Review Article

Leray–Schauder Fixed Point Theorems for Block Operator Matrix with an Application

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In this paper, we establish some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix defined on nonempty, closed, and convex subsets Ω of Banach spaces. Note here that Ω need not be bounded. These results are formulated in terms of weak sequential continuity and the technique of De Blasi measure of weak noncompactness on countably subsets. We will also prove the existence of solutions for a coupled system of nonlinear equations with an example.

1. Introduction

With the development of new problems in diverse fields of sciences as well as in physical, biological, and social sciences, the theory of fixed point and its applications are very diverse and continuously growing. Also, the theory of block operator matrix is a subject of great interest thanks to the useful applications for studying some systems of integral equations as well as systems of partial or ordinary differential equations. Recent work has employed the fixed point technique for the operator matrix with nonlinear entries acting on Banach spaces or Banach algebras for studying the existence of solutions for several classes of systems of nonlinear integral equations, see, for example, [1–5]. These operators are defined by a 2×2 block operator matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (1)$$

Based on new generalized Schauder and Krasnoselskii fixed point theorems for the block operator matrix (1), Ben Amar et al., in [6], have established some results for a coupled system of differential equations on $L_p \times L_p$ for $p \in (1, \infty)$, under abstract boundary conditions of Rotenberg's model type. These last equations were proposed by M. Rotenberg and model the evolution of a cell population

[7]. Due to the lack of compactness on L_1 spaces, the study in [6] did not cover the case $p = 1$. Later, Jeribi et al. [1] proposed to extend the results of Amar et al. [6] to the case $p = 1$ by establishing new variants of fixed point theorems for (1), and their analysis was carried out via arguments of weak topology and, particularly, the technique of measures of weak noncompactness. In the above quoted works, the assumptions that $I - A$ or $I - D$ are invertible play a fundamental role in the arguments. Jeribi et al., in [8], were interested in studying the case when $I - A$ is not injective and established some fixed point theorems for operator (1), involving multivalued maps acting on Banach spaces. This way, their results were formulated in terms of weak sequential continuity and the technique of De Blasi measure of weak noncompactness. The results obtained are then applied to the two-dimensional nonlinear functional integral equation:

$$\begin{cases} x(t) = k(t, x(t)) + \left[q(t) + \int_0^{\sigma(t)} g(t, s, y(s)) ds \right] \cdot u, \\ y(t) = \phi \left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds \right) \cdot v + a(t) y(t), \end{cases} \quad (2)$$

for all $t \in J = [0, 1]$, where $u, v \in X$ with $u \neq 0$ and $v \neq 0$; here, X is a Banach space. And, $\sigma: J \rightarrow J$, $g: J \times J \times X \rightarrow X$, $k, w: J \times J \rightarrow \mathbb{R}$, and $b: J \rightarrow X$ are suitably defined functions.

The main purpose of this paper is to obtain some new variants of Leray–Schauder-type fixed point theorem for operator (1) on a Banach space. From application, we discuss the existence of solutions to problem (2) in a suitable Banach space and an example of a nonlinear integral equation in the Banach space $\mathcal{C}([0, 1], \mathbb{R})$.

Note that system (2) can be written as a fixed point problem:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} k(t, \cdot) & G(t, \cdot) \\ \varphi(t, \cdot) & a(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad (3)$$

where

$$\begin{cases} G(t, y(t)) = \left[q(t) + \int_0^{\sigma(t)} g(t, s, y(s)) ds \right] \cdot u, \\ \varphi(t, x(t)) = \phi \left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds \right) \cdot v. \end{cases} \quad (4)$$

$$\omega(M) = \inf \{ r > 0, \text{ there exists a weakly compact set } K \text{ such that } M \subseteq K + B_r \}. \quad (5)$$

The De Blasi measure of weak noncompactness satisfies the following properties. For a proof, we refer the reader to [9, 10].

Lemma 1. *Let M_1 and M_2 be in $P_{bd}(X)$, and we have*

- (i) *If $M_1 \subset M_2$, then $\omega(M_1) \leq \omega(M_2)$*
- (ii) *$\omega(M_1) = 0$ if and only if M_1 is relatively weakly compact*
- (iii) *$\omega(\overline{M_1^w}) = \omega(M_1)$, where $\overline{M_1^w}$ is the weak closure of M_1*
- (iv) *$\omega(\lambda M_1) = |\lambda| \omega(M_1)$, for all $\lambda \in \mathbb{R}$*
- (v) *$\omega(\overline{\text{co}}(M_1)) = \omega(M_1)$, where $\overline{\text{co}}(M_1)$ is the closed convex hull of M_1*
- (vi) *$\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$*
- (vii) *$\omega(M_1 \cup \{x\}) = \omega(M_1)$, for all $x \in X$*

In [10], Appell and De Pascale proved that, in L^1 -spaces, the maps $\omega(\cdot)$ has the following form:

$$\omega(M) = \limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\psi \in M} \left[\int_D \|\psi(t)\|_X dt : \text{meas}(D) \leq \varepsilon \right] \right\}. \quad (6)$$

For all bounded subsets M of $L^1(\Omega, X)$, where X is a finite dimensional Banach space and $\text{meas}(\cdot)$ denotes the Lebesgue measure, we recall the following definitions.

The present paper is built up as follows. In Section 2, we introduce the necessary definitions and preliminary concepts. Section 3 is devoted to present some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix maps acting on Banach spaces. Finally, in Section 4, we apply Corollary 1 in order to discuss the existence of solutions for problem (2).

2. Basic Definitions and Preliminary Concepts

In this section, we give some essential definitions, properties, and theorems of fixed point theory, which should be used in the present paper. Throughout this paper, unless otherwise mentioned, X denotes a Banach space endowed with the norm $\|\cdot\|$ and with the zero element θ , B_r denotes the closed ball in X centered at θ with radius $r > 0$, and $P_{bd}(X)$ denotes the collection of all nonempty bounded subsets of X . Moreover, we write $x_n \rightarrow x$ and $x_n \rightharpoonup x$ to denote, respectively, the strong convergence and the weak convergence of a sequence $\{x_n\}_n$ to x . We say that a map $T: X \rightarrow X$ is weakly sequentially continuous if, for every sequence $\{x_n\}_n \subset X$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$.

The De Blasi measure of weak noncompactness [9] is the map $\omega: P_{bd}(X) \rightarrow [0, \infty)$ defined by

Definition 1. Let Ω be a subset of a Banach space X , and $k \in [0, 1)$. Let $T: \Omega \rightarrow X$ be a mapping, we say that

- (1) T is k -contractive if $\omega(T(M)) \leq k\omega(M)$ for any bounded set $M \subset \Omega$
- (2) T is condensing if $\omega(T(M)) < \omega(M)$ for any bounded set $M \subset \Omega$ with $\omega(M) > 0$
- (3) T is countably k -contractive if $\omega(T(M)) \leq k\omega(M)$ for any countable bounded set $M \subset \Omega$
- (4) T is countably condensing if $\omega(T(M)) < \omega(M)$ for any countable bounded set $M \subset \Omega$ with $\omega(M) > 0$

Clearly, every k -contractive is countably k -contractive. Now, T is said to be k -Lipschitzian if $\|Tx - Ty\| \leq k\|x - y\|$ with $k \in [0, \infty)$. If $k \in [0, 1)$, T is called a contraction.

Definition 2. A mapping $T: X \rightarrow X$ is said to be weakly compact if $T(M)$ is relatively weakly compact for every nonempty bounded subset $M \subset X$.

Following [11], we recall the next definition.

Definition 3. A mapping $T: X \rightarrow X$ is called to be a separate contraction if there exist two functions $\varphi, \psi: [0, \infty) \rightarrow [0, \infty)$, satisfying

- (1) ψ is strictly increasing and $\psi(0) = 0$
- (2) $\|Tx - Ty\| \leq \varphi(\|x - y\|)$ for all $x, y \in X$
- (3) $\psi(r) + \varphi(r) \leq r$ for any $r > 0$

Note that every contraction mapping is a separate contraction mapping.

Definition 4 (see [12]). A mapping $T: X \longrightarrow X$ is a nonlinear contraction mapping if there exists a continuous nondecreasing function $\varphi: [0, \infty) \longrightarrow [0, \infty)$ satisfying

- (1) $\varphi(r) < r$ for any $r > 0$
- (2) $\|Tx - Ty\| \leq \varphi(\|x - y\|)$ for all $x, y \in X$

Definition 5. An operator $T: X \longrightarrow X$ is said to be ψ -expansive if there exists a function $\psi: [0, \infty) \longrightarrow [0, \infty)$ such that

- (1) $\psi(0) = 0$
- (2) $\psi(r) > r$, for any $r > 0$
- (3) ψ is either continuous or nondecreasing
- (4) $\|Tx - Ty\| \geq \psi(\|x - y\|)$ for all $x, y \in X$

In particular, for $\psi(t) = ht$ with $h > 1$, then T is an expansive mapping (see [13]). The following results are the nonlinear alternatives of Leray–Schauder-type states in [14].

Theorem 1. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$ such that $\overline{U^w}$ is a weakly compact subset of Ω and $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous mapping.

Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Remark 1. In Theorem 1, the condition “ $\overline{U^w}$ is a weakly compact” can be replaced by “ $F(\overline{U^w})$ is relatively weakly compact,” for the proof, see Remark 3.2 in [14].

Theorem 2. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Assume that $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous and condensing map with $F(\overline{U^w})$ bounded. Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Amar et al., in [15], showed that the condition “condensing” in Theorem 2 can be relaxed by the assumption “countably condensing.”

Theorem 3. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Assume that $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous and countably condensing map with $F(\overline{U^w})$ bounded. Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

The following results are crucial for our purposes.

Lemma 2 (see [16]). Let Ω be a subset of a Banach space X and let $T: \Omega \longrightarrow X$ be a k -Lipschitzian map. Assume that T is a sequentially weakly continuous map. Then, $\omega(T(M)) \leq k\omega(M)$ for each bounded subset M of Ω ; here, $\omega(\cdot)$ stands for the De Blasi measure of weak noncompactness.

Lemma 3 (see [17]). Let K be a Hausdorff compact space and X be a Banach space. A bounded sequence $\{f_n\}_n \subset \mathcal{C}(K, X)$ converges weakly to $f \in \mathcal{C}(K, X)$ if and only if, for every $t \in K$, the sequence $\{f_n(t)\}_n$ converges weakly (in X) to $f(t)$.

3. Main Result

In this section, we give some new variants of Leray–Schauder for the operator (1). The first result is formulated as follows.

Theorem 4. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \longrightarrow X$ and $B, D: X \longrightarrow X$ be four operators such that

- (i) A is linear and bounded, and there is $p \in \mathbb{N}^*$ such that A^p is a separate contraction
- (ii) B and C are weakly sequentially continuous and $C(\overline{U^w})$ is relatively weakly compact
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $(I - A)^{-1}B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (7)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. If we refer to Lemma 1.2 in [11] and to page 39 in [18], we can prove that $(I - A)^{-1}$ and $(I - D)^{-1}$ exist and are weakly continuous. Hence, we can define the mapping $F: \overline{U^w} \longrightarrow \Omega$ by

$$Fx = (I - A)^{-1}B(I - D)^{-1}Cx. \quad (8)$$

In view of Theorem 1, it suffices to establish that F is weakly sequentially continuous and $F(\overline{U^w})$ is relatively weakly compact.

We have $(I - A)^{-1}$ and $(I - D)^{-1}$, and B and C are weakly sequentially continuous; then, F is weakly

sequentially continuous, and because $C(\overline{U^w})$ is relatively weakly compact, we get $F(\overline{U^w})$ is relatively weakly compact. Hence, either

- (1) F has a fixed point or
- (2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$.

In the first case, the vector $y = (I - D)^{-1}Cx$ solves the problem, whereas in the second case we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. $\square$

Remark 2

- (1) Theorem 4 remains true if we suppose that A^p or D^p is a nonlinear contraction
- (2) Theorem 4 is a generalization of Theorem 4 and Theorem 3.2 in [19]

We also have the following result.

Theorem 5. *Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) A, B , and C are weakly sequentially continuous
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (9)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. The reasoning in the proof of Theorem 4 yields that $(I - D)^{-1}$ exists and is weakly continuous. Hence, we can define the mapping $F: \overline{U^w} \rightarrow \Omega$ by

$$Fx = Ax + B(I - D)^{-1}Cx. \quad (10)$$

We can see that F is weakly sequentially continuous because A, B, C , and $(I - D)^{-1}$ are weakly sequentially continuous. The assumption (i) proves that $F(\overline{U^w})$ is relatively weakly compact. Hence, by Theorem 1, we have

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}x$ solves the problem, whereas in the second case we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. $\square$

Remark 3

- (1) In Theorem 5, we can replace the assumption “ $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$,” by “ $x = Ax + B(I - D)^{-1}Cy \in \Omega$, for all $y \in \overline{U^w}$ implies $x \in \Omega$ ”
- (2) Theorem 5 is a generalization of Theorem 3.3 in [19]

Thus, we give the following result

Theorem 6. *Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) A, B , and C are weakly sequentially continuous
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ \lambda C & \lambda D \end{pmatrix} \begin{pmatrix} u \\ \frac{v}{\lambda} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (11)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. Reasoning as in the proof of Theorem 5, we obtain

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}x$ solves the problem, whereas in the second case, we use the vector $v = \lambda(I - D)^{-1}Cu$ to achieve the proof. $\square$

Remark 4. Theorem 6 is a generalization of Theorem 3.4 in [19].

Theorem 7. *Let Ω be a nonempty closed and convex subset of Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) D is a contraction with $C(\overline{U^w}) \subset (I - D)(\Omega)$
- (iii) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (12)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. By assumption (ii), we can see that $(I - D)^{-1}$ exists and continuous; hence, from Theorem 5, we deduce the desired result.

Inspired by [20], we deduce the following result. $\square$

Corollary 1. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) D is a contraction with $C(\overline{U^w}) \subset (I - D)(\Omega)$
- (iii) $Ax + B(I - D)^{-1}Cx \in \Omega$ for all $x \in \overline{U^w}$

In addition, assume that

$$\lambda Au + \lambda Bv \neq u, \quad (13)$$

for all $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$. Then, the set of fixed points of the block operator matrix (1) in $\overline{U^w} \times X$ is nonempty.

In the next results, we explore the case where A and D are nonlinear.

Theorem 8. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $(I - D)^{-1}$ exists on $C(\Omega)$
 - (ii) $A(\overline{U^w})$ is relatively weakly compact and B is countably β -contractive
 - (iii) C is countably γ -contractive and D is countably δ -contractive with $\gamma + \delta < 1$
 - (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω
- Then, either
- (A1) the block operator matrix (1) has a fixed point or
 - (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (14)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. We defend the operator $F: \overline{U^w} \rightarrow \Omega$ by

$$Fx = Ax + B(I - D)^{-1}Cx. \quad (15)$$

We show that F is weakly sequentially continuous and countably condensing.

Let $\{x_n\}_n$ be a sequence in Ω such that $x_n \rightarrow x$; since C is weakly sequentially continuous, then the set $\{Cx_n; n \in \mathbb{N}\}$ is relatively weakly compact. Note that

$$(I - D)^{-1}C = C + D(I - D)^{-1}C. \quad (16)$$

We have D is countably δ -contractive; then,

$$\begin{aligned} \omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) &\leq \omega(\{Cx_n; n \in \mathbb{N}\}) + \omega(\{D(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &\leq \omega(\{D(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &\leq \delta \omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &< \omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}), \end{aligned} \quad (17)$$

which implies that $\omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) = 0$; then, $\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $\{x_{\sigma(n)}\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $(I - D)^{-1}Cx_{\sigma(n)} \rightarrow y$. From (16) and because C and D are weakly sequentially continuous, we get $Cx + Dy = y$; then, $(I - D)^{-1}Cx = y$; consequently, $(I - D)^{-1}Cx_{\sigma(n)} \rightarrow (I - D)^{-1}Cx$. Now, we show that $(I - D)^{-1}Cx_n \rightarrow (I - D)^{-1}Cx$. Suppose the contrary; then, there exists a subsequence $\{x_{\phi(n)}\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ and a weak neighborhood V of $(I - D)^{-1}Cx$ such that $(I - D)^{-1}Cx_{\phi(n)} \notin V$ for all $n \in \mathbb{N}$. Since $x_{\phi(n)} \rightarrow x$, then arguing as before, we may extract a subsequence $\{x_{\psi(n)}\}_{n \in \mathbb{N}}$ of $\{x_{\phi(n)}\}_{n \in \mathbb{N}}$ such that $(I - D)^{-1}Cx_{\psi(n)} \rightarrow (I - D)^{-1}Cx$, which is a contraction. As a result, $(I - D)^{-1}C$ is weakly sequentially continuous. Because A and B are weakly

sequentially continuous, we get $A + B(I - D)^{-1}C$ is weakly sequentially continuous.

Now, we show that $A + B(I - D)^{-1}C$ is countably condensing. Let M be a countably subset of $\overline{U^w}$ such that $\omega(M) > 0$; by (16), we have

$$\begin{aligned} \omega((I - D)^{-1}C(M)) &\leq \omega(C(M)) + \omega(D(I - D)^{-1}C(M)) \\ &\leq \gamma \omega(M) + \delta \omega((I - D)^{-1}C(M)). \end{aligned} \quad (18)$$

Then,

$$\omega((I - D)^{-1}C(M)) \leq \frac{\gamma}{1 - \delta} \omega(M) \leq \omega(M). \quad (19)$$

Using the subadditivity of the De Blasi measure of weak noncompactness, we obtain

$$\begin{aligned}\omega(F(M)) &\leq \omega(A(M)) + \omega(B(I-D)^{-1}C(M)) \\ &\leq \beta\omega(M) \\ &< \omega(M).\end{aligned}\quad (20)$$

Hence, F is countably condensing.
Then, by Theorem 3, we obtain

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}Cx$ solves the problem, whereas in the second case, we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. $\square$

Remark 5. In Theorem 8, we can replace the De Blasi measure of weak noncompactness ω by any subadditive measures of weak noncompactness on X .

In Theorem 8, condition (i) is difficult to verify; in the following, we will change it under weaker conditions.

Theorem 9. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $(I - D)$ is ψ -expansive and $C(\Omega) \subset (I - D)(X)$
- (ii) $A(\overline{U^w})$ is relatively weakly compact and B is countably β -contractive
- (iii) C is countably γ -contractive and D is countably δ -contractive with $\gamma + \delta < 1$
- (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (21)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. Let $x, y \in X$ such that $x \neq y$; since $(I - D)$ is ψ -expansive, we obtain

$$\|(I - D)x - (I - D)y\| \geq \psi(\|x - y\|) > 0, \quad (22)$$

Thus, $(I - D): X \rightarrow (I - D)(X)$ is bijective, and because $C(\Omega) \subset (I - D)(X)$, we get $(I - D)^{-1}$ is well defined on $C(\Omega)$. Now, reasoning as in the proof of Theorem 8, we get the desired result. $\square$

Theorem 10. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) B is countably β -contractive and $A(\overline{U^w})$ is relatively weakly compact
- (ii) T_y^p is expansive for some $p \in \mathbb{N}$ and each $y \in X$, where $T_y x = Dx + y$ for $x \in X$
- (iii) D is a contraction with constant $0 < \delta < 1$ and C is countably γ -contractive with $\gamma + \delta < 1$
- (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω

Then, either

- (A1) the block operator matrix (1) has a fixed point, or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (23)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. By assumptions (i) and (ii) with Lemma 2.3 in [21], we get $(I - D)^{-1}C$ is well defined. And, by assumption (i) and Lemma 2, we have that D is countably δ -contractive. Now, reasoning as in the proof of Theorem 8, we get the desired result. $\square$

Remark 6. Theorem 10 remains true if we suppose that T_y^p is contractive.

4. Application

Let X be a Banach space. As usual, we will denote by $E = \mathcal{C}(J, X)$ the Banach space of all X -valued continuous functions defined on $J = [0, 1]$. We equip the space E with its standard norm:

$$\|x\|_\infty = \sup_{t \in J} \|x(t)\|. \quad (24)$$

The goal of this section is to apply Corollary 1 to study the existence of continuous solutions to the nonlinear functional integral equations (2).

Let us now introduce the following assumptions.

(H0)

- (i) The function $\sigma: J \rightarrow J$ is continuous and nondecreasing
- (ii) The function $q: J \rightarrow \mathbb{R}$ is continuous
- (iii) The function $a: J \rightarrow X$ is continuous, and $\|a\|_\infty < 1$

(H1) The operator $k: J \times X \rightarrow X$ is such that

- (i) For all $t \in J$, the operator $k(t, \cdot): X \rightarrow X$ is weakly sequentially continuous and weakly compact.
- (ii) For each $x \in X$, the operator $k(\cdot, x): J \rightarrow X$ is continuous.
- (iii) There exists a continuous function $\delta: J \rightarrow [0, +\infty)$ with bound $\Delta = \|\delta\|_\infty$ such that

$$\|k(t, x) - k(t, y)\| \leq \delta(t)\|x - y\|, \quad \text{for all } x, y \in X \text{ and } t \in J. \quad (25)$$

(H2) The operator $g: J \times J \times X \rightarrow \mathbb{R}$ is continuous such that, for each $s \in J$ and $x \in X$, and the operator $g(\cdot, s, t): J \rightarrow \mathbb{R}$ is continuous uniformly.

(H3) The operator $\phi: J \times X \rightarrow \mathbb{R}$ is such that

(i) For each $t \in J$, the operator $\phi(t, \cdot): X \rightarrow \mathbb{R}$ is weakly sequentially continuous.

(ii) For each $x \in X$, the operator $\phi(\cdot, x): J \rightarrow \mathbb{R}$ is continuous.

(iii) There exists a constant L such that

$$|\phi(t, x) - \phi(t, y)| \leq L\|x - y\|, \quad \text{for all } x, y \in X \text{ and } t \in J. \quad (26)$$

(H4) The operators $w: J \times X \rightarrow \mathbb{R}$ is such that

(i) For each $x \in X$, the operator $w(\cdot, x): J \rightarrow \mathbb{R}$ is measurable.

(ii) For each $t \in J$, the operator $w(t, \cdot): X \rightarrow \mathbb{R}$ is weakly sequentially continuous.

(iii) There exists a constant b and a function $m \in L^1([0, 1])$ such that

$$|w(t, x)| \leq m(t) \text{ and } \int_0^1 \frac{1}{t+s} m(s) ds \leq b. \quad (27)$$

(H5) Assume that there exists $r_0 > 1$ such that $|g(\cdot, \cdot, \cdot)| \leq r_0 - \|q\|_\infty$, $\sup_{t \in J} |w(t, 0)| \leq ((1 - \|a\|_\infty)r_0 - L\|m\|_{L^1}\|v\|_\infty)/\|v\|_\infty$, and $\sup_{t \in J} |k(t, 0)| \leq r_0(1 - \Delta - \|u\|_\infty)$.

Let us define the subset Ω of $\mathcal{C}(J, X)$ by

$$\Omega = \{x \in \mathcal{C}(J, X): \|x\| \leq r_0\}. \quad (28)$$

We can see that Ω is a nonempty closed and convex subset of E . Let U be a weakly open subset of Ω such that $0 \in \bar{U}$. Notice that (2) is equivalent to the system:

$$\begin{cases} x(t) = Ax(t) + By(t), \\ y(t) = Cx(t) + Dy(t), \end{cases} \quad (29)$$

where the operators A, B, C , and D defined by

$$Ax(t) = k(t, x(t)), \quad \text{for all } t \in J,$$

$$Bx(t) = \left[q(t) + \int_0^{\sigma(t)} g(t, s, x(s)) ds \right] \cdot u, \quad \text{for all } t \in J,$$

$$Cx(t) = \phi\left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds\right) \cdot v, \quad \text{for all } t \in J,$$

$$Dy(t) = a(t)y(t), \quad \text{for all } t \in J.$$

(30)

Now, we have come to a place where we give the main result of this section.

Theorem 11. Assume that (H_0) , (H_1) , (H_2) , (H_3) , (H_4) , and (H_5) hold. In addition, assume that

$$\lambda Au_1 + \lambda Bu_2 \neq u_1, \quad (31)$$

for all $(u_1, u_2) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$.

Then, system (2) has, at least, one solution in $\mathcal{C}(J, X) \times \mathcal{C}(J, X)$.

Proof. In order to apply Corollary 1, we divided the proof into four steps.

Step 1: in this step, we prove in (a), (b), and (c), respectively, that the operators A , B , and C are well defined and weakly sequentially continuous.

(a) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Ax(t_n)(t_n) - Ax(t)\| &= \|k(t_n, x(t_n)) - k(t, x(t))\| \leq \|k(t_n, x(t_n)) - k(t_n, x(t))\| + \|k(t_n, x(t)) - k(t, x(t))\| \\ &\leq \delta(t_n)\|x(t_n) - x(t)\| + \|k(t_n, x(t)) - k(t, x(t))\| \leq \Delta\|x(t_n) - x(t)\| + \|k(t_n, x(t)) - k(t, x(t))\|. \end{aligned} \quad (32)$$

Since $t_n \rightarrow t$ and taking into account the assumptions (H_1) (ii), we obtain $\|Ax(t_n) - Ax(t)\| \rightarrow 0$. Accordingly, $Ax \in E$. Now, we show that A is weakly sequentially continuous. To

see this, let $\{x_n\}_n$ be a sequence in Ω such that $x_n \rightarrow x \in \Omega$. Hence, $\{x_n\}_n$ is bounded; then, by Lemma 3, we have $x_n(t) \rightarrow x(t)$, for all $t \in J$; consequently,

$$Ax_n(t) = k(t, x_n(t)) \rightarrow k(t, x(t)) = Ax(t). \quad (33)$$

Using again Lemma 3, we get $Ax_n \rightarrow Ax$ because $\{Ax_n\}_n$ is bounded by $\Delta r_0 + \sup_{t \in J} \|k(t, 0)\|$; hence, A is weakly sequentially continuous.

(b) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Bx(t_n) - Bx(t)\| &\leq \left| \int_0^{\sigma(t_n)} g(t_n, s, x(s)) ds - \int_0^{\sigma(t)} g(t, s, x(s)) ds \right| \|u\|_\infty + |q(t_n) - q(t)| \|u\|_\infty \\ &\leq \left[\int_0^{\sigma(t_n)} |g(t_n, s, x(s)) - g(t, s, x(s))| ds \right] \|u\|_\infty + \left[\left| \int_{\sigma(t_n)}^{\sigma(t)} g(t, s, x(s)) ds \right| + |q(t_n) - q(t)| \right] \|u\|_\infty \\ &\leq \left[\int_0^1 |g(t_n, s, x(s)) - g(t, s, x(s))| ds \right] \|u\|_\infty + [r_0 |\sigma(t_n) - \sigma(t)| + |q(t_n) - q(t)|] \|u\|_\infty. \end{aligned} \quad (34)$$

By using assumption (H_2) with the dominated convergence theorem, we obtain

$$\int_0^1 |g(t_n, s, x(s)) - g(t, s, x(s))| ds \rightarrow 0. \quad (35)$$

Hence,

$$\|Bx(t_n) - Bx(t)\| \rightarrow 0, \quad (36)$$

This implies that the function Bx is continuous. Now, let $\{x_n\}_n$ be a sequence in E such that $x_n \rightarrow x \in E$. For all $t \in J$, we have $Bx_n(t) = r_n(t) \cdot u$, where

$$r_n(t) = q(t) + \int_0^{\sigma(t)} g(t, s, x_n(s)) ds. \quad (37)$$

Because $\{r_n(t), n \in \mathbb{N}\}$ is a real bounded sequence, we deduce that there is a renamed subsequence such that $r_n(t) \rightarrow r(t)$. Since $\{Bx_n, n \in \mathbb{N}\}$ is bounded, we obtain $Bx_n \rightarrow Bx$; hence, B is weakly sequentially continuous.

(c) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Cx(t_n) - Cx(t)\| &\leq \left| \phi\left(t_n, \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq \left| \phi\left(t_n, \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds\right) - \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq L \left| \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds - \int_0^t \frac{t}{t + s} w(s, x(s)) ds \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq L \left| \int_0^{t_n} \frac{s(t_n - t)}{(t_n + s)(t + s)} w(s, x(s)) ds - \int_{t_n}^t \frac{t}{t + s} w(s, x(s)) ds \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq 2Lb|t_n - t| \|v\|_\infty + \|\phi(t_n, z) - \phi(t, z)\| \|v\|_\infty, \end{aligned} \quad (38)$$

where

$$z(t) = \int_0^t \frac{t}{t+s} w(s, x(s)) ds, \quad \text{for all } t \in J. \quad (39)$$

If we refer to assumption $(H_3)(ii)$, we deduce Cx is continuous.

Now, let $\{x_n\}_n$ be a sequence in E such that $x_n \rightarrow x \in E$; using assumption $(H_4)(ii)$, we have

$$\frac{t}{t+s} w(s, x_n(s)) \rightarrow \frac{t}{t+s} w(s, x(s)), \quad \text{in } \mathbb{R}. \quad (40)$$

Using the dominated convergence theorem, we obtain

$$(Cx_n)(t) \rightarrow (Cx)(t), \quad \text{in } X. \quad (41)$$

Since $\{Cx_n, n \in \mathbb{N}\}$ is bounded, then we can apply Lemma 3, and we get $Cx_n \rightarrow Cx$; then, C is weakly sequentially continuous.

Step 2: in this step, in (a) and (b), respectively, we prove that $C(\overline{U^w})$ and $A(\overline{U^w})$ are relatively weakly compact.

(a) Let $\{x_n\}_n$ be any sequence in Ω . By $(H_3)(iii)$, we have

$$\begin{aligned} \|Cx_n(t)\| &= \left\| \left[\phi\left(t, \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds\right) - \phi(t, 0) \right] + |\phi(t, 0)| \right\| \|v\|_\infty \\ &\leq \left[L \left| \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds \right| + \sup_{t \in J} |w(t, 0)| \right] \|v\|_\infty \\ &\leq \left[L \|m\|_{L^1} + \sup_{t \in J} |w(t, 0)| \right] \|v\|_\infty. \end{aligned} \quad (42)$$

For all $t \in [0, 1]$, this proves that $\{Cx_n, n \in \mathbb{N}\}$ is a uniformly bounded sequence in $C(\Omega)$. As a result, $C(\Omega)(t)$ is sequentially relatively weakly compact. Now, we proceed to show that it is also weakly equicontinuous. If we take $\varepsilon > 0$, $x \in \Omega$, and $t, t' \in [0, 1]$ (without loss of generality assume that $t < t'$), then we have

$$|x^*((Cx_n)(t) - (Cx_n)(t'))| \leq [L|M(\varepsilon)| + |\Phi(\varepsilon)|] \|x^*(v)\|, \quad (43)$$

where

$$\begin{aligned} M(\varepsilon) &= \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds - \int_0^{t'} \frac{t'}{t'+s} w(s, x_n(s)) ds, \quad |t - t'| < \varepsilon, \text{ and,} \\ \Phi(\varepsilon) &= \phi\left(t, \int_0^{t'} \frac{t'}{t'+s} w(s, x(s)) ds\right) - \phi\left(t', \int_0^{t'} \frac{t'}{t'+s} w(s, x(s)) ds\right), \quad |t - t'| < \varepsilon. \end{aligned} \quad (44)$$

Using assumption (H_4) , we have

$$\begin{aligned} |M(\varepsilon)| &\leq |t - t'| \int_0^1 \frac{1}{t+s} w(s, x_n(s)) ds + \int_{t'}^t \frac{t'}{t'+s} w(s, x_n(s)) ds, \\ &\leq a|t - t'| + \int_{t'}^t m(s) ds. \end{aligned} \quad (45)$$

Since $M(\varepsilon) \rightarrow 0$ and $\Phi(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$, we obtain

$$|x^*((Cx_n)(t) - (Cx_n)(t'))| \rightarrow 0, \quad \text{as } t \rightarrow t'. \quad (46)$$

Applying the Arzelà-Ascoli's theorem [22], we get $C(\Omega)$ is sequentially relatively weakly compact in X , and an application of Eberlein-Smulian's theorem [23] shows that $C(\Omega)$ is relatively weakly compact.

(b) We have $A(\Omega) = \{Ax: \|x\| \leq r_0\}$; this subset is nothing else than

$$k(\cdot, \Omega) = \{k(\cdot, x): \|x\| \leq r_0\}, \quad (47)$$

Hence, by hypothesis (H_1) (i), we can see that $A(\Omega)$ is relatively weakly compact.

Step 3: in this step in (a), we show that $C(\overline{U^w}) \subset (I - D)(\Omega)$, and in (b), we prove that $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$.

(a) Let $x \in \overline{U^w}$, we defined the mapping $\phi_x: \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ by $\phi_x(y) = Cx + Dy$. Because D is a contraction, we can see that ϕ_x is a contraction; then, an application of Banach's fixed point theorem yields there is a unique point $y \in \mathcal{C}(J, X)$ such that $y = Cx + Dy$, and this implies that $C(\overline{U^w}) \subset (I - D)(\mathcal{C}(J, X))$. Since $y \in \mathcal{C}(J, X)$, then there is $t^* \in J$ such that

$$\begin{aligned} \|y\|_\infty &= \|y(t^*)\| \\ &= Cx(t^*) + Dy(t^*) \\ &\leq \left| \phi \left(t^*, \int_0^{t^*} \frac{t^*}{t^* + s} w(s, x_n(s)) ds \right) \right| \|v\|_\infty + \|a\|_\infty \|y(t^*)\|. \end{aligned} \quad (48)$$

This implies that

$$\|y\|_\infty \leq \frac{[L\|m\|_{L^1} + \sup_{t \in J} |w(t, 0)|] \|v\|_\infty}{1 - \|a\|_\infty} \leq r_0. \quad (49)$$

Then, $C(\overline{U^w}) \subset (I - D)(\Omega)$.

(b) Let $y \in \mathcal{C}(J, X)$ and $x \in \overline{U^w}$ such that

$$y(t) = Ax(t) + B(I - D)^{-1}Cx(t). \quad (50)$$

or, equivalently, for all $t \in J$,

$$y(t) = Ax(t) + B(I - D)^{-1}Cx(t). \quad (51)$$

We have

$$\begin{aligned} \|y(t)\| &= \|Ax(t) + B(I - D)^{-1}Cx(t)\| \\ &\leq \|Ax(t)\| + \|B(I - D)^{-1}Cx(t)\| \\ &\leq \Delta \|x(t)\| + \sup_{t \in J} |k(t, 0)| + r_0 \|u\|_\infty \\ &\leq \Delta r_0 + \sup_{t \in J} |k(t, 0)| + r_0 \|u\|_\infty \\ &\leq r_0. \end{aligned} \quad (52)$$

This implies that

$$Ax + B(I - D)^{-1}Cx \in \Omega, \quad \text{for all } x \in \overline{U^w}. \quad (53)$$

Thus, all the hypotheses of Corollary 1 are satisfied, and therefore, system (2) has, at least, one solution in $\mathcal{C}(J, X) \times \mathcal{C}(J, X)$. $\square$

5. Example

Consider the Banach space $E = \mathcal{C}([0, 1], \mathbb{R})$ of all continuous real-valued functions on $J = [0, 1]$, with norm $\|x\|_\infty = \sup_{t \in [0, 1]} |x(t)|$. In this case, $X = \mathbb{R}$, and E is a reflexive Banach space. We consider the following coupled nonlinear integral equation in E :

$$\begin{cases} x(t) = \frac{1}{4}t^3 \cos\left(\frac{x(t)}{2}\right) + \frac{1}{8} \left[\frac{1}{4} + \int_0^t \frac{|y(s)|}{1 + |y(s)|} ds \right], \\ y(t) = t \int_0^t \frac{ts|x(s)|}{(t+s)e^{|x(s)|}} ds + \frac{1}{3} \sin(t)y(t). \end{cases} \quad (54)$$

To show that (54) has a solution in E , we will verify that all conditions of Theorem 11 are satisfied.

Here, for all $t, s \in J$ and $x, y \in E$, we have

$$\begin{aligned} \sigma(t) &= t, \\ q(t) &= \frac{1}{4}, \\ a(t) &= \frac{1}{3} \sin(t), \\ u &= \frac{1}{8}, \\ v &= 1, \end{aligned} \quad (55)$$

$$\phi(t, x(t)) = tx(t),$$

$$k(t, x(t)) = \frac{1}{4}t^3 \cos\left(\frac{x(t)}{2}\right),$$

$$g(t, s, y(s)) = \frac{|y(s)|}{1 + |y(s)|},$$

$$w(s, x(s)) = \frac{s(|x(s)| + 1)}{2e^{|x(s)| + 1}}.$$

For each $t \in J$, the operator $k(t, \cdot): X \longrightarrow X$ is continuous (then weakly sequentially continuous), and for each $x \in X$, the operator $k(\cdot, x): J \longrightarrow X$ is continuous. Now, let $x, y \in E$ and $t \in J$, and we have

$$|k(t, x(t)) - k(t, y(t))| \leq \frac{1}{8}t^3 |x(t) - y(t)|, \quad (56)$$

where the function $\delta: t \longrightarrow 1/8t^3$ is continuous with bound $\Delta = 1/8$.

Next, we have g is continuous, and for each $s \in J$ and $x \in \mathbb{R}$, the operator $g(\cdot, s, t): J \longrightarrow \mathbb{R}$ is continuous uniformly.

Moreover, for all $x, y \in \mathbb{R}$ and $t \in J$, we have

$$|\phi(t, x(t)) - \phi(t, y(t))| \leq |x(t) - y(t)|. \quad (57)$$

Thus, $L = 1$.

Next, we have

$$|w(t, x(t))| \leq \frac{t}{2}, \quad (58)$$

$$\text{and } \int_0^1 \frac{1}{t+s} m(s) ds \leq 1.$$

where $m(t) = t$, for all $t \in J$.

Choose $b = 1$ and $r_0 = 2$, we obtain $r_0 - \|q\|_\infty = 7/4$, $((1 - \|a\|_\infty)r_0 - L\|m\|_{L^1}\|v\|_\infty)/\|v\|_\infty = 5/6$, and $r_0(1 - \Delta - \|u\|_\infty) = 3/2$. Then, the inequalities $|g(\cdot, \cdot, \cdot)| \leq 7/4$, $\sup_{t \in J} |w(t, 0)| \leq 5/6$, and $\sup_{t \in J} |k(t, 0)| \leq 3/2$ are verified.

Now, let $U = \{x \in \mathcal{C}([0, 1], \mathbb{R}) : \|x\| < 2\}$, we have $\partial U = \{x \in \mathcal{C}([0, 1], \mathbb{R}) : \|x\| = 2\}$. Suppose that there exists $u_1 \times u_2 \in \partial U \times X$ and $\lambda \in (0, 1)$ such that $\lambda Au_1 + \lambda Bu_2 = u_1$. We have $\|\lambda Au_1 + \lambda Bu_2\| < 13/32$. And, $\|u_1\| = 2$ which is a contraction. Hence, for all $u_1 \times u_2 \in \partial U \times X$ and $\lambda \in (0, 1)$, we have $\lambda Au_1 + \lambda Bu_2 \neq u_1$. Hence, all conditions of Theorem 11 are verified.

6. Conclusion

In recent years, some works were devoted to the investigation of fixed point theorems for operator matrices with entries acting on Banach spaces and Banach algebras. The aim of the present paper was to establish some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix. The second aim of this study was to use our results to prove the existence of solutions for a coupled system of nonlinear equations. An example to illustrate our theory is included.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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