Delta_p(x^k, x^1), \\ &\leq \limsup_{k \rightarrow \infty} \Delta_p(x^k, x^1) \leq \Delta_p(\bar{y}, x^1). \end{aligned} \quad (79)$$

Combining (78) and (79), we have $\bar{y} = \bar{x}$. This completes the proof. $\square$

4. Numerical Example

In this section, we present one numerical example to compare the performance of Algorithm 3.1 with Algorithm (3).

Example 1. Let $E_1 = E_2 = R^n$ and A be a $n \times n$ randomly generated matrix. Let $C = \{x \in R^n: \langle a, x \rangle \geq 1\}$, where $a = (-5, -5, 0, 0, \dots, 0) \in R^n$. Then,

$$\Pi_C(x) = P_C(x) = \frac{1 - \langle a, x \rangle}{\|a\|_2^2} a + x. \quad (80)$$

Let $B: R^n \rightarrow 2^{R^n}$ be defined by $B(x) = \{2x\}$, and $T = P_C$. We take $\theta_k = (3/7k)$, $r_k = (1/2k)$, and $\alpha_k = (k/3k+1)$. We select $x^0 = (-1, -1, -1, -1, \dots, -1) \in R^n$ and $x^1 = (-2, -2, -2, -2, \dots, -2) \in R^n$ as initial two points and select $r = 1 = \gamma_k$ in the two algorithms. The iterative process of Algorithm 3.1 for the example is as follows:

$$\left\{ \begin{array}{l} w^k = x^k + \frac{3}{7k}(x^k - x^{k-1}), \\ v^k = P_C(w^k - \mu_k A^*(I - Q_{r_k}^B)Aw^k), \\ u^k = \frac{k}{3k+1}v^k + \frac{2k+1}{3k+1}P_C(v^k), \\ C_k = \{u \in E_1: \langle w^k - u^k, u \rangle \leq (\|w^k\|^2 - \|u^k\|^2)\}, \\ H_k = \{u \in E_1: \langle x^k - u, x^1 - x^k \rangle \geq 0\}, \\ x^{k+1} = P_{C_k \cap H_k}(x^1), \quad \forall k \in \mathbb{N}, \end{array} \right. \quad (81)$$

where μ_k is chosen as defined by (37) and $Q_{r_k}^B(Aw^k) = (k/k+1)Aw^k$ for all $k \geq 1$. We choose various values of n as follows:

CaseI: $n = 10$, CaseII: $n = 20$, CaseIII: $n = 40$, and CaseIV: $n = 50$, and use $(\|x^{k+1} - x^k\|_2^2 / \|x^2 - x^1\|_2^2) < 10^{-6}$ as the stopping criterion. We, thus, plot the graph of $\|x^{k+1} - x^k\|_2^2$ against the number of iteration in each case and compare the computation results of our algorithm with Algorithm (3) of Alsulami and Takahashi [13]. The computation results can be seen in Figure 1 and Table 1.

From Table 1, we can find that Algorithm 3.1 performs better in terms of number of iterations and CPU time-taken for computation than Algorithm (3). From Figure 1, we can see that the error generated by Algorithm 3.1 in the previous iterative steps is ascending; then, it goes down quickly in the latter iterative steps and converges to zero, which is just the effect of the inertial technique, while the error generated by Algorithm (3) always decreases and converges slowly to zero. The results manifest that inertial technique is an effective method for improving the convergence.

FIGURE 1: Example cases. (a) Top-Left Case I $n = 10$; (b) Top-Right Case II $n = 20$; (c) Bottom-Left Case III $n = 40$; (d) Bottom-Right Case IV $n = 50$.

TABLE 1: Computation result for the example.

Case	CPU time (sec.)		Iter.	
	Algo. (1.3)	Algo. 3.1	Algo. (1.3)	Algo. 3.1
I: $n = 10$	0.1960	0.157	247	56
II: $n = 20$	0.2030	0.1630	315	94
III: $n = 40$	0.2840	0.2280	422	156
IV: $n = 50$	0.3260	0.2490	472	183

5. Conclusion

In this paper, we introduced an inertial iterative algorithm for approximating a common solution of the split feasibility problem, monotone inclusion problem, and fixed-point problem for the class of Bregman weak relative non-expansive mapping in p -uniformly convex and uniformly smooth Banach spaces. Our algorithm is designed in such a way that its implementation does not require to compute the spectral radius of $A^T A$. We also proved a strong convergence theorem under some suitable conditions. Finally, a numerical example is given to test the accuracy and efficiency of our algorithm. The results in this paper improve and extend many related results in the literature.

Data Availability

No data were used to support this study.

Disclosure

Opinions expressed and conclusions arrived are those of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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