Equilibrium States with Finite Amplitudes at Exactly and Nearly Class-I Bragg Resonances

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The exactly and nearly class-I Bragg resonances of strongly nonlinear waves are studied analytically by the homotopy analysis method. Two types of equilibrium states with time-independent wave spectra and different energy distributions are obtained. Effects of the incident wave height, the seabed height, and the frequency detuning on resonant waves are investigated. Bifurcation points of the equilibrium states are found and tend to greater value of relatively incident wave height for a steeper wave. The wave steepness of the whole wave system grows linearly with the seabed height. Meanwhile, the resonant peak can shift to up or down side when the near resonance is considered. This work provides us a deeper understanding on class-I Bragg resonance and enlightens further studies of higher-order wave-bottom interactions.

1. Introduction

When an incident wave propagates over a monochromatically undulated and fixed seabed, a new resonant wave will be generated if the following resonance conditions are fulfilled:

\[ k_2 = k_1 \pm k_b, \]  
\[ \omega_2 = \omega_1 + \delta \omega, \]

where \( k_1 \) and \( k_2 \) represent the vector wavenumbers of the incident and reflected (i.e., resonant) waves, respectively, \( k_b \) is the vector wavenumber of the undulated seabed, and \( \omega_1, \omega_2 \) is the angular frequency related with vector wavenumber \( k_i \) by the linear dispersion relation:

\[ \mathcal{D}(\omega; k_i) \equiv \omega - \sqrt{\rho g d \tanh(k_i d)} = 0, \]

where \( d \) denotes the mean water depth and \( \delta \omega \) is the frequency detuning measuring how far is the near resonance (\( \delta \omega \neq 0 \)) away from the exact resonance (\( \delta \omega = 0 \)). This resonance is called class-I Bragg resonance [1].

The class-I Bragg resonance was first studied by a flume experiment [2]. The experiment verified the condition of class-I Bragg resonance proposed by Davies [3]. Multiscale perturbation method was applied to derive the analytical solution of Bragg resonance and obtained the formula of reflected and transmission coefficient at any point in the far area [4].

When an evolutionary wave system was considered, namely, that wave amplitudes are assumed to be functions of time and space; periodically, energy exchanges were found between incident and reflected waves through the envelope equation for weakly nonlinear waves and small seabed undulations [4]. Mei first proposed the application of artificial sand bars to coastal protection [5]. Then, the research on Bragg resonance turned to applications. Based on the modified mild slope equation (MMSE), the Frobenius series solution and Taylor series solution were constructed, which were used to derive the Bragg resonance of the surface gravity wave on the bottom of the single-period sine and the Bragg resonance of the surface wave on the finite trapezoidal block array [6, 7]. The class-I Bragg resonance mechanism was applied to the shore protection [8]. When the incident wave obliquely propagated over a patch of corrugations which had a plane of symmetry in the incident direction, waves were deflected to the downstream and left a protected wake near shore.
In addition, equilibrium states of waves at class-I Bragg resonance with constant spectra also exist [9]. The wave amplitudes and angular frequencies in such wave systems are time independent. And, bifurcations in energy distributions of such waves at class-I Bragg resonance were found using the homotopy analysis method (HAM). It should be emphasized that these equilibrium states are only for the exact resonance \((\delta \omega = 0)\). Meanwhile, these waves have relatively small amplitudes. The correspondingly angular frequencies of the incident and resonant waves are also quite small. However, nonlinearity is an essential characteristic of waves, and the wave-bottom interaction of nonlinear waves is a focus of research. Previously, the studies of Bragg resonance mainly investigate linearized waves or weakly nonlinear waves (Mei et al. [4, 5], Kirby [10], and Liu and Yue [1]). Research studies on strongly nonlinear waves are a focus of research. Previously, the studies of Bragg resonance are investigated in Section 3. Concluding remarks and discussions come in Section 4.

2. Problem Formulation

The fluid is assumed to be inviscid and the flow irrotational. Thus, the wave field can be described by the velocity potential \(d\), where \(t\) is the time, \((x, y, z)\) are the Cartesian coordinates, and \(z\)-axis points upwards from the mean water level \((z = 0)\). The surface gravity wave with its elevation of \(\zeta(x, y, t)\) propagates over seabed bars at \(z = -d + h(x, y)\) with \(d\) as the mean water depth. The seabed corrugation \(h(x, y) = b \cos (k_y, r)\) is monochromatic and has finite slope \(bk_y\), where \(k_y = |k_y|\) and \(r = (x, y)\).

The velocity potential \(\phi(x, y, z, t)\) satisfies Laplace equation,

\[
\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \tag{4}
\]

in the domain of \(-d + h(x, y) < z < \zeta(x, y, t)\) and subjects to the boundary conditions,

\[
\zeta = \varphi_x h_x + \varphi_y h_y, \tag{5}
\]

\[
\varphi_z = \varphi_x h_x + \varphi_y h_y, \tag{7}
\]

on the unknown free surface \(z = \zeta(x, y, t)\) and

on the corrugated bottom \(z = -d + h(x, y)\) with no flux, where \(g\) is the gravitational acceleration, \(\nabla = \partial_x \mathbf{i} + \partial_y \mathbf{j} + \partial_z \mathbf{k}\) with \(i, j, k\) as the unit vectors in \((x, y, z)\) directions, respectively. Note that subjecting boundary condition (5) into (6) gives an alternative set of boundary conditions on the free surface as follows:

\[
\varphi_{tt} + g\varphi_z + (\nabla \phi \cdot \nabla \phi)_t + \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi \cdot \nabla \phi) = 0, \tag{8}
\]

\[
\zeta = -\frac{1}{g} \left( \varphi_t + \frac{1}{2} \nabla \phi \cdot \nabla \phi \right). \tag{9}
\]

Thus, the boundary condition (8) is decoupled and only depends on the potential function \(\varphi(x, y, z, t)\). After introducing the dimensionless variables,

\[
\begin{align*}
(X, Y, Z, D) &= (x, y, z, d)k_1, \\
(K_1, K_b) &= (k_1, k_b)k_1^{-1}, \\
T &= t\omega_1, \\
(\Omega_1, \Delta \omega) &= (\alpha_1, \delta \omega)\omega_1^{-1}, \\
\zeta(X, Y, T) &= \zeta(x, y, t)k_1^{-1}, \\
\bar{\varphi}(X, Y, Z, T) &= \alpha_1^{-1} \Omega_1^{-1} k_1 \phi(x, y, z, t), \\
\bar{h}(X, Y) &= h(x, y)b_1^{-1},
\end{align*}
\]

and two phase functions

\[
\begin{align*}
\xi_1 &= k_1 \cdot r - \alpha_1 t = X \cos \alpha_1 + Y \sin \alpha_1 - \Omega_1 T, \\
\xi_2 &= k_b \cdot r = K_b X \cos \alpha_b + K_b Y \sin \alpha_b,
\end{align*}
\]

the governing equation (4) and boundary conditions (7)–(9) can be nondimensionalized and rewritten in the new variables of \(\xi_1\), \(\xi_2\) as follows:

\[
\hat{\nabla}^2 \hat{\phi} = 0, \quad -\varepsilon_1 + \varepsilon_3 H < Z < \varepsilon_2 \eta, \tag{12}
\]

\[
\Omega_1^2 \phi_{\xi_1} + \frac{\phi_Z}{\tanh(\varepsilon_1)} = 2\varepsilon_2 \Omega_1 F_{\xi_1} + \varepsilon_2 \hat{\nabla} \phi \cdot \hat{\nabla} F = 0, \tag{13}
\]

\[
\eta = \Omega_1 \tan(\varepsilon_1) \phi_{\xi_1} - \varepsilon_2 \varepsilon_1 \tan(\varepsilon_1) F, \quad Z = \varepsilon_2 \eta, \tag{14}
\]

\[
\phi_Z = \hat{\nabla} \phi \cdot \hat{\nabla} (-\varepsilon_1 + \varepsilon_5 H), \quad Z = -\varepsilon_1 + \varepsilon_5 H, \tag{15}
\]

where
\[\begin{align*}
\epsilon_1 &= k_1 d, \\
\epsilon_2 &= a_1 k_1, \\
\epsilon_3 &= b k_1, \\
\phi (\xi_1, \xi_2, Z) &= \tilde{\phi} (X, Y, Z, T), \\
\eta (\xi_1, \xi_2) &= \tilde{\eta} (X, Y, T), \\
H (\xi_2) &= \tilde{h} (X, Y), \\
\nabla &= (\cos \alpha_1 \partial_{\xi_1} + K_b \cos \alpha_2 \partial_{\xi_2}) i + (\sin \alpha_1 \partial_{\xi_1} + K_b \sin \alpha_2 \partial_{\xi_2}) j + \partial_Z k, \\
F (\xi_1, \xi_2, Z) &= \frac{(\phi_1^2 + 2K_b \cos (a_1 - a_2) \phi_1 \phi_2 + K_b^2 \phi_1^2 + \phi_2^2)}{2},
\end{align*}\]

where \(\alpha_1\) and \(\alpha_2\) are the angles between \(k_1\) and \(k_b\) and the \(x\)-axis, respectively, \(\sigma_1\) is the dimensional angular frequency in the nonlinear theory, and \(a_1\) is the dimensional amplitude of the incident wave.

### 2.1. HAM-Based Approach

The amplitude-based HAM approach is first used in the steady-state resonant waves where the wave amplitude is given and the frequency is to-be-determined \([11]\). Here, it is used to solve the class-I Bragg resonant waves with finite amplitudes. The first step is to establish the zeroth-order deformation equations. They are constructed by introducing the embedding parameter \(q \in [0, 1]\), which does not have any physical meaning at all, into the boundary conditions \((13)-(14)\),

\[\begin{align*}
(1 - q) &\mathcal{L}_1 (q) \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right] = q \mathcal{N}_1 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right] \Omega_1 (q), \\
(1 - q) &\mathcal{L}_2 (q) \left[ \hat{\eta} (\xi_1, \xi_2; q) \right] = q \mathcal{N}_2 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right] \Omega_1 (q),
\end{align*}\]

on \(Z = \epsilon_2 \hat{\eta} (\xi_1, \xi_2; q)\) and the boundary condition \((15)\),

\[\begin{align*}
(1 - q) &\mathcal{L}_1 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) - \phi_0 (\xi_1, \xi_2, Z) \right] = q \mathcal{N}_1 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right], \\
(1 - q) &\mathcal{L}_2 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) - \phi_0 (\xi_1, \xi_2, Z) \right] = q \mathcal{N}_2 \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right],
\end{align*}\]

on \(Z = -\epsilon_1 + q \epsilon_2 H (\xi_2)\), where \(c_0 \neq 0\) is the convergence-control parameter and

\[\begin{align*}
\mathcal{L}_1 (q) &= \mu_n \Omega_1^2 (q) \frac{\partial}{\partial \xi_1} + \frac{1}{\tanh (\epsilon_1)} \frac{\partial}{\partial Z}, \\
\mathcal{L}_2 &= \frac{\partial}{\partial Z},
\end{align*}\]

which suggests that, as long as the convergence-control parameter \(c_0\) is properly chosen, the continuous variations of \(\hat{\phi} (\xi_1, \xi_2, Z; q)\), \(\hat{\eta} (\xi_1, \xi_2; q)\) and \(\hat{\Omega}_1 (q)\) occur from the initial guesses \(\hat{\phi} (\xi_1, \xi_2, Z; 0) = \phi_0 (\xi_1, \xi_2, Z)\), \(\hat{\eta} (\xi_1, \xi_2; 0) = \eta_0 (\xi_1, \xi_2) = 0\) and \(\hat{\Omega}_1 (0) = \bar{\omega}_{1,0}\) to their exact solutions

\[\begin{align*}
\mathcal{N}_1 \left[ \hat{\phi}, \Omega_1 \right] &= \Omega_1^2 \phi_1 + \frac{\partial Z}{\tanh (\epsilon_1)} - 2\epsilon_1 \Omega_1 \tilde{F}_1 + \epsilon_3 \nabla \phi \cdot \nabla \tilde{F}, \\
\mathcal{N}_2 \left[ \hat{\phi}, \eta, \Omega_1 \right] &= \eta_1 - \Omega_1 \tanh \epsilon_1 \phi_1 + \epsilon_2 \tanh \epsilon_1 \tilde{F}, \\
\tilde{F} (\xi_1, \xi_2, Z) &= \left( \frac{\phi_1^2 + 2K_b \cos (a_1 - a_2) \phi_1 \phi_2 + K_b^2 \phi_1^2 + \phi_2^2}{2} \right), \\
\mathcal{N}_3 [\hat{\phi}] &= \hat{\phi}_Z - \nabla \phi \cdot \nabla (-\epsilon_1 + \epsilon_3 H).
\end{align*}\]

Here, the homotopy-Maclaurin series is defined as

\[\hat{\phi} (\xi_1, \xi_2, Z; q) = \phi_0 (\xi_1, \xi_2, Z) + \sum_{n=1}^{\infty} \phi_n (\xi_1, \xi_2, Z) q^n,\]

\[\hat{\eta} (\xi_1, \xi_2; q) = \eta_0 (\xi_1, \xi_2) + \sum_{n=1}^{\infty} \eta_n (\xi_1, \xi_2) q^n,\]

\[\hat{\Omega}_1 (q) = \omega_{1,0} + \sum_{n=1}^{\infty} \omega_{1,n} q^n,\]

with the definitions of \(n\)th-order homotopy-derivative,

\[\phi_n (\xi_1, \xi_2, Z) = \mathcal{D}_n \left[ \hat{\phi} (\xi_1, \xi_2, Z; q) \right],\]

\[\eta_n (\xi_1, \xi_2) = \mathcal{D}_n [\hat{\eta} (\xi_1, \xi_2; q)],\]

\[\omega_{1,n} = \mathcal{D}_n [\hat{\Omega}_1 (q)],\]

and the homotopy-derivative operator,

\[\mathcal{D}_n [f (q)] = \frac{1}{n!} \left. \frac{d^n f (q)}{dq^n} \right|_{q=0}.\]

Thus, when \(q\) varies continuously from 0 to 1, the above zeroth-order deformation equations lead to the continuous variation from equations

\[\begin{align*}
\mathcal{L}_1 (0) \left[ \hat{\phi} (\xi_1, \xi_2, Z; 0) \right] &= 0, \\
\mathcal{L}_2 (0) \left[ \hat{\eta} (\xi_1, \xi_2; 0) - \eta_0 (\xi_1, \xi_2) \right] &= 0, \\
\mathcal{L}_2 \left[ \hat{\phi} (\xi_1, \xi_2, Z; 0) - \phi_0 (\xi_1, \xi_2, Z) \right] &= 0,
\end{align*}\]

to

\[\begin{align*}
\mathcal{N}_1 \left[ \hat{\phi} (\xi_1, \xi_2, Z; 1), \Omega_1 (1) \right] &= 0, \\
\mathcal{N}_2 \left[ \hat{\phi} (\xi_1, \xi_2, Z; 1), \hat{\eta}_1, \Omega_1 (1) \right] &= 0, \\
\mathcal{N}_3 \left[ \hat{\phi} (\xi_1, \xi_2, Z; 1) \right] &= 0,
\end{align*}\]
\( \hat{\phi}(\xi_1, \xi_2, Z; 1) = \phi(\xi_1, \xi_2, Z), \quad \hat{\eta}(\xi_1, \xi_2; 1) = \eta(\xi_1, \xi_2), \) and \( \Omega_1(1) = \Omega_1 \), whose \( M \)th-order approximations are

\[ \phi(\xi_1, \xi_2, Z) = \phi_0(\xi_1, \xi_2, Z) + \sum_{n=1}^{M} \phi_\nu(\xi_1, \xi_2, Z), \quad (35) \]

\[ \eta(\xi_1, \xi_2) = \sum_{n=1}^{M} \eta_\nu(\xi_1, \xi_2), \quad (36) \]

\[ \Omega_1 = \hat{\omega}_{1,0} + \sum_{n=1}^{M} \hat{\omega}_{1,n}. \quad (37) \]

The above \( n \)th-order homotopy derivatives can be obtained through the \( n \)th-order deformation equations,

\[ \mathcal{F}_1[\phi_n] = -U_n - T_n + \chi_n s_{n-1} + c_0 \chi_n \Delta_n^\phi, \quad (38) \]

\[ \eta_n = \eta_{n-1} + c_0 \Delta_n^\eta, \quad (39) \]
on \( Z = 0 \) and

\[ \mathcal{F}_2[\phi_n] = \chi_{n-1} V_{n-1} + c_0 \Delta_n^b - \nabla_n, \quad (40) \]
on \( Z = -\epsilon_1 \), which are determined by taking the \( n \)th-order homotopy-derivative on both sides of the zeroth-order deformation equations (17)–(20) or by substituting the Maclaurin series (26)–(28) into the zeroth-order deformation equations (17)–(20) and setting the coefficient of \( q^n \) to be zero. The derivations of equations (38)–(40) are shown in detail in Appendix.

2.1.1. Solution Expressions. The elevation and potential function of surface gravity waves governed by equations (12)–(15) can be expressed by

\[ \eta(\xi_1, \xi_2) = \sum_{n=1}^{M} \eta_n(\xi_1, \xi_2) = \sum_{n=1}^{\infty} \sum_{n=0}^{\infty} A_{1(n)} \cos(\xi_1 + n \xi_2) \]

\[ = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} A_{1(n)} \cos(\xi_1 + n \xi_2), \quad (41) \]

\[ \phi(\xi_1, \xi_2, Z) = \sum_{n=1}^{M} \phi_n(\xi_1, \xi_2, Z) \]

\[ = \sum_{n=1}^{\infty} \sum_{n=0}^{\infty} \left[ B_{1(n)}^\phi P_{u,v}(\xi_1, \xi_2, Z) + B_{2(n)}^\phi Q_{u,v}(\xi_1, \xi_2, Z) \right], \quad (42) \]

where

\[ P_{u,v} = \sin(\xi_1 + v \xi_2) \frac{\cosh[K_{u,v}(Z + \xi_1)]}{\cosh[K_{u,v} \xi_1]}, \]

\[ Q_{u,v} = \sin(\xi_1 + v \xi_2) \frac{\sinh(K_{u,v}Z)}{K_{u,v} \cosh[K_{u,v} \xi_1]}, \]

\[ K_{u,v} = \sqrt{u^2 + K_1^2 v^2 + 2uvK_0 \cos(\alpha - \delta)}, \quad (43) \]

\[ A_{1(n)}^{u,v} = \sum_{n=0}^{\infty} A_{1(n)}^{u,v}, \]

\[ B_{1(n)}^{u,v} = \sum_{n=0}^{\infty} B_{1(n)}^{u,v}, \]

\[ B_{2(n)}^{u,v} = \sum_{n=0}^{\infty} B_{2(n)}^{u,v}. \]

Note that \( A_{1(n)}^{u,v}, B_{1(n)}^{u,v}, \) and \( B_{2(n)}^{u,v} \) are all constants for waves with time-independent spectra. Except the amplitude of the incident wave \( A_{1(0)}^{1,0} = 1 \) (i.e., \( A_{1(0)}^{1,0} = \hat{a}_1 \)), \( A_{1(n)}^{u,v}, B_{1(n)}^{u,v}, \) and \( B_{2(n)}^{u,v} \) are to be determined. We remark that the form of solution expressions (41)–(42) satisfies the governing equation (12) automatically. Thus, only boundary conditions (13)–(15) need to be solved to get the unknown variables \( \Omega_1, \eta(\xi_1, \xi_2), \) and \( \phi(\xi_1, \xi_2, Z). \)

2.1.2. Auxiliary Linear Operator. The auxiliary linear operator,

\[ \mathcal{F}_1 = \left[ \mu_{u,v} \Omega_{1,0}^2 \frac{\partial^2}{\partial \xi_1^2} + \frac{1}{\tanh(\xi_1)} \frac{\partial}{\partial Z} \right] \bigg|_{Z=0}. \quad (44) \]

has the following property:

\[ \mathcal{F}_1[P_{u,v}] = \lambda_{u,v} \sin(\xi_1 + v \xi_2), \quad (45) \]

where

\[ \lambda_{u,v} = -u^2 \mu_{u,v} + \frac{K_{u,v} \tanh(K_{u,v} \xi_1)}{\tanh(\xi_1)}. \quad (46) \]

If we define

\[ \mu_{u,v} = \begin{cases} \frac{K_{u,v} \tanh(K_{u,v} \xi_1)}{u^2 \tanh(\xi_1)}, & u = u^*, v = v^*, \\ 1, & \text{others,} \end{cases} \quad (47) \]

where \( u^* \) and \( v^* \) satisfy the resonant conditions,

\[ u^* k_1 - v^* k_0 = k_3, \]

\[ u^* \omega_1 = \omega_2 + \delta \omega_{u,v,v}, \quad (48) \]

then we have
\[\lambda_{uv} = \begin{cases} 0, & u = u^*, v = v^*, \\ -u^2 + K_{uv} \tanh(K_{uv} \xi_1), & \text{others.} \end{cases} \] (49)

Thus,
\[\tilde{\mathcal{F}}_1[|P_{1,0}(\xi_1, \xi_2, Z)|] = 0, \quad \tilde{\mathcal{F}}_1[|P_{u,v}(\xi_1, \xi_2, Z)|] = 0. \] (50)

Meanwhile, the linear operator \(\tilde{\mathcal{F}}_2\) has the following property:
\[\tilde{\mathcal{F}}_2[P_{u,v}(\xi_1, \xi_2, Z)] = 0, \quad \tilde{\mathcal{F}}_2[Q_{u,v}(\xi_1, \xi_2, Z)] = \sin(u \xi_1 + v \xi_2). \] (51)

2.1.3. The nth-Order Homotopy Variables. The initial guesses of the potential function and nonlinear frequency are
\[
\phi_0(\xi_1, \xi_2, Z) = B_{1,0}^{1,0} P_{1,0}(\xi_1, \xi_2, Z) + B_{1,0}^{u,v} P_{u,v}(\xi_1, \xi_2, Z),
\] (52)
\[
\Omega_{1,0} = 1.
\] (53)

When \(n = 1\), substituting (52) into the right-hand side of (40), we have
\[
\mathcal{L}_2[\phi_1(\xi_1, \xi_2, Z)] = \sum_u \sum_v B_{2,1}^{u,v} \sin(u \xi_1 + v \xi_2),
\] (54)
where \(B_{2,1}^{u,v}\) is a function of \(B_{1,0}^{1,0}\) and \(B_{1,0}^{u,v}\). Since the linear operator has the property of (50), the solution of (54) can be written as
\[
\phi_1(\xi_1, \xi_2, Z) = B_{1,1}^{u,v} P_{u,v}(\xi_1, \xi_2, Z) + B_{2,1}^{u,v} Q_{u,v}(\xi_1, \xi_2, Z),
\] (55)
\[
\phi_1(\xi_1, \xi_2, Z) = \phi_1(\xi_1, \xi_2; B_{1,0}^{1,0}, B_{1,0}^{u,v}).
\] (56)

Then, after substituting (55) into the left-hand side of (38) and (2) and (56) into the right-hand side of (38), we have
\[
\tilde{\mathcal{F}}_1[\phi_1(\xi_1, \xi_2, Z; B_{1,1}^{u,v})] = \mathcal{A}(\xi_1, \xi_2; B_{1,1}^{1,0}, B_{1,0}^{u,v}, \Omega_{1,1}),
\] (57)
where
\[
C^{u,v} = \text{a function of } B_{1,0}^{1,0}, B_{1,0}^{u,v}, \text{ and } \Omega_{1,1}. \]

Balancing the coefficient of term \(\sin(u \xi_1 + v \xi_2)\) on both sides of (57) gives a series of equations concerning \(B_{1,1}^{u,v}\) (except \(B_{1,0}^{1,0}\) and \(B_{1,0}^{u,v}\)), \(B_{1,0}^{1,0}\), \(B_{1,0}^{u,v}\), and \(\Omega_{1,1}\). An additional algebraic equation can be established from the boundary condition of \(A_{1,0}^{1,0} = 1\). Thus, the problem is closed. Unknowns can be successfully determined. Till now, \(\eta_1(\xi_1, \xi_2), \eta_2(\xi_1, \xi_2, Z), \text{ and } \Omega_{1,1}\) are obtained. And, those coefficients in (55) are all known except \(B_{1,1}^{1,0}\) and \(B_{1,0}^{u,v}\), which will be determined when \(n = 2\).

When \(n \geq 2\), the same solving procedure is used order by order. And, finally, the Mth-order HAM approximations of (35)–(37) can be obtained.

It should be mentioned that the convergent-control parameter \(c_0\) is determined when residual squares of equations (13)–(15) decrease. The optimal value of \(c_0\) corresponds to the fastest dropping of the residual squares. In this paper, the residual squares for the Mth-order HAM approximations are at least in order of \(10^{-10}\).

3. Results

Here, we study the equilibrium states of waves at class-I Bragg resonance. From the point of view of energy distribution, waves can be classified into two types. For Type 1, the incident and resonant wave components have the same wave energy, and they together share nearly the whole wave energy. In other words, each of them has about half of the total wave energy. However, in Type 2, the incident and resonant components contain different wave energy. Effects of physical parameters on these two types of waves are studied in detail.

3.1. Oblique Incidence. We focus on the oblique incidence, namely, that the angle \(\alpha = \alpha_1 - \alpha_0\) between vector wave-numbers \(k_1\) and \(k_0\) is nonzero, i.e., \(\alpha \neq 0\).

Figure 1 shows the energy distributions of incident \((a_1^2/II)\) and resonant \((a_2^2/II)\) components in both Type 1 and 2 when \(k_1d = 2.5, bk_1 = 0.05, \text{ and } \delta_0 = 0\), where \(II\) is the sum of squares of all wave amplitudes. As \(a_1\) decreases, resonant wave energy gradually grows until it is equal to...
the incident part. Bifurcations are found between Type 1 and 2. For type 2, as the incident wave slope $a_1k_1$ increases, the values of $\alpha$ corresponding to the bifurcation points tend to a fixed one, i.e., $\alpha = 63.7^\circ$, as shown in Figure 1 and Table 1. Thus, this second type of equilibrium state of waves at class-I Bragg resonance exists in the range of $63.7^\circ < \alpha < 90^\circ$. Without loss of generality, we choose $\alpha = 3\pi/8 = 67.5^\circ$ as the obliquely incident angle when the effects of other physical parameters on the class-I Bragg resonant wave system are studied.

3.2. Exact Resonance. According to the solving procedures shown in Section 2, the resonant component in the wave elevation should contain the term of $\sin(\xi_1 - \xi_2)$, where

$$\xi_1 - \xi_2 = (k_1 - k_2) \cdot r - a_1t = k_2 \cdot r - a_2t.$$  \hspace{1cm} (59)

Thus, here in this paper, the nonlinear frequencies of incident and resonant waves are always equal to each other, i.e.,

$$\sigma_1 = \sigma_2,$$  \hspace{1cm} (60)

so are the dimensionless nonlinear frequencies

$$\frac{\sigma_1}{\omega_1} = \frac{\sigma_2}{\omega_2} = \frac{\sigma_2}{\omega_2},$$  \hspace{1cm} (61)

when the conditions of the class-I Bragg exact resonance (1), (2), and (60) with $\delta_\omega = 0$ are satisfied.

3.2.1. Effect of Relatively Incident Wave Height $a_1/d$. Now, we consider the effect of relatively incident wave height ($a_1/d$) on exact resonant wave systems. As mentioned before, two types of equilibrium states can be found. We assume that the seabed is infinitely extended in space. Full interactions between sea and bottom result in the equilibrium state of Type 1, where the incident and resonant waves equally share the whole wave energy. Now, we investigate the effects of relatively incident wave height ($a_1/d$) on the equilibrium state of Type 2.

When ($a_1/d$) grows, less waves would be reflected by the seabed. An extreme case is that when ($a_1/d$) tends to 0 or the mean water depth $d$ tends to infinity, no reflection occurs since the surface waves cannot feel the seabed. We take the case of $a_1k_1 = 0.15$ in Figure 2(a) as an example. The energy of incident and resonant waves are represented by $a_1^2/\Pi$ and $a_2^2/\Pi$, respectively. When ($a_1/d$) grows, more wave energy flows to the resonant component, while the incident one has less energy. This situation ends at $a_1/d = 0.083$. And, after that, this second type of equilibrium state merges into the first type. The point of $a_1/d = 0.083$ is called the bifurcation point.

Meanwhile, various values of the incident wave slope $a_1k_1$, are also presented in Figure 2(a). For those with smaller wave slopes $a_1k_1$, bifurcations occur at smaller ($a_1/d$). On the contrary, the equilibrium state of Type 2 with greater nonlinearity combines with Type 1 at shallower area with larger ($a_1/d$). Thus, as shown by Figure 2(a) and Table 2, the bifurcation points shift to greater ($a_1/d$) when the incident wave becomes more and more steep.

It is mentioned by Stiassnie [12] that a single wave tends to break when its wave steepness $a_1k_1$ of the class-I Bragg exact resonant wave system in the case of $k_1d = 2.5$, $bk_1 = 0.05$, and $\delta_\omega = 0$. Solid line: $a_1k_1 = 0.01$; dashed line: $a_1k_1 = 0.02$; dotted line: $a_1k_1 = 0.03$; dash-dotted line: $a_1k_1 = 0.04$; dash-double-dotted line: $a_1k_1 = 0.05$; square: $a_1k_1 = 0.06$; circle: $a_1k_1 = 0.07$; triangle: $a_1k_1 = 0.08$; cross: $a_1k_1 = 0.09$; diamond: $a_1k_1 = 0.1$.

Table 1: Bifurcation points of incident angle $\alpha$ versus incident wave steepness $a_1k_1$ of the class-I Bragg exact resonant wave system in the case of $k_1d = 2.5$, $bk_1 = 0.05$, and $\delta_\omega = 0$.

<table>
<thead>
<tr>
<th>$a_1k_1$</th>
<th>$\alpha$ (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>70.00</td>
</tr>
<tr>
<td>0.02</td>
<td>65.20</td>
</tr>
<tr>
<td>0.03</td>
<td>64.30</td>
</tr>
<tr>
<td>0.04</td>
<td>63.95</td>
</tr>
<tr>
<td>0.05</td>
<td>63.82</td>
</tr>
<tr>
<td>0.06</td>
<td>63.75</td>
</tr>
<tr>
<td>0.07</td>
<td>63.72</td>
</tr>
<tr>
<td>0.08</td>
<td>63.71</td>
</tr>
<tr>
<td>0.09</td>
<td>63.70</td>
</tr>
<tr>
<td>0.10</td>
<td>63.70</td>
</tr>
</tbody>
</table>

Figure 1: Wave energy distribution ($a_1^2/\Pi$ and $a_2^2/\Pi$) versus $\alpha$ of the class-I Bragg resonant wave system in the case of $k_1d = 2.5$, $bk_1 = 0.05$, and $\delta_\omega = 0$. Solid line: $a_1k_1 = 0.01$; dashed line: $a_1k_1 = 0.02$; dotted line: $a_1k_1 = 0.03$; dash-dotted line: $a_1k_1 = 0.04$; dash-double-dotted line: $a_1k_1 = 0.05$; square: $a_1k_1 = 0.06$; circle: $a_1k_1 = 0.07$; triangle: $a_1k_1 = 0.08$; cross: $a_1k_1 = 0.09$; diamond: $a_1k_1 = 0.1$.

The wave steepness $a_1k_1$ of the wave system is defined same as Liu [13], i.e.,

$$h_s = \frac{k_1(\eta_{\max} - \eta_{\min})}{2},$$  \hspace{1cm} (62)

where $k_1$ is the wavenumber of the dominant wave. In addition, for a wave system with some fixed $a_1k_1$, it becomes
steeper at greater $a_1/d$ according to the nonlinear angular frequencies and wave steepness, as depicted in Figures 2(b) and 2(c).

3.2.2. Effect of Seabed Height $b/d$. Figure 3(a) shows the energy distributions as $b/d$ changes when $k_1d = 2.0$, $a_1k_1 = 0.15$, $\alpha = 3\pi/8$, and $\delta \omega = 0$. When $b/d = 0$, the bottom is flat. There will be no resonant wave. As $b/d$ grows, the resonant wave component has more and more energy, while the incident one gets less and less. And, finally, these two components share nearly the whole wave energy equally. The bifurcation point of these two types of wave systems is at about $b/d = 0.105$.

The corresponding angular frequencies $\sigma_i/\omega_i$ (i = 1, 2) and wave steepness $h_i$ are presented in Figure 3(b). As $b/d$ grows, both the frequency and wave height have turning points. These turning points locate exactly at the above bifurcation point $b/d = 0.105$.

When $b/d < 0.105$, the dimensionless frequencies $\sigma_i/\omega_i$ (i = 1, 2) have approximately linear growth with the relative seabed height $b/d$ and so is the equivalent wave height $h_i$. When $b/d > 0.105$, the wave height $h_i$ does not increase further. Meanwhile, the angular frequencies $\sigma_i/\omega_i$ even decrease. Since $h_i > 0.3$ at $b/d = 0.125$, wave breaking may also occur.
3.3. Near Resonance. Waves propagating over a flat seabed in various water depths with time-independent spectra have been studied by homotopy analysis method when the near resonance occurs [13–15]. In this section, we consider the linearly near resonance of class-I Bragg which satisfies conditions (1), (2), and (60) with $\delta \omega \neq 0$. Only the equilibrium-state waves of Type 2 are found.

The peak of wave steepness at near resonance shifts from the exact resonance ($\delta \omega = 0$), as shown by Figure 4. Both up and down shifts of the peaks are found. In other words, resonances occur in both domains of positive and negative frequency detunings. For example, the resonance occurs in the domain of $\delta \omega < 0$ when $a_1k_1 = 0.17$, while in the domain of $\delta \omega > 0$, when $a_1k_1 = 0.19$ and 0.20. Moreover, the effects of positive and negative detunings on the wave system are not symmetric.

4. Concluding Remarks and Discussions

We study the equilibrium states of waves at both exact and near Bragg resonances of the first kind. Using the amplitude-based homotopy analysis method, waves with time-independent spectra are considered. The waves are classified into two types. The incident and resonant waves have the same energy in Type 1, while in Type 2 these two wave components have different energy. In both types, the incident and resonant components share almost the whole wave energy. In addition, waves with strong nonlinearities are obtained. The wave steepness $h_1$ of some resonant wave system is up to 0.3, which is believed to be large enough for a nonlinear wave.

Effects of relatively incident wave height $a_1/d$ and seabed height $b/d$ on the equilibrium states of wave systems at exact resonance are studied when the incident wave with different wave slopes propagate over the seabed obliquely. Bifurcations of the two types of waves are found. The bifurcation point shifts to greater value of $a_1/d$ for a steeper incident wave. Meanwhile, the wave steepness $h_1$ grows linearly with the seabed height $b/d$.

When linearly near resonance is considered, both up and down shifts of the resonant peak are found. And, more than one local peaks are obtained for some resonances, which implies that multiple resonances occur although the lower order of components are found to have tiny wave energy and the whole wave energy is still occupied by the incident and resonant component in the leading order.

This work on class-I Bragg resonance can be extended to class II, III, and even higher-order resonances. Moreover, interactions of internal waves among multilayer fluids over undulated seabed can also be studied.
Appendix

A. Derivations of $n$th-Order Deformation Equations (38)–(40)

We derive the $n$th-order deformation equations (38)–(40) by substituting Maclaurin series (26)–(28) into the zeroth-order deformation equations (17)–(20) and setting the coefficient of $q^n$ to be zero. Following this procedure, first of all, we show the derivation of equation (38) in detail. In the left-hand side of equation (17), we have

\[ L_1(q) \left[ \tilde{\phi}(\xi_1, \xi_2, Z; q) \right] = \mu_{uv} \Omega_1^2 \tilde{\phi}_{\xi_1, \xi_1} + \frac{\tilde{\phi}_Z}{\tanh(\epsilon_1)} \bigg|_{Z=0} \]

\[ = \sum_{n=0}^{\infty} q^n \left[ \mu_{uv} \left( \sum_{i=0}^n \sum_{l=0}^i \Omega_{1,j} \Omega_{1,i-j} \tilde{\phi}_{\tilde{\phi}^2,0} \right) + \frac{1}{\tanh(\epsilon_1)} \tilde{\phi}^0,n \right]. \]

Let

\[ S_n = \mu_{uv} \left( \sum_{i=0}^n \sum_{l=0}^i \Omega_{1,j} \Omega_{1,i-j} \tilde{\phi}_{\tilde{\phi}^2,0} \right) + \frac{1}{\tanh(\epsilon_1)} \tilde{\phi}^0,n \]

(A.1)

Then, the left-hand side of equation (17) hence reads

\( (1 - q) L_1(q) [\tilde{\phi}] = (1 - q) \sum_{n=0}^{\infty} q^n S_n = \sum_{n=0}^{\infty} (S_n - x_n S_{n-1}) q^n, \)

\[ (A.2) \]

\[ (A.3) \]

where

\[ x_n = \begin{cases} 0, & n = 0, \\ 1, & n \geq 1. \end{cases} \]

\[ (A.4) \]

The expression of $S_n$ can be simplified furthermore as follows:

\[ S_n = \mu_{uv} \left( \sum_{i=0}^n \sum_{l=0}^i \Omega_{1,j} \Omega_{1,i-j} \tilde{\phi}_{\tilde{\phi}^2,0} \right) + \frac{1}{\tanh(\epsilon_1)} \tilde{\phi}^0,n \]

\[ = \mu_{uv} \Omega_{1,0}^2 \tilde{\phi}_{\tilde{\phi}^2,0} + \frac{1}{\tanh(\epsilon_1)} \tilde{\phi}^0,n + \mu_{uv} \Omega_{1,0}^2 \sum_{m=1}^n \phi_{m-m,m}^2 + \frac{1}{\tanh(\epsilon_1)} \sum_{m=1}^n \gamma_{m,m}^n + T_n \]

\[ = T_1[\phi_n] + U_n + T_n, \]

\[ (A.5) \]

where

\[ T_1[\phi_n] = \left[ \mu_{uv} \Omega_{1,0}^2 \phi_{n,\xi_1,\xi_1} + \frac{\phi_n, Z}{\tanh(\epsilon_1)} \right] \bigg|_{Z=0}, \]

\[ U_n = \mu_{uv} \Omega_{1,0}^2 \sum_{m=1}^n \phi_{m-m,m}^2 + \frac{1}{\tanh(\epsilon_1)} \sum_{m=1}^n \gamma_{m,m}^n, \]

\[ T_n = \mu_{uv} \left( \sum_{i=0}^n \sum_{l=0}^i \Omega_{1,j} \Omega_{1,i-j} \tilde{\phi}_{\tilde{\phi}^2,0} \right). \]

\[ (A.6) \]

Meanwhile, in the right-hand side of equation (17),
\[ N_1 \left[ \phi (\xi_1, \xi_2, Z; q), \Omega_1 (q) \right] = \sum_{n=0}^{\infty} q^n \left( \sum_{i=0}^n \sum_{j=0}^i \Omega_{i,j} \phi_{n-i}^2 \right) + \frac{1}{\tanh (\epsilon_1)} \sum_{n=0}^{\infty} \phi_z q^n - 2 \epsilon_2 \sum_{n=0}^{\infty} \phi_z q^n + \epsilon_2^2 \sum_{n=0}^{\infty} \Lambda_n q^n \]

\[ = \sum_{n=0}^{\infty} q^n \Delta_n^\phi, \]

where

\[ \Delta_n^\phi = \left( \sum_{i=0}^n \sum_{j=0}^i \Omega_{i,j} \phi_{n-i} \phi_{n-j} \right) + \frac{1}{\tanh (\epsilon_1)} \phi_z q^n - 2 \epsilon_2 \left( \sum_{i=0}^n \Omega_{i,n-i} \right) + \epsilon_2^2 \Lambda_n, \]

\[ \Gamma_{m,1} = \sum_{n=0}^{m} \left[ \phi_n^1 \phi_{m-n}^0 + K_b \phi_n^2 \phi_{m-n}^0 + \phi_{Z,n}^0 \phi_{Z,m-n}^0 \right] + K_b \cos (\alpha_1 - \alpha_2) \left( \phi_n^0, \phi_{m-n}^0, \phi_{m-n}^0 \right), \]

\[ \Lambda_n = \sum_{m=0}^{n} \left[ \phi_m^1 \Gamma_{n-m,1} + K_b \phi_m^2 \Gamma_{n-m,2} + \phi_{Z,n}^0 \Gamma_{n-m,3} \right] + K_b \cos (\alpha_1 - \alpha_2) \left( \phi_m^0, \phi_{n-m,2}^0, \phi_{n-m,3}^0 \right), \]

\[ \Gamma_{m,2} = \sum_{n=0}^{m} \left[ \phi_n^1 \phi_{m-n}^1 + K_b \phi_n^2 \phi_{m-n}^1 + \phi_{Z,n}^0 \phi_{Z,m-n}^0 \right] + K_b \cos (\alpha_1 - \alpha_2) \left( \phi_n^0, \phi_{m-n,1}^0, \phi_{m-n,2}^0 \right), \]

\[ \Gamma_{m,3} = \sum_{n=0}^{m} \left[ \phi_n^1 \phi_{Z,m-n}^0 + K_b \phi_n^2 \phi_{Z,m-n}^0 + \phi_{Z,n}^0 \phi_{Z,Z,m-n}^0 \right] + K_b \cos (\alpha_1 - \alpha_2) \left( \phi_n^0, \phi_{Z,m-n}^0, \phi_{Z,n}^0 \right). \]

Thus, the zeroth-order deformation equation (17) can be written as follows:

\[ \sum_{n=0}^{\infty} (S_n - \chi_n S_{n-1}) q^n = c_0 \sum_{n=0}^{\infty} q^{n+1} \Delta_n^\phi, \]  \[ \text{(A.9)} \]

which gives the corresponding nth-order deformation equation

\[ S_n - \chi_n S_{n-1} = c_0 \chi_n \Delta_n^\phi, \]  \[ \text{(A.10)} \]

or in another form of

\[ \mathcal{F}_1 [\phi_n] = -U_n - T_n + \chi_n S_{n-1} + \chi_n c_0 \Delta_n^\phi, \]  \[ \text{at } Z = 0. \]  \[ \text{(A.11)} \]

Secondly, substituting the Maclaurin series (27) into the zeroth-order deformation equation (18) and considering the initial guess (32), the left-hand side of equation (18) reads

\[ (1 - q)\tilde{u} (\xi_1, \xi_2; q) = (1 - q) \sum_{n=1}^{\infty} \eta_n q^n = \sum_{n=1}^{\infty} (\eta_n - \eta_{n-1}) q^n. \]  \[ \text{(A.12)} \]

Meanwhile, the right-hand side of equation (18) can be written in the following form:
where

\[ q_{q_0,q_2}^\eta = q_{0} \sum_{n=0}^{\infty} \Delta_n^\eta q^n \]

\[ = c_0 \sum_{n=1}^{\infty} \Delta_{n-1}^\eta q^n, \]

(A.13)

Thus, equation (18) can be rewritten in the form of the Maclaurin series:

\[ \sum_{n=1}^{\infty} (\eta_n - \eta_{n-1})q^n = c_0 \sum_{n=1}^{\infty} \Delta_{n-1}^\eta q^n, \]  

(A.15)

which gives the \( n \)th-order deformation equation with respect to \( \eta \):

\[ \phi_n(\xi_1,\xi_2, Z) = \phi_n(\xi_1,\xi_2, -\epsilon_1) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \phi_n(\xi_1,\xi_2, Z)}{\partial Z^m} |_{Z=-\epsilon_1} (Z - (-\epsilon_1))^m \]

(A.17)

and then, we have

\[ \phi_n(\xi_1,\xi_2, Z) = \psi^n_{0,0} + \sum_{m=1}^{\infty} \psi^n_{0,m} (q\epsilon_3 H)^m, \]  

(A.20)

Define

\[ \psi_{i,j}^{n,m} = \left. \frac{\partial^i j}{\partial \xi_1^i \partial \xi_2^j} (\frac{1}{m!} \frac{\partial^m \phi_n(\xi_1,\xi_2, Z)}{\partial Z^m}) \right|_{Z=-\epsilon_1}, \]  

(A.19)

Substitute (26) and (A.20) into the left-hand side of equation (20), and we have
\[(1 - q)\mathcal{D} \left[ \phi - \phi_0 \right] = (1 - q) \frac{\partial (\phi - \phi_0)}{\partial Z} = (1 - q) \sum_{n=1}^{\infty} \frac{\partial \phi_n}{\partial Z} q^n \]

\[= (1 - q) \sum_{n=1}^{\infty} q^n \left[ \psi_{0,0}^{m,n} + \sum_{m=1}^{n} (m + 1) \psi_{0,0}^{n,m,m+1} (\varepsilon_3 H)^m \right] \]

\[= \sum_{n=1}^{\infty} (V_n - \chi_{n-1} V_{n-1}) q^n , \]

(A.23)

where

\[V_n = \psi_{0,0}^{n,1} + \sum_{m=1}^{n-1} (m + 1) \psi_{0,0}^{n,m,m+1} (\varepsilon_3 H)^m \]

\[= \frac{\partial \phi_1}{\partial Z} |_{Z = -\varepsilon_3} + \nabla_n , \]

(A.24)

Meanwhile, in right-hand side of equation (20),

\[\nabla \phi \cdot \nabla (\varepsilon_1 + \varepsilon_3 H) = \varepsilon_3 K_b H_{\xi_3} \left[ \cos (\alpha_1 - \alpha_0) \phi_{\xi_3} + K_b \phi_{\xi_3} \right] , \]

(A.25)

where \( H = H (\xi_3) = \cos (\xi_3) \). Thus, the right-hand side of equation (20) equals

\[q \phi(\phi + \varepsilon_3 K_b H_{\xi_3} \left[ \cos (\alpha_1 - \alpha_0) \phi_{\xi_3} + K_b \phi_{\xi_3} \right]) . \]

(A.26)

Substituting (A.20)–(A.22) into the above expression, we have

\[\hat{\phi}_Z + \nabla \phi \cdot \nabla (\varepsilon_1 + \varepsilon_3 H) = \sum_{n=0}^{\infty} q^n \Delta_n^b , \]

where

\[\Delta_n^b = \sum_{m=0}^{n} (\varepsilon_3 H)^m \left[ (m + 1) \psi_{0,0}^{n,m,m+1} + \varepsilon_3 K_b H_{\xi_3} \cos (\alpha_1 - \alpha_0) \psi_{1,0}^{n,m,m} + \varepsilon_3 K_b^2 H_{\xi_3} \psi_{0,1}^{n,m,m} \right] . \]

(A.28)

\[\text{Conflicts of Interest} \]

The authors declare that they have no conflicts of interest.

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\[\text{References} \]


