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Research Article

Affine Graphs and their Topological Indices

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Graphs are essential tools to illustrate relationships in given datasets visually. Therefore, generating graphs from another concept is very useful to understand it comprehensively. This paper will introduce a new yet simple method to obtain a graph from any finite affine plane. Some combinatorial properties of the graphs obtained from finite affine planes using this graph-generating algorithm will be examined. The relations between these combinatorial properties and the order of the affine plane will be investigated. Wiener and Zagreb indices, spectrums, and energies related to affine graphs are determined, and appropriate theorems will be given. Finally, a characterization theorem will be presented related to the degree sequences for the graphs obtained from affine planes.

1. Introduction

In this section, we start with some definitions and fundamental notions regarding affine planes from [1].

Definition 1. An affine plane $\mathscr{A}$ is an ordered pair $(\mathscr{P}, \mathscr{L})$ which we call the elements of $\mathscr{P}$ as points and the elements of $\mathscr{L}$ as lines, with the following properties:

(A1) Any two distinct points lie on a unique line

(A2) For each point p not on a line l, there is exactly one line l' passing through p such that l is parallel to l'

(A3) There exists a set of three noncollinear points.

The lines l and l' are called as parallel if $l \cap l' = \phi$ or $l = l'$, and we denote this by

$$l \parallel l'. \tag{1}$$

Condition (A2) is also called as *parallel axiom*.

If $\mathscr{P} \cup \mathscr{L}$ is a finite set, then the affine plane $\mathscr{A} = (\mathscr{P}, \mathscr{L})$ is called as *finite affine plane*.

Let us suppose $\mathscr{A}$ is an affine plane. We know from [1] that, in the affine planes, the number of points on each line is the same. Letting k be the number of points per line in $\mathscr{A}$, we call k the *order* of $\mathscr{A}$. Accordingly any point is on $k + 1$ lines.

If $\mathscr{A}$ has order k, then

(1) $\mathscr{A}$ has k^2 points

(2) Each line is parallel to k lines

(3) $\mathscr{A}$ has $k^2 + k$ lines

(4) Each line l meets k^2 other lines

(5) There are $k + 1$ parallel classes

Example 1. In [1], it is shown that the smallest affine plane has four points and six lines and is described synthetically as

$$\begin{aligned} &\mathscr{P} = \{1, 2, 3, 4\}, \\ &\mathscr{L} = \{l_1 = \{1, 2\}, l_2 = \{3, 4\}, l_3 = \{1, 3\}, l_4 = \{2, 4\}, \\ &\quad l_5 = \{1, 4\}, l_6 = \{2, 3\}\}. \end{aligned} \tag{2}$$

The lines can be grouped into three sets of parallel lines:

$$\begin{aligned} l_1 &\parallel l_2, \\ l_3 &\parallel l_4, \\ l_5 &\parallel l_6. \end{aligned} \tag{3}$$

$(\mathscr{P}, \mathscr{L})$ is the affine plane presented in Figure 1.

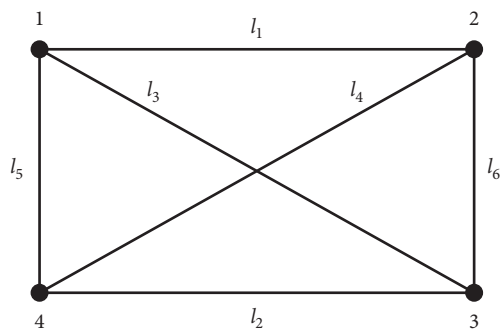

Figure 1: Affine plane of order 2.

When a finite affine plane is given, there is a number $k \geq 2$ as the order. However, for a given number k, it is not necessary to have an affine plane of order k. It is a matter of ongoing studies that whether or not there are affine planes in which order.

Let us now recall some fundamental notions of graph theory from [2–6] to make the paper more self-contained.

Let $G = (V, E)$ be a multigraph with order n and size m, defined as the number of vertices and the number of edges, respectively. Let u and v be two vertices in graph G. If $e = \{u, v\}$ is an edge, then it can be written as $e = uv$, and the vertices u and v are called as *adjacent vertices* $(u \sim v)$ and so *the ends of e*. The edge $e = \{u, v\}$ is said to be incident with u and v. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. Two or more links with the same pair of ends are said to be *parallel edges*.

A graph G is called *simple* if it has neither loops nor parallel edges. During this work, we are going to deal with simple graphs.

A graph G is said to be *regular* if every vertex in G has the same degree. More precisely, G is said to be *k-regular* if $d(v) = k$ for each vertex v in G, where $k \geq 0$. *Adjacency matrix* A of G defined as follows: $A = A(G) = (a_{ij})$, where

$$\begin{aligned} a_{ij} &= 1, \text{if } i \sim j, \\ a_{ij} &= 0, \text{otherwise.} \end{aligned} \tag{4}$$

Obviously, A is a symmetric matrix with zero diagonal; entries consist of 0's and 1's. The eigenvalues of A are the n roots of the *characteristic polynomial* $\det(xI - A)$, so they are algebraic integers. The characteristic polynomial of G is denoted by $P_G(x)$.

Definition 2. The spectrum of a finite graph G is the spectrum of the adjacency matrix A, that is, its set of eigenvalues together with their multiplicities.

The eigenvalues represented as $\lambda_1, \lambda_2, \ldots, \lambda_n$, and unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If G has distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ with multiplicities $k_1, k_2, \ldots, k_m$, respectively, we shall write $\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_m^{k_m}$ for the spectrum of G.

Let G be a graph with n vertices and m edges $(m > 0)$. The *energy* E_G *of* G is defined by

$$E_G = \sum_{i=1}^{n} |\lambda_i|, \tag{5}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G.

A topological index related to a graph is a real number that must be a structural invariant. The topological indices are important for numerical relationships with the structure.

Definition 3. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the minimum number of edges on a path in G between u and v. The Wiener index $W(G)$ of G is defined by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v). \tag{6}$$

The distance $d_G(u)$ of a vertex u is the sum of all distances between u and all other vertices of G. Thus, we can define the Wiener index as below:

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d_G(u). \tag{7}$$

Two of the most useful topological graph indices are the *first and second Zagreb indices* which have been introduced by Gutman and Trinajstic in [7]. They are denoted by $M_1(G)$ and $M_2(G)$ and were defined as

$$\begin{aligned} M_1(G) &= \sum_{u \in V(G)} (d(u))^2, \\ M_2(G) &= \sum_{uv \in E(G)} d(u) d(v), \end{aligned} \tag{8}$$

respectively.

2. The Relation of Graphs with Finite Affine Planes

Affine planes are one of the fundamental examples of incidence geometry. Considering the interpretation of graphs to social sciences and their application to network technologies, with the regularity and parallelism classes of affine planes, it is an interesting issue to obtain graphs from affine planes as one of the transitionalities that can yield results that are most logical and suitable for real-life practices.

We are going to be in an investigation for the answer to the following question: what if someone perceives the lines of a geometric structure as an element of graph theory and what could be obtained from this? To do so, we are introducing a new perspective. In this study, we work only with finite affine planes.

2.1. The Method of Obtaining Graphs from Finite Affine Planes. In this part, a method will be presented to obtain a graph from a given finite affine plane. We note that obtaining a graph $G = (V, E)$ from a finite affine plane $\mathscr{A} = (\mathscr{P}, \mathscr{L})$, which itself is already a graph $(G = \mathscr{A})$, is pointless, but still the method we are going to introduce remains valid for such affine planes.

Definition 4 (1) Let $\mathscr{A} = (\mathscr{P}, \mathscr{L})$ be a finite affine plane of order k. If l_i is a line, then it must have k points. We take this line as an ordered k-tuple $l_i = (l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ik})$. It is obvious that $k \geq 2$. We establish a new set "$s(l_i)$" for any line $l_i = (l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ik})$ in $\mathscr{A}$ such that if $k = 2$, then $l_i = (l_{i1}, l_{i2})$, and we define

$$s(l_i) \coloneqq l_i, \tag{9}$$

and if $k \geq 3$, then first we define

$$l_{i,j} \coloneqq \begin{cases} \{l_{ij}, l_{i(j+1)}\}, & 1 \leq j < k, \\ \{l_{ik}, l_{i1}\}, & j = k, \end{cases} \tag{10}$$

and second

$$\begin{aligned} s(l_i) &\coloneqq \{l_{i,j} \mid j = 1, 2, \ldots, k\} \\ &\coloneqq \{l_{i,1}, l_{i,2}, l_{i,3}, \ldots, l_{i,k-1}, l_{i,k}\} \\ &\coloneqq \{\{l_{i1}, l_{i2}\}, \{l_{i2}, l_{i3}\}, \ldots, \{l_{i(k-1)}, l_{ik}\}, \{l_{ik}, l_{i1}\}\}. \end{aligned} \tag{11}$$

Throughout this paper $s(l_i)$ is called as ordered k-gon corresponding to line l_i when $k \geq 3$.

If we try to obtain a graph from a finite affine plane $\mathscr{A} = (\mathscr{P}, \mathscr{L})$, then we have to change our perception of lines. To do so, we use the method introduced in Definition 4.

Let us consider a line $l_i = \{l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ik}\}$. If $k \geq 3$, for $1 \leq r \leq k$, any point l_{ir} is incident with only $l_{i,r-1}$ and $l_{i,r}$.

For a line $l_i = \{l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ik}\}$, if $k \geq 3$, $s(l_i)$ can be considered as a C_k. Also, we know that a C_3 is often called a triangle, a C_4 is a quadrilateral, a C_5 is a pentagon, a C_6 is a hexagon, and so on.

In the following theorem, we obtain a graph from a finite affine plane $\mathscr{A}$ of any order by considering every element of obtained $s(l_i)$ for a line l_i, as an edge. It is obvious that every element of $s(l_i)$ obtained from l_i is a set of exactly two points.

Theorem 1. *If $\mathscr{A} = (\mathscr{P}, \mathscr{L})$ is a finite affine plane of order k, then we have a graph $G = (V, E)$ such that*

$$\begin{aligned} V(G) &= \mathscr{P}, \\ E(G) &= \bigcup_{l_i \in \mathscr{L}} s(l_i). \end{aligned} \tag{12}$$

Proof. For any line $l_i = \{l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ik}\} \in \mathscr{L}$, it is trivial that $|l_i| = k \geq 2$ and every set $l_{i,j}$ consists of exactly 2 vertices.

Let $x, y \in V(G) = \mathscr{P}$ be distinct elements. From (A1), there exists only one line

$$l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}, \tag{13}$$

passing through the points x and y. Therefore, for $1 \leq r, t \leq k$, there exists

$$\begin{aligned} x &= l_{ir}, \\ y &= l_{it}, \end{aligned} \tag{14}$$

in l_i. Since $r \neq t$, we take $r < t$ without loss of generality.

When $t = r + 1$, there is an edge,

$$l_{i,r} = \{l_{ir}, l_{i(r+1)}\} = \{l_{ir}, l_{it}\} = \{x, y\}, \tag{15}$$

incident with x and y. In other words, when $t = r + 1$, there is only one edge passing through x and y and that is $l_{i,r}$.

When $t > r + 1$ if $t = k$ and $r = 1$, then there is an edge,

$$l_{i,k} = \{l_{ik}, l_{i1}\} = \{l_{it}, l_{ir}\} = \{y, x\}, \tag{16}$$

incident with y and x. In other words, when $t > r + 1$ if $t = k$ and $r = 1$, there is only one edge passing through y and x and that is $l_{i,k}$.

In other cases, it is obvious from the construction of $s(l_i)$ that there is no edge incident with x and y. Thus, we show that $G = (V, E)$ is a graph.

Throughout this work, we are going to use the method introduced by Theorem 1 to obtain a graph from a finite affine plane unless otherwise noted.

We should point out that we have already developed a new and different algorithm to create a new graph generating system from various incidence structures. □

Corollary 1. *If $\mathscr{A} = (\mathscr{P}, \mathscr{L})$ is a finite affine plane of order k and the graph $G = (V, E)$ is obtained from $\mathscr{A}$, then $E(G) = \mathscr{L}$ if and only if $\mathscr{A}$ is the smallest affine plane.*

Proof. We know that if graph $G = (V, E)$ is obtained from $\mathscr{A}$, then

$$\begin{aligned} V(G) &= \mathscr{P}, \\ E(G) &= \bigcup_{l_i \in \mathscr{L}} s(l_i). \end{aligned} \tag{17}$$

Firstly, let us assume $E(G) = \mathscr{L}$. For every line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$ of $\mathscr{A}$, if $k \neq 2$, then we have an ordered k-gon corresponding to that l_i which causes a differentiation (an increase on the number of edges) between the number of edges of $E(G)$ and the number of lines of $\mathscr{L}$. For that reason, k must be 2 for every line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$ of $\mathscr{A}$. It means that $\mathscr{A}$ is the affine plane of order 2 which is the smallest one.

Secondly, let us assume $\mathscr{A}$ is the smallest affine plane. For every line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$ of $\mathscr{A}$, we know that $k = 2$. From Definition 4, we always get the $k = 2$ condition which implies for every line, and we get the result

$$s(l_i) = l_i. \tag{18}$$

Since $E(G) = \cup_{l_i \in \mathscr{L}} s(l_i)$, we obtain

$$E(G) = \{l_i \mid l_i \in \mathscr{L}\} = \mathscr{L}. \tag{19}$$

□

Corollary 2. *For a graph $G = (V, E)$ which is obtained from an affine plane $\mathscr{A} = (\mathscr{P}, \mathscr{L})$,*

$$E(G) = \mathscr{L} \Leftrightarrow G = \mathscr{A}. \tag{20}$$

Proof. From Theorem 1, we know that

$$\begin{aligned} V(G) &= \mathscr{P}, \\ E(G) &= \bigcup_{l_i \in \mathscr{L}} s(l_i). \end{aligned} \tag{21}$$

If $E(G) = \mathscr{L}$, then $G = \mathscr{A}$ since $(V(G), E(G)) = (\mathscr{P}, \mathscr{L})$. We found that

$$\begin{aligned} G = \mathscr{A} &\Rightarrow (V, E) = (\mathscr{P}, \mathscr{L}), \\ &\Rightarrow V(G) = \mathscr{P} \wedge E(G) = \mathscr{L}, \\ &\Rightarrow E(G) = \mathscr{L}, \end{aligned} \tag{22}$$

since the equality of ordered pairs dictates one by one equality.

If there were no condition $V(G) = \mathscr{P}$ in Theorem 1, then the equality of $E(G)$ and $\mathscr{L}$ would not imply the equality of G and $\mathscr{A}$. □

Definition 5. A graph obtained from an affine plane $\mathscr{A}$ of order k is called as "affine graph of order k corresponding to $\mathscr{A}$," or if there will not be any confusion, "affine graph of order k " can be used.

Now, we will examine how affine graphs are obtained when the affine plane of order 2 and the affine plane of order 3 is taken as $\mathscr{A} = (\mathscr{P}, \mathscr{L})$.

Example 2. We are going to investigate the graph obtained from the smallest affine plane given in Example 1.

In this situation, as a result of Corollary 1 and Corollary 2, we do not need to make any further arrangement for the set of the points and lines because the smallest affine plane is also a graph itself:

$$\begin{aligned} V(G) &= \{1, 2, 3, 4\} = \mathscr{P}, \\ E(G) &= \{l_1 = \{1, 2\}, l_2 = \{3, 4\}, l_3 = \{1, 3\}, l_4 = \{2, 4\}, l_5 = \{1, 4\}, l_6 = \{2, 3\}\} = \mathscr{L}, \end{aligned} \tag{23}$$

where $G = (V, E)$ is a graph and has 4 vertex and 6 edges. If we determine the vertex degrees for each vertex, we obtain

$$\begin{aligned} d(1) = d(2) = d(3) = d(4) = 3, \\ \{3, 3, 3, 3\} = \{3^{(4)}\}. \end{aligned} \tag{24}$$

Thus, G is a 3-regular graph (cubic graph) with the degree sequence as above.

Adjacency matrix for G is given below:

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \tag{25}$$

where G has spectrum 3, $(-1)^3$ and the energy of G is $E_G = 6$.

The Wiener index of G is

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d_G(u) = \frac{1}{2} \cdot (3 + 3 + 3 + 3) = 6. \tag{26}$$

Zagreb indices of G is

$$\begin{aligned} M_1(G) &= \sum_{u \in V(G)} (d(u))^2 = 3^2 + 3^2 + 3^2 + 3^2 = 4 \cdot 3^2 = 36, \\ M_2(G) &= \sum_{uv \in E(G)} d(u) d(v) = 3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 = 6 \cdot 3^2 = 54. \end{aligned} \tag{27}$$

Example 3. Let us take the affine plane of order 3 with points and lines given, respectively,

$$\begin{aligned} \mathscr{P}' &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \\ \mathscr{L}' &= \left\{ \begin{array}{l} l_1 = \{1, 2, 3\}, l_2 = \{4, 5, 6\}, l_3 = \{7, 8, 9\}, l_4 = \{1, 4, 7\}, l_5 = \{2, 5, 8\}, l_6 = \{3, 6, 9\}, \\ l_7 = \{1, 5, 9\}, l_8 = \{2, 6, 7\}, l_9 = \{3, 4, 8\}, l_{10} = \{3, 5, 7\}, l_{11} = \{1, 6, 8\}, l_{12} = \{2, 4, 9\} \end{array} \right\}, \end{aligned} \tag{28}$$

and illustrated by Figure 2.

With the method given earlier, we know that $V(G) = \mathscr{P}$. Every given line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$ in this affine plane of order 3 consists of 3 elements. So, k is 3 in every step of obtaining $s(l_i)$. In other words, every line of this affine plane considered as an ordered 3-gon which is denoted as C_3 in graph theory. See Figure 3.

The vertex and edge sets of the graph G' that is obtained from the affine plane of order 3 are as follows:

$$V(G') = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$
$$E(G') = \left\{ \begin{array}{c} l_{1,1} = \{1,2\}, l_{1,2} = \{2,3\}, l_{1,3} = \{3,1\}, l_{2,1} = \{4,5\}, l_{2,2} = \{5,6\}, l_{2,3} = \{6,4\}, \\ l_{3,1} = \{7,8\}, l_{3,2} = \{8,9\}, l_{3,3} = \{9,7\}, l_{4,1} = \{1,4\}, l_{4,2} = \{4,7\}, l_{4,3} = \{7,1\}, \\ l_{5,1} = \{2,5\}, l_{5,2} = \{5,8\}, l_{5,3} = \{8,2\}, l_{6,1} = \{3,6\}, l_{6,2} = \{6,9\}, l_{6,3} = \{9,3\}, \\ l_{7,1} = \{1,5\}, l_{7,2} = \{5,9\}, l_{7,3} = \{9,1\}, l_{8,1} = \{2,6\}, l_{8,2} = \{6,7\}, l_{8,3} = \{7,2\}, \\ l_{9,1} = \{3,4\}, l_{9,2} = \{4,8\}, l_{9,3} = \{8,3\}, l_{10,1} = \{3,5\}, l_{10,2} = \{5,7\}, l_{10,3} = \{7,3\}, \\ l_{11,1} = \{1,6\}, l_{11,2} = \{6,8\}, l_{11,3} = \{8,1\}, l_{12,1} = \{2,4\}, l_{12,2} = \{4,9\}, l_{12,3} = \{9,2\} \end{array} \right\}, \tag{29}$$

where G' is a graph and has 9 vertex and 36 edges. If we determine the vertex degrees for each vertex, we obtain

$$\begin{gathered} d(1) = d(2) = d(3) = d(4) = d(5) = d(6) = d(7) = d(8) = d(9) = 8, \\ \{8,8,8,8,8,8,8,8,8\} = \left\{8^{(9)}\right\}. \end{gathered} \tag{30}$$

Thus, G' is a 8-regular graph with the degree sequence as above. Adjacency matrix for G' is given below:

$$A(G') = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \tag{31}$$

where G' has spectrum $8, (-1)^8$ and the energy of G' is $E_{G'} = 16$. See Figure 4.

The Wiener index of G' is

$$W(G') = \frac{1}{2} \sum_{u \in V(G')} d_{G'}(u) = \frac{1}{2} \cdot (8 + 8 + 8 + \cdots + 8) = 36. \tag{32}$$

Zagreb indices of G' is

$$\begin{gathered} M_1(G') = \sum_{u \in V(G')} (d(u))^2 = 8^2 + 8^2 + 8^2 + \cdots + 8^2 = 9 \cdot 8^2 = 36 \cdot (4)^2, \\ M_2(G') = \sum_{uv \in E(G')} d(u)d(v) = 8 \cdot 8 + 8 \cdot 8 + 8 \cdot 8 + \cdots + 8 \cdot 8 = 36 \cdot 8^2 = 36 \cdot (4)^3. \end{gathered} \tag{33}$$

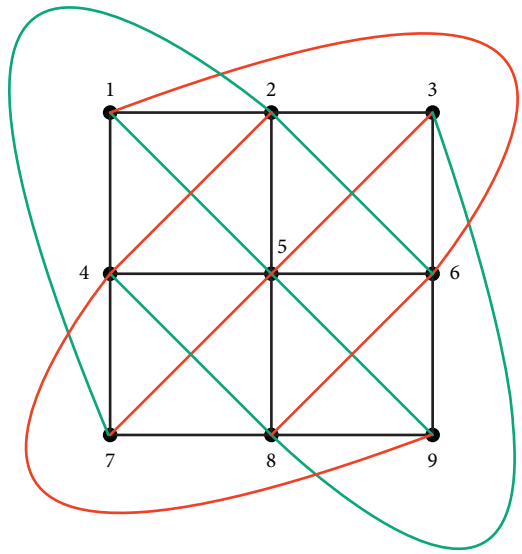

Figure 2: Affine plane of order 3.

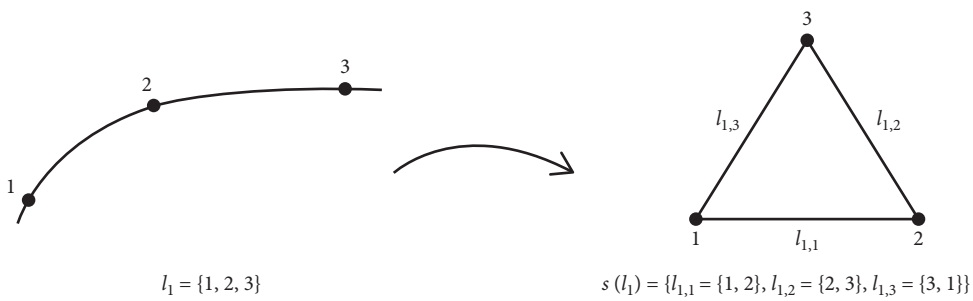

Figure 3: Corresponding triangle for the line l_1 of the affine plane of order 3.

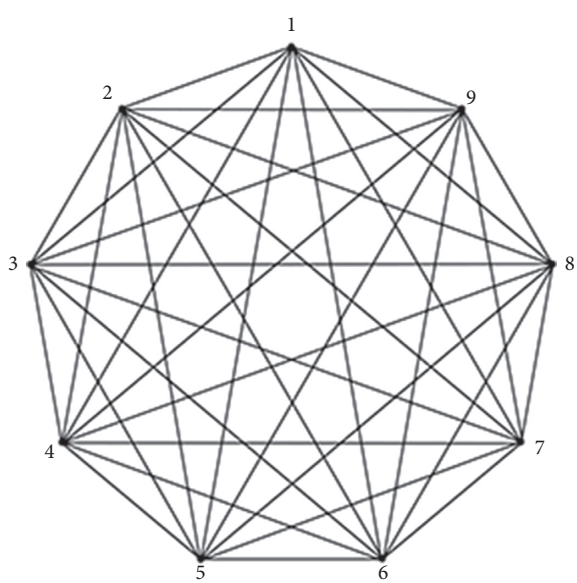

Figure 4: The 8-regular graph obtained from the affine plane of order 3.

Example 4. Let us take the affine plane of order 4 with points and lines given, respectively:

$$\mathscr{P}'' = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\},$$
$$\mathscr{L}'' = \left\{ \begin{array}{c} l_1 = \{1,2,3,4\}, l_2 = \{5,6,7,8\}, l_3 = \{9,10,11,12\}, \\ l_4 = \{13,14,15,16\}, l_5 = \{1,5,9,13\}, l_6 = \{2,6,10,14\}, \\ l_7 = \{3,7,11,15\}, l_8 = \{4,8,12,16\}, l_9 = \{1,6,11,16\}, \\ l_{10} = \{2,5,12,15\}, l_{11} = \{3,8,9,14\}, l_{12} = \{4,7,10,13\}, \\ l_{13} = \{1,7,12,14\}, l_{14} = \{2,8,11,13\}, l_{15} = \{3,5,10,16\}, \\ l_{16} = \{4,6,9,15\}, l_{17} = \{1,8,10,15\}, l_{18} = \{2,7,9,16\}, \\ l_{19} = \{3,6,12,13\}, l_{20} = \{4,5,11,14\} \end{array} \right\}. \tag{34}$$

With the method given earlier, we know that $V(G) = \mathscr{P}$. Every given line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$ in this affine plane of order 4 consists of 4 elements. So, k is 4 in every step of obtaining $s(l_i)$. In other words, every line of this affine plane is considered as an ordered 4-gon, which is denoted as C_4 in graph theory. The vertex and edge sets of the graph G'' that is obtained from the affine plane of order 4 are as follows:

$$V(G'') = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\},$$
$$E(G'') = \left\{ \begin{array}{c} l_{1,1} = \{1,2\}, l_{1,2} = \{2,3\}, l_{1,3} = \{3,4\}, l_{1,4} = \{4,1\}, \\ l_{2,1} = \{5,6\}, l_{2,2} = \{6,7\}, l_{2,3} = \{7,8\}, l_{2,4} = \{8,5\}, \\ l_{3,1} = \{9,10\}, l_{3,2} = \{10,11\}, l_{3,3} = \{11,12\}, l_{3,4} = \{12,9\}, \\ l_{4,1} = \{13,14\}, l_{4,2} = \{14,15\}, l_{4,3} = \{15,16\}, l_{4,4} = \{16,13\}, \\ l_{5,1} = \{1,5\}, l_{5,2} = \{5,9\}, l_{5,3} = \{9,13\}, l_{5,4} = \{13,1\}, \\ l_{6,1} = \{2,6\}, l_{6,2} = \{6,10\}, l_{6,3} = \{10,14\}, l_{6,4} = \{14,2\}, \\ l_{7,1} = \{3,7\}, l_{7,2} = \{7,11\}, l_{7,3} = \{11,15\}, l_{7,4} = \{15,3\}, \\ l_{8,1} = \{4,8\}, l_{8,2} = \{8,12\}, l_{8,3} = \{12,16\}, l_{8,4} = \{16,4\}, \\ l_{9,1} = \{1,6\}, l_{9,2} = \{6,11\}, l_{9,3} = \{11,16\}, l_{9,4} = \{16,1\}, \\ l_{10,1} = \{2,5\}, l_{10,2} = \{5,12\}, l_{10,3} = \{12,15\}, l_{10,4} = \{15,2\}, \\ l_{11,1} = \{3,8\}, l_{11,2} = \{8,9\}, l_{11,3} = \{9,14\}, l_{11,4} = \{14,3\}, \\ l_{12,1} = \{4,7\}, l_{12,2} = \{7,10\}, l_{12,3} = \{10,13\}, l_{12,4} = \{13,4\}, \\ l_{13,1} = \{1,7\}, l_{13,2} = \{7,12\}, l_{13,3} = \{12,14\}, l_{13,4} = \{14,1\}, \\ l_{14,1} = \{2,8\}, l_{14,2} = \{8,11\}, l_{14,3} = \{11,13\}, l_{14,4} = \{13,2\}, \\ l_{15,1} = \{3,5\}, l_{15,2} = \{5,10\}, l_{15,3} = \{10,16\}, l_{15,4} = \{16,3\}, \\ l_{16,1} = \{4,6\}, l_{16,2} = \{6,9\}, l_{16,3} = \{9,15\}, l_{16,4} = \{15,4\}, \\ l_{17,1} = \{1,8\}, l_{17,2} = \{8,10\}, l_{17,3} = \{10,15\}, l_{17,4} = \{15,1\}, \\ l_{18,1} = \{2,7\}, l_{18,2} = \{7,9\}, l_{18,3} = \{9,16\}, l_{18,4} = \{16,2\}, \\ l_{19,1} = \{3,6\}, l_{19,2} = \{6,12\}, l_{19,3} = \{12,13\}, l_{19,4} = \{13,3\}, \\ l_{20,1} = \{4,5\}, l_{20,2} = \{5,11\}, l_{20,3} = \{11,14\}, l_{20,4} = \{14,4\} \end{array} \right\}, \tag{35}$$

where G'' is a graph and has 16 vertex and 80 edges. If we determine the vertex degrees for each vertex, we obtain

$$\begin{gathered} d(1) = d(2) = d(3) = d(4) = d(5) = d(6) = d(7) = 10, \\ d(8) = d(9) = d(10) = d(11) = d(12) = d(13) = d(14) = d(15) = d(16) = 10, \\ \{10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10\} = \left\{10^{(16)}\right\}. \end{gathered} \tag{36}$$

Thus, G'' is a 10-regular graph with the degree sequence as above. The adjacency matrix for G'' is given below:

$$A(G'') = \begin{bmatrix} 0&1&0&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&0&1&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&1&0&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&0&1&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&0&1&0&1&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&0&1&0&1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&1&0&1&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&0&1&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&1&0&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&0&1&0&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&1&0&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&0&1&0&1&1&1&1 \\ 1&1&1&1&0&0&0&0&1&1&1&1&0&1&0&1 \\ 1&1&1&1&0&0&0&0&1&1&1&1&1&0&1&0 \\ 1&1&1&1&0&0&0&0&1&1&1&1&0&1&0&1 \\ 1&1&1&1&0&0&0&0&1&1&1&1&1&0&1&0 \end{bmatrix}, \quad (37)$$

where G'' has spectrum $10, (-6), 2^2, (-2)^4, 0^8$ and the energy of G'' is $E_{G''} = 28$. See Figure 5.

The Wiener index of G'' is

$$\begin{aligned} W(G'') &= \frac{1}{2} \sum_{u \in V(G'')} d_{G''}(u) \\ &= \frac{1}{2} \cdot ((1 \cdot 10 + 2 \cdot 5) + (1 \cdot 10 + 2 \cdot 5) \\ &\quad + \cdots + (1 \cdot 10 + 2 \cdot 5)) \\ &= \frac{1}{2} \cdot (20 \cdot 16) = 160. \end{aligned} \quad (38)$$

Zagreb indices of G'' is

$$\begin{aligned} M_1(G'') &= \sum_{u \in V(G'')} (d(u))^2 = 10^2 + 10^2 + 10^2 + \cdots + 10^2 = 16 \cdot 10^2 = 64 \cdot (5)^2, \\ M_2(G'') &= \sum_{uv \in E(G'')} d(u)d(v) = 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + \cdots + 10 \cdot 10 = 80 \cdot 10^2 = 64 \cdot (5)^3. \end{aligned} \quad (39)$$

When we obtain the affine graph for the affine plane of order 5, the affine graph of order 5, we calculate it as 25 vertex and 150 edges, and it is 12-regular. The adjacency matrix for this graph is given below.

The Wiener index for affine graph of order 5, namely, G''' is

$$\begin{aligned} W(G''') &= \frac{1}{2} \sum_{u \in V(G''')} d_{G'''}(u) \\ &= \frac{1}{2} \cdot ((1 \cdot 12 + 2 \cdot 12) + (1 \cdot 12 + 2 \cdot 12) \\ &\quad + \cdots + (1 \cdot 12 + 2 \cdot 12)) \\ &= \frac{1}{2} \cdot (36 \cdot 25) = 450. \end{aligned} \quad (40)$$

Zagreb indices of G''' is

$$M_1\left(G'''\right) = \sum_{u\in V\left(G'''\right)} (d(u))^2 = 12^2 + 12^2 + 12^2 + \cdots + 12^2 = 25\cdot 12^2 = 100\cdot (6)^2,$$

$$M_2\left(G'''\right) = \sum_{uv\in E\left(G'''\right)} d(u)d(v) = 12\cdot 12 + 12\cdot 12 + 12\cdot 12 + \cdots + 12\cdot 12 = 150\cdot 12^2 = 100\cdot (6)^3,$$

$$A\left(G'''\right) = \begin{bmatrix}
0&1&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
1&0&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
1&1&1&1&1&0&1&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&1&0&0&1&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&0&0&1&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&0\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&1&0
\end{bmatrix}. \tag{41}$$

Also, it has spectrum $12, ((-(1/2)) + (5\sqrt{5}/2))^2, ((-(1/2)) + (\sqrt{5}/2))^{10}, ((-(1/2)) - (\sqrt{5}/2))^{10}, ((-(1/2)) - (5\sqrt{5}/2))^2$ and the energy of this graph is $12 + 20\sqrt{5}$.

The results regarding the spectra and energies of affine graphs are consistent with the lower and upper boundaries, given for regular graphs in [8–10].

Corollary 3. *Affine graphs of order k consist of k^2 vertices and $(k^2 + k)k = k^3 + k^2$ edges.*

Now, we give a theorem and a corollary for the characterization of the graphs that are obtained from affine planes.

Theorem 2. *Affine graphs of order k for $k \geq 3$ are $2(k+1)$-regular and has the degree sequence in the following form:*

$$\{2(k+1), 2(k+1), \ldots, 2(k+1)\} = \left\{(2(k+1))^{(k^2)}\right\}. \tag{42}$$

Proof. Let $\mathscr{A} = (\mathscr{P}, \mathscr{L})$ be an affine plane of order k for $k \geq 3$ and the graph $G = (V, E)$ is obtained from $\mathscr{A}$.

We know that, in affine planes of order k, all lines has k points and every point is exactly on $k+1$ distinct lines. As mentioned just before Theorem 1 for a point l_{ir} on a line $l_i = \{l_{i1}, l_{i2}, \ldots, l_{ik}\}$, there are exactly two edges $l_{i,r-1}$ and $l_{i,r}$ for that l_i.

The point l_{ir} is on exactly $k+1$ distinct lines, so if it is incident with two edges for every line, then we have $2(k+1)$ edges incident with the vertex l_{ir}.

Therefore, every vertex in G has the degree $2(k+1)$. There are k^2 vertices in G since $V(G) = \mathscr{P}$ and $|\mathscr{P}| = k^2$. Thus, we have

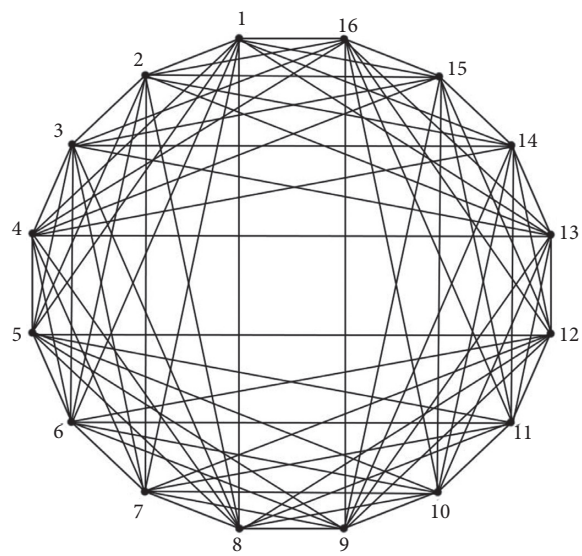

Figure 5: The 10-regular graph obtained from the affine plane of order 4.

$$\{2(k+1), 2(k+1), \ldots, 2(k+1)\} = \left\{ (2(k+1))^{(k^2)} \right\}, \tag{43}$$

as the degree sequence. □

Theorem 3. *Let G be an affine graphs of order k for $3 \le k \le 5$. The Wiener index for these graphs can be calculated as*

$$W(G) = (k+1) \cdot (k-2) \cdot (k)^2. \tag{44}$$

Proof. Proof can be done by following simple calculations. □

Theorem 4. *Let G be an affine graphs of order k for $k \ge 3$. Zagreb indices for these graphs can be calculated as*

$$\begin{aligned} M_1(G) &= 4 \cdot k^2 \cdot (k+1)^2, \\ M_2(G) &= 4 \cdot k^2 \cdot (k+1)^3. \end{aligned} \tag{45}$$

Proof. Proof can be done by the following simple calculations. □

3. Conclusion

With this study, we start giving some relations between the incidence structures and graph theory from our new perspective. It is easily seen that there are some differences and similarities between the affine graphs of order k and the affine planes of order k, although the relations between them have not been examined thoroughly yet.

There are also some open problems in this subject that we are constantly studying on and which we think we are going to be able to examine and answer in the future. Some of them are in the following paragraphs.

We know that it is possible to obtain *projective graphs* as the geometric structure shift to projective planes, from [11] and more from [12]. What is the relation between affine and projective graphs which have the same order? For example, how does the existence and absence of the parallel axiom in affine and projective planes affect the graphs that are obtained from these planes?

We know that an affine plane is obtained when a line, with the points, is thrown away from a projective plane. Under which conditions can we find an affine graph of order k by deleting a C_{k+1}, with the vertices, from a projective graph of order k?

We know that affine planes of order k can be embedded into projective planes of order k. Can affine graphs of order k be embedded in projective graphs of order k?

How can we determine whether a given graph is an affine graph?

Is there a relation between the isomorphism of geometric structures and associated graphs?

How can we determine the chromatic index of affine graphs? Is there a relationship between the parallel groups of affine planes and the chromatic index of corresponding affine graphs?

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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