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Research Article

A Class of Sextic Trigonometric Bézier Curve with Two Shape Parameters

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In this paper, we present a new class of sextic trigonometric Bernstein (ST-Bernstein, for short) basis functions with two shape parameters along with their geometric properties which are similar to the classical Bernstein basis functions. A sextic trigonometric Bézier (ST-Bézier, for short) curve with two shape parameters and their geometric characteristics is also constructed. The continuity constraints for the connection of two adjacent ST-Bézier curves segments are discussed. Shape control parameters can provide an opportunity to modify the shape of curve as designer desired. Some open and closed curves are also part of this study.

1. Introduction

A Bézier curve is a parametric curve that is used to draw the shapes in the fields of computer graphics and computer-aided geometric design. The Bézier curve is usually followed by the defining polygon. The tangent vectors direction at the end is the same as the vector defined by the first and last segment of the Bézier curves. It is useful in many industrial and engineering fields with various applications. Since Bézier curve always mimics the shape of its control polygon, designers can easily attain the required shape for designing purposes.

The trigonometric Bézier curves got a lot of attention in the fields of computer graphics and computer-aided geometric design due to their construction of conic section. Bézier curve with two parameters and control point was introduced by Kun [1], but the behavior of curve was not symmetric. The generalized Bézier-like curve with all of its geometric characteristics and continuity conditions is described by Yan and Liang [2]. They also applied the generalized Bézier-like curve for the tensor product surfaces to gain access for triangular surfaces as well. The modeling of innovative surfaces based on stream curves was described by Liu et al. [3]. By extending the concept of Bézier curve, Li [4] defined alpha-Bézier-like curves of degree n with shape control parameters. The properties and applications of alpha-Bézier-like curves are also given. Han et al. [5–7] introduced some cubic and quartic trigonometric Bézier curves. They created an ellipse by using a cubic trigonometric Bézier curve and some designing and geometric modeling also made by continuity conditions. The behavior of shape control parameters also examined on these Bézier curves. Xiujuan et al. [8] investigated special revolution surfaces and their dramatic improvement. Bashir et al. [9] derived a class of quasi-quantic trigonometric Bézier curves with two shape parameters and proved their geometric features. The properties of the basis functions and curves are established, and the effect of the shape control parameters is also discussed. Practical applications of Bézier curves in geometric modeling and engineering are limited due to their shortcomings, and much work has been done to resolve these shortcomings [10–16].

BiBi et al. [17, 18] suggested a new method for solving the problem of constructing symmetric curves and surfaces by using GHT-Bézier curves with four different shape parameters. The shape of curves can easily be modified by using different values of shape parameters. They generated some free-form complex curves with parametric continuity conditions by using GHT-Bézier curves to demonstrate the efficiency of modeling. Yan and Liang [2] used the recursive technique to create the rectangular Bézier curve and surface based on a new class of polynomial basis functions with one shape parameter. Hu et al. [19] presented a novel scheme to generate free-form complex figures using shape-adjustable generalized (SG) Bézier curves with some geometric continuities conditions. They constructed the necessary and sufficient constraints for G^1 and G^2 continuity for connection of two adjacent SG-Bézier curves to overcome the difficulty that most of the composite curves in engineering cannot often be constructed by using only a single curve. Majeed and Qayyum [20] presented the cubic and rational cubic trigonometric B-spline curves using new trigonometric functions with shape parameter. The proposed curves inherit the basic properties of classical B-spline and have been proved. Misro et al. [21] developed the general technique to construct S- and C-shaped transition curves using cubic trigonometric Bézier Curve with two shape parameters which satisfy G^2 Hermite condition. Misro et al. [22] constructed a new quintic trigonometric Bézier curve that has the potential to estimate the maximum driving speed allowed for safe driving on roads. The shape parameters used in this trigonometric Bézier function provided more flexibility for users in designing highways. The trigonometric Bézier curve of fifth degree with two shape parameters has been presented by Misro et al. [23]. Shape parameters provided more control on the shape of the curve compared to the classical Bézier curve. Juhász and Róth [24] presented a scheme for interpolating the given set of data points with C^n continuous trigonometric spline curves of order $n+1$ which are produced by blending elliptical arcs with global parameter $\alpha \in (0, \pi)$. Zhu and Han [25] constructed four new trigonometric Bernstein-like basis functions with two exponential shape parameters, based on which a class of trigonometric Bézier-like curves, similar to the cubic Bézier curves, have also been developed. The trigonometric Bézier-like curves corner cutting algorithm was also constructed. Yan and Liang [26] presented a new kind of algebraic-trigonometric blended spline curve, called xyB curves. The proposed curves not only inherit most properties of classical cubic B-spline curves in polynomial space but also enjoy some other advantageous properties for modeling.

In this article, the research begins with the development of new ST-Bernstein basis functions with two shape control parameters. This study also provides a guarantee to construct a new ST-Bézier curve with two shape parameters. The newly constructed curves share all geometric properties of classical Bézier curves except the shape modification property, which is superior to the classical Bézier curve. The C^2 and G^2 continuity constraints are constructed to connect the two adjacent ST-Bézier curves segments. Moreover, in contrast with classical Bézier curves, our proposed scheme gives more shape adjustability in curve designing. Several examples are presented to show that the proposed method has high applied values in geometric modeling in terms of some closed and open curves.

The remainder of the paper is organized as follows. In Section 2, the new ST-Bernstein basis functions with two parameters are presented, which possess all geometric properties. In Section 3, the graphical representation of ST-Bézier curve with all geometric properties is given. The parametric and geometric continuity conditions with shape control parameters are given in Section 5. Shape control on ST-Bézier curve via shape parameters is given in Section 4. In Section 6, some application to construction of some closed and open curves is given with multiple shape control parameters. Finally, concluding remarks of this work are given in Section 7.

2. Sextic Trigonometric Bernstein Basis Functions

In this section, the ST-Bernstein basis function with two shape parameters μ, ω and their geometric properties is discussed.

Definition 1. For $\eta \in [0, \pi/2]$, the ST-Bernstein basis function with two shape parameters $\mu, \omega \in [-4, 1]$ is defined by

$$\begin{cases} \Phi_0(\eta) = (1-\sin\eta)^4(1-\mu\sin\eta), \\ \Phi_1(\eta) = \sin\eta(1-\sin\eta)^3(4+\mu(1-\sin\eta)), \\ \Phi_2(\eta) = \sin^2\eta(1-\sin\eta)^2(6+\mu(1-\sin\eta)), \\ \Phi_3(\eta) = 1-\Phi_0(\eta)-\Phi_1(\eta)-\Phi_2(\eta)-\Phi_4(\eta)-\Phi_5(\eta)-\Phi_6(\eta), \\ \Phi_4(\eta) = \cos^2\eta(1-\cos\eta)^2(6+\omega(1-\cos\eta)), \\ \Phi_5(\eta) = \cos\eta(1-\cos\eta)^3(4+\omega(1-\cos\eta)), \\ \Phi_6(\eta) = (1-\cos\eta)^4(1-\omega\cos\eta). \end{cases} \tag{1}$$

The graphical representation of ST-Bernstein basis functions is given in Figure 1. By changing the values of shape control parameters, the variation in graphs of ST-Bernstein basis function is obvious. In Figure 1(a), the ST-Bernstein basis functions are drawn as $\mu = \omega = 1$ (thick red), $\mu = \omega = 0.5$ (dotted, blue), $\mu = \omega = 0$ (dot dashed, green), $\mu = \omega = -1$ (dashed, pink), $\mu = \omega = -2$ (thick, purple), $\mu = \omega = 3$ (thick, black), and $\mu = \omega = -4$ (thick, yellow). In Figure 1(b), the ST-Bernstein basis functions are drawn as $\mu = \omega = 1$ (thick, red), $\mu = 1, \omega = 0.5$ (dotted, blue), $\mu = 1, \omega = 0$ (dot dashed, green), $\mu = 1, \omega = -1$ (dashed, pink), $\mu = 1, \omega = -2$ (thick, purple), $\mu = 1, \omega = -3$ (thick, black), and $\mu = 1, \omega = -4$ (thick, yellow). In Figure 1(c), the ST-Bernstein basis functions are drawn as $\mu = \omega = 1$ (thick, red), $\mu = 0.5, \omega = 1$ (dotted, blue), $\mu = 0, \omega = 1$ (dot dashed, green), $\mu = -1, \omega = 1$ (dashed, pink), $\mu = -2, \omega = 1$ (thick, purple), $\mu = -3, \omega = 1$ (thick, black), and $\mu = -4, \omega = 1$ (thick, gray).

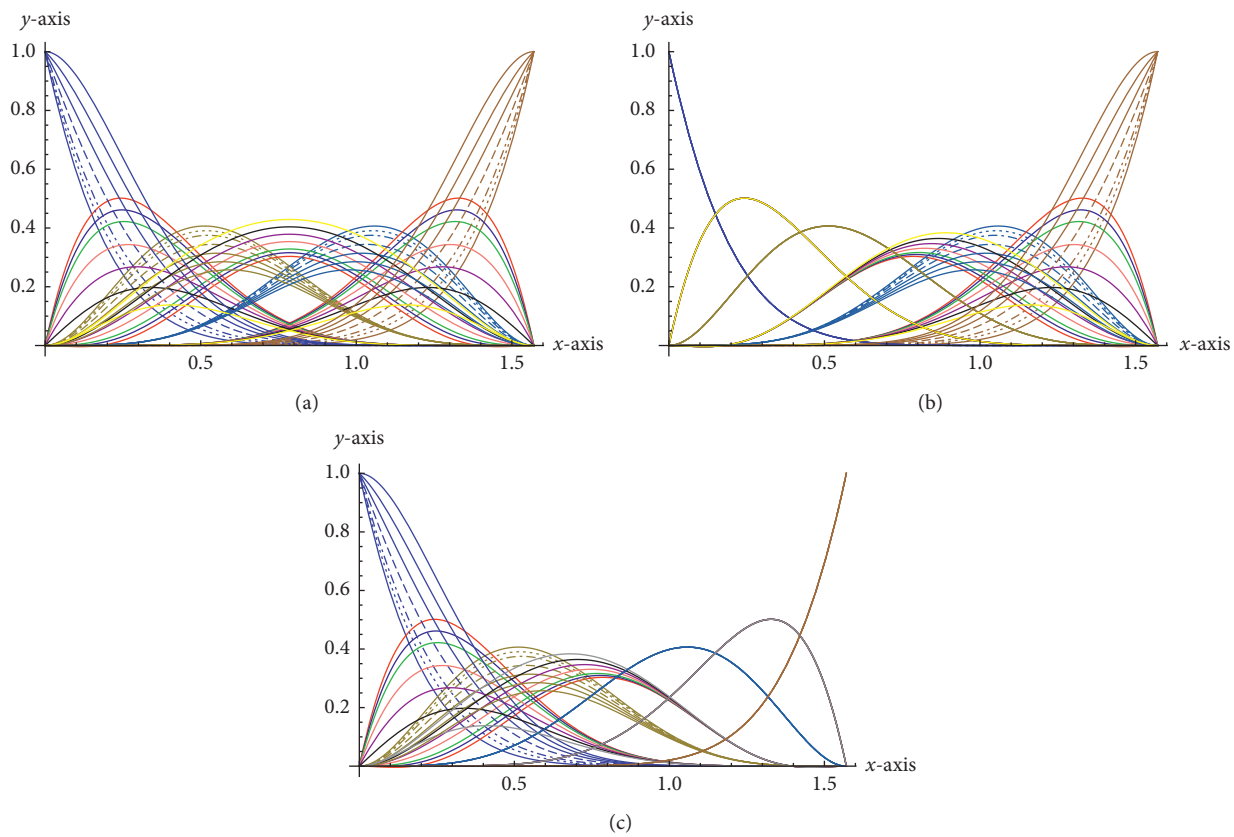


Figure 1: ST-Bernstein basis functions with shape parameters.

Theorem 1. *The ST-Bernstein basis functions in equation (1) have the following geometric characteristics:*

(1) Nonnegativity: $\Phi_i(\eta) \geq 0\,(i = 0, 1, \ldots, 6)$

(2) Partition of unity: $\sum_{i=0}^{6} \Phi_i(\eta) = 1$

(3) Symmetry: $\Phi_i(\eta; \mu, \omega) = \Phi_{6-i}(\pi/2 - \eta; \mu, \omega)\,(i = 0, 1, \ldots, 6)$

(4) Monotonicity: for the given value of the shape parameter μ, ω, $\Phi_0(\eta)$ *is monotonically decreasing and* $\Phi_6(\eta)$ *is monotonically increasing*

(5) Terminal property: $\Phi_0(0) = 1, \Phi_i(0) = 0, \Phi_i(\pi/2) = 0, \Phi_6(\pi/2) = 1,\ i = 0, 1, 2, \ldots, 5$

Proof

(1) For $\eta \in [0, \pi/2]$ and $-4 \leq \mu, \omega \leq 1$, since $(1 \pm \sin\eta) \geq 0$, $(1 - \mu \sin\eta) \geq 0$, $(1 \pm \cos\eta) \geq 0$ and $(1 - \omega \cos\eta) \geq 0$, $\sin\eta \geq 0$, $\cos\eta \geq 0$, $\sin^2\eta \geq 0$, $\cos^2\eta \geq 0$, this shows that $\Phi_i(\eta) \geq 0$, $(i = 0, 1, 2, \ldots, 6)$.

(2) It is obvious by Definition 1.

(3) For $i = 0$,

$$\Phi_0(\eta) = (1 - \sin\eta)^4 (1 - \mu \sin\eta) = \left(1 - \cos\left(\frac{\pi}{2} - \eta\right)\right)^4 \left(1 - \omega \cos\left(\frac{\pi}{2} - \eta\right)\right) = \Phi_6(\eta). \tag{2}$$

For $i = 1$,

$$\Phi_1(\eta) = \sin\eta (1 - \sin\eta)^3 (4 + \omega(1 - \sin\eta)) = \cos\left(\frac{\pi}{2} - \eta\right)\left(1 - \cos\left(\frac{\pi}{2} - \eta\right)\right)^3 \left(4 + \omega\left(1 - \cos\left(\frac{\pi}{2} - \eta\right)\right)\right) = \Phi_5(\eta). \tag{3}$$

For $i = 2$,

$$\Phi_2(\eta) = \sin^2\eta(1-\sin\eta)^2(6+\mu(1-\sin\eta)) = \cos^2\left(\frac{\pi}{2}-\eta\right)\left(1-\cos\left(\frac{\pi}{2}-\eta\right)\right)^2\left(6+\omega\left(1-\cos\left(\frac{\pi}{2}-\eta\right)\right)\right) = \Phi_4(\eta). \tag{4}$$

(4) For any $\eta_0, \eta_1 \in [0, \pi/2]$ such that if $\eta_0 \le \eta_1$ then $\Phi_0(\eta_0) \ge \Phi_0(\eta_1)$, this means that $\Phi_0(\eta)$ is monotonically decreasing similarly when $\eta_0 \le \eta_1$ and $\Phi_6(\eta_0) \ge \Phi_6(\eta_1)$, which shows that $\Phi_6(\eta)$ is monotonically increasing. Let $f_1(\eta) = \Phi_0'(\eta) = -\mu\cos\eta(1-\sin\eta)^4 - 4\cos\eta(1-\sin\eta)^3(1-\mu\sin\eta)$ and $f_2(\eta) = \Phi_6'(\eta) = \omega(1-\cos\eta)^4\sin\eta + 4(1-\cos\eta)^3(1-\omega\cos\eta)\sin\eta$, where $\eta \in [0, \pi/2]$. From the figures of the functions $f_1(\eta)$ and $f_2(\eta)$, we can see $f_1(\eta) \le 0$ and $f_2(\eta) \ge 0$ when $\eta \in [0, \pi/2]$. Therefore, $\Phi_0(\eta)$ and $\Phi_6(\eta)$ are monotonically decreasing and increasing about η, respectively. This can also be shown graphically in Figure 1.

(5) When we put $\eta = 0$ and $\eta = \pi/2$ in Definition 1, we get $\Phi_0(0) = 1, \Phi_i(0) = 0\,(i = 1, 2, \ldots, 6), \Phi_i(\pi/2) = 0\,(i = 0, 1, 2, \ldots,), \Phi_6(\pi/2) = 1$ and the first derivatives of these basis functions at their end points are given as follows:

$$\Phi_i'(0) = \begin{cases} -4-\mu, & i = 0, \\ 4+\mu, & i = 1, \\ 0, & i = 2, \ldots, 6, \end{cases}$$
$$\Phi_i'\left(\frac{\pi}{2}\right) = \begin{cases} 4+\omega, & i = 6, \\ -4-\omega, & i = 5, \\ 0, & i = 0, \ldots, 4. \end{cases} \tag{5}$$

Similarly, the second derivatives of these basis functions at their end points are given as follows (see Figure 2):

$$\Phi_i''(0) = \begin{cases} 12+8\mu, & i = 0, \\ -12-8\mu, & i = 1, \\ 2(6+\mu), & i = 2, \\ -2\mu, & i = 3, \\ 0, & i = 4, 5, 6, \end{cases}$$
$$\Phi_i'\left(\frac{\pi}{2}\right) = \begin{cases} 12+8\omega, & i = 6, \\ -12-8\omega, & i = 5, \\ 2(6+\omega), & i = 4, \\ -2\omega, & i = 3, \\ 0, & i = 0, 1, 2. \end{cases} \tag{6}$$

□

3. Sextic Trigonometric Bézier Curves with Two Shape Parameters

In this section, the ST-Bézier curves with two shape parameters μ, ω and their geometric properties are discussed.

Definition 2. For the given control points $S_i\,(i = 0, 1, 2, \ldots, 6)$, the curve

$$\alpha(\eta) = \sum_{i=0}^{6} S_i\Phi_i(\eta), \quad \eta \in \left[0, \frac{\pi}{2}\right], \mu, \omega \in [-4, 1], \tag{7}$$

is called ST-Bézier curve, where $\Phi_i(\eta)\,(i = 0, 1, 2, \ldots, 6)$ are called ST-Bernstein basis functions and μ, ω are the shape parameters.

Some graphical results of ST-Bézier curve are discussed as follows: when shape parameters vary equally, Figure 3 is generated, while, by keeping one parameter fixed to 1, Figure 4 is generated. In Figure 4(a), when $\mu = 1$ and ω varies, then influence of shape parameters can be seen on the left side of the figure. Meanwhile when we consider $\omega = 1$ and parameter μ varies, the influence of these parameters can be observed on the right side of Figure 4(b).

Theorem 2. *The ST-Bézier curves in equation (7) have the following geometric properties:*

(1) End point properties:

$$\begin{cases} \alpha(0) = S_0, \\ \alpha\left(\frac{\pi}{2}\right) = S_6, \\ \alpha'(0) = (4+\mu)(S_1 - S_0), \\ \alpha'\left(\frac{\pi}{2}\right) = (4+\omega)(S_6 - S_5), \\ \alpha''(0) = 2(2(3+2\mu)S_0 - 2(3+2\mu)S_1 + (6+\mu)S_2 - \mu S_3), \\ \alpha''\left(\frac{\pi}{2}\right) = 2(2(3+2\omega)S_6 - 2(3+2\omega)S_5 + (6+\omega)S_4 - \omega S_3). \end{cases} \tag{8}$$

(2) Symmetry: the control points S_i define the same ST-Bézier curve having symmetric influence in different parameterizations, such as

$$\alpha(\eta; \mu, \omega; S_0, \ldots, S_6) = \alpha\left(\frac{\pi}{2} - \eta; \mu, \omega; S_6, \ldots, S_0\right). \tag{9}$$

(3) Geometric invariance: the shape of ST-Bézier curve is independent from the coordinate axis, which means

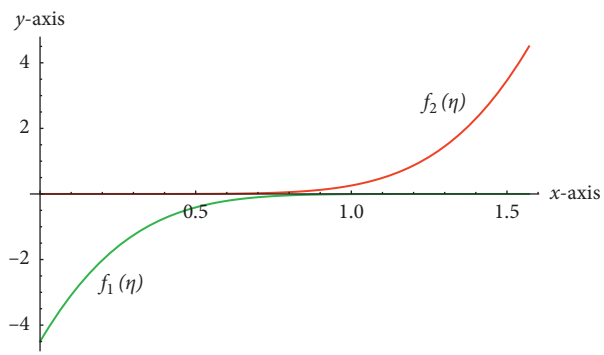


FIGURE 2: Functions $f_1(\eta)$ and $f_2(\eta)$ on $\eta \in [0, \pi/2]$.

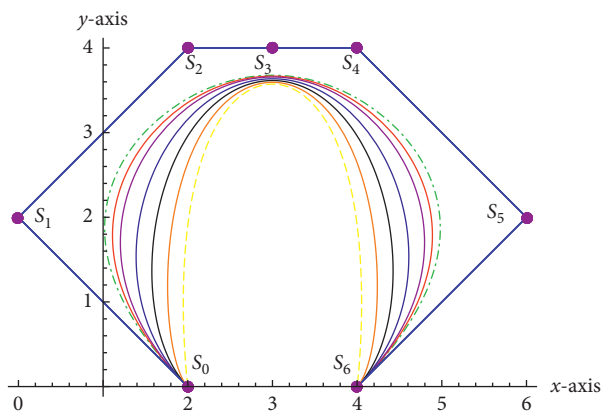


FIGURE 3: ST-Bézier curves with some shape parameters when $\mu = \omega = 1, 0.5, 0, -1, -2, -3, -4$.

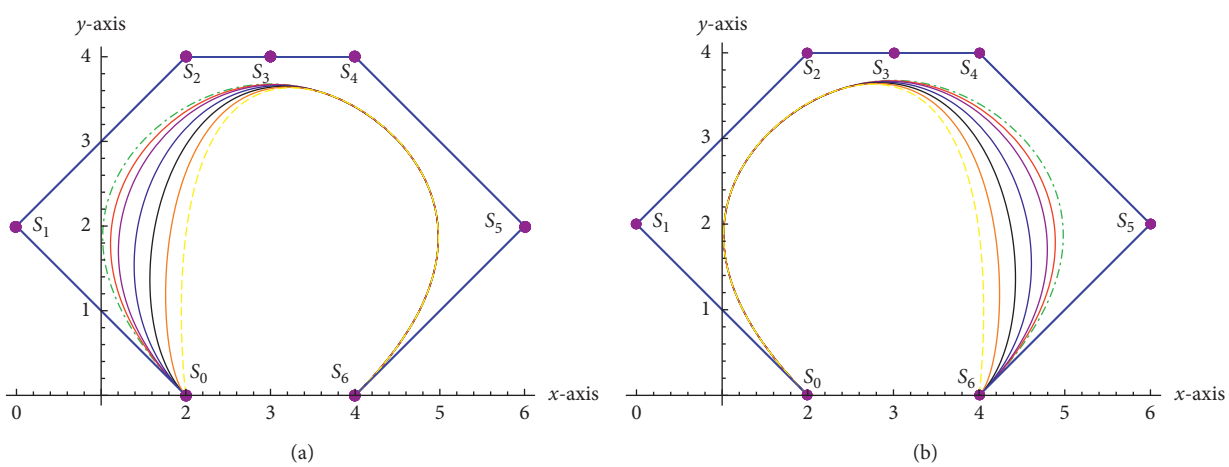


FIGURE 4: ST-Bézier curves with different shape parameters. (a) $\mu = 1, \omega = 1, 0.5, 0, -1, -2, -3, -4$. (b) $\omega = 1, \mu = 1, 0.5, 0, -1, -2, -3, -4$.

that the curve defined in equation (7) satisfies the two following equations:

$$\alpha(\eta; \mu, \omega; S_0 + \vartheta, \ldots, S_6 + \vartheta) = \alpha(\eta; \mu, \omega; S_0, \ldots, S_6) + \vartheta,$$
$$\alpha(\eta; \mu, \omega; S_0 \times \rho, \ldots, S_6 \times \rho) = \alpha(\eta; \mu, \omega; S_0, \ldots, S_6) \times \rho. \quad (10)$$

(4) *Shape control property: the shape of classical Bézier curve cannot be modified due to absence of shape control parameters, while the ST-Bézier curve possesses two shape parameters by which we can modify the curve easily.*

(5) *Convex hull property: the ST-Bézier curve must be confined inside the convex polygon spanned by its control points.*

(6) *Linearly independent: for any $b_i \in R\,(i = 0, 1, \ldots, 6)$, consider a linear combination as follows:*

$$\sum_{i=0}^{6} b_i \Phi_i(\eta) = 0, \quad \eta \in \left[0, \frac{\pi}{2}\right]. \tag{11}$$

Taking 1st, 2nd, 3rd ,..., 6th derivatives of the above equation with respect to η on each side yields the following equations:

$$\begin{cases} \sum_{i=0}^{6} b_i \Phi_i'(\eta) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i''(\eta) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i''(\eta) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(iv)}(\eta) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(v)}(\eta) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(vi)}(\eta) = 0. \end{cases} \tag{12}$$

When $\eta = 0$, we have

$$\begin{cases} \sum_{i=0}^{6} b_i \Phi_i'(0) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i''(0) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i''(0) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(iv)}(0) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(v)}(0) = 0, \\ \sum_{i=0}^{6} b_i \Phi_i^{(vi)}(0) = 0. \end{cases} \tag{13}$$

By using the values of the 1st, 2nd, 3rd,..., 6th-order derivatives of ST-Bernstein basis functions at $\eta = 0$, we get $b_i = 0\,(i = 0, 1, \ldots, 6)$; this fact shows that $\Phi_i(\eta)\,(i = 0, 1, \ldots, 6)$ are linearly independent.

4. Shape Control of the ST-Bézier Curve

As we know, the shape parameters help us to modify the shape of the curve within which lies inside the control polygon. So, by using these shape control parameters, the ST-Bézier curves can be modified wherever the shape does not change at all. The shape parameters μ, ω cause local change in the curve; see Figure 5. In Figure 5(a), the effect on shape of the curve is observed when $\mu = \omega =$ {1 (dot dashed, green), 0.5 (thick, red), 0 (thick, purple), −1 (thick, blue), −2 (thick, black), −3 (thick, orange), and −4 (dashed, yellow)}. In Figure 5(b), the influence of ST-Bézier curve can be seen on the left-hand side when $\omega = 1$ is fixed and μ varies in the interval [−4,1] as $\mu = 1$ (thick, pink), $\mu = 0.5$ (thick, red), $\mu = 0$ (thick, green), $\mu = -1$ (thick, black), $\mu = -3$ (thick, orange), and $\mu = -4$ (dashed, yellow). Similarly, Figure 5(c) is determined with influence on the right-hand side of the figure when $\mu = 1$ is fixed and ω varies as $\omega = 1$(thick, pink), $\omega = 0.5$ (thick, red), $\omega = 0$ (thick, green), $\omega = -1$ (thick, black), $\omega = -3$ (thick, orange), and $\omega = -4$ (dashed, yellow).

5. Continuity Conditions for ST-Bézier Curve Segments

In this section, the continuity conditions are derived for smooth joining of two ST-Bézier curves segments.

Lemma 1 (see [15]). *The necessary and sufficient conditions for smooth joining of two ST-Bézier curve segments* $\alpha(\eta) = \sum_{i=0}^{6} S_i \Phi_i(\eta)$ *with control points* $S_0, \ldots, S_6$ *and shape parameters* μ, ω *and* $\alpha_1(\eta) = \sum_{i=0}^{6} T_i \Phi_i(\eta)$ *with control points* $T_0, \ldots, T_6$ *and shape parameters* $\mu 1, \omega 1$ *via parametric continuity are as follows:*

(1) $S_6 = T_0$, *for* C^0 *continuity*

(2) $S_6 = T_0, \alpha'(\pi/2) = \alpha_1'(0)$, *for* C^1 *continuity*

(3) $S_6 = T_0, \alpha'(\pi/2) = \alpha_1'(0), \alpha''(\pi/2) = \alpha_1''(0)$, *for* C^2 *continuity*

Lemma 2 (see [15]). *The necessary and sufficient conditions for smooth joining of two ST-Bézier curve segments* $\alpha(\eta) = \sum_{i=0}^{6} S_i \Phi_i(\eta)$ *with control points* $S_0, \ldots, S_6$ *and shape parameters* μ, ω *and* $\alpha(\eta) = \sum_{i=0}^{6} T_i \Phi_i(\eta)$ *with control points* $T_0, \ldots, T_6$ *and shape parameters* $\mu 1, \omega 1$ *via geometric continuity are as follows:*

(1) $S_6 = T_0$, *for* G^0 *continuity*

(2) $S_6 = T_0, \alpha' = \lambda \alpha_1'(\pi/2), \lambda > 0$, *for* G^1 *continuity*

(3) $S_6 = T_0, \alpha'(\pi/2) = \lambda \alpha_1'(0), \lambda > 0$ *and the curvature* $\kappa(\pi/2) = |\alpha'(\pi/2) \times \alpha''(\pi/2)| / |\alpha'(\pi/2)|^3 = |\alpha_1'(0) \times \alpha_1''(0)| / |\alpha_1'(0)|^3 = \kappa_1(0)$, *for* G^2 *continuity*

Theorem 3. *Suppose that any two adjacent segments of ST-Bézier curves* $\alpha(\eta) = \sum_{i=0}^{6} S_i \Phi_i(\eta)$ *and* $\alpha(\eta) = \sum_{i=0}^{6} T_i \Phi_i(\eta)$ *with control points* $S_0, \ldots, S_6$ *and* $T_0, \ldots, T_6$ *reach* $C^k, k = 0, 1, 2$ *smooth continuities at joint point if they match the following necessary and sufficient conditions:*

(1) $S_6 = T_0$ *for* C^0 *continuity.*

(2) For C^1 *continuity,*

$$S_6 = T_0, T_1 = \frac{-4S_5 - \omega S_5 + 8S_6 + \mu 1 S_6 + \omega S_6}{4 + \mu 1}. \tag{14}$$

(3) For C^2 *continuity,*

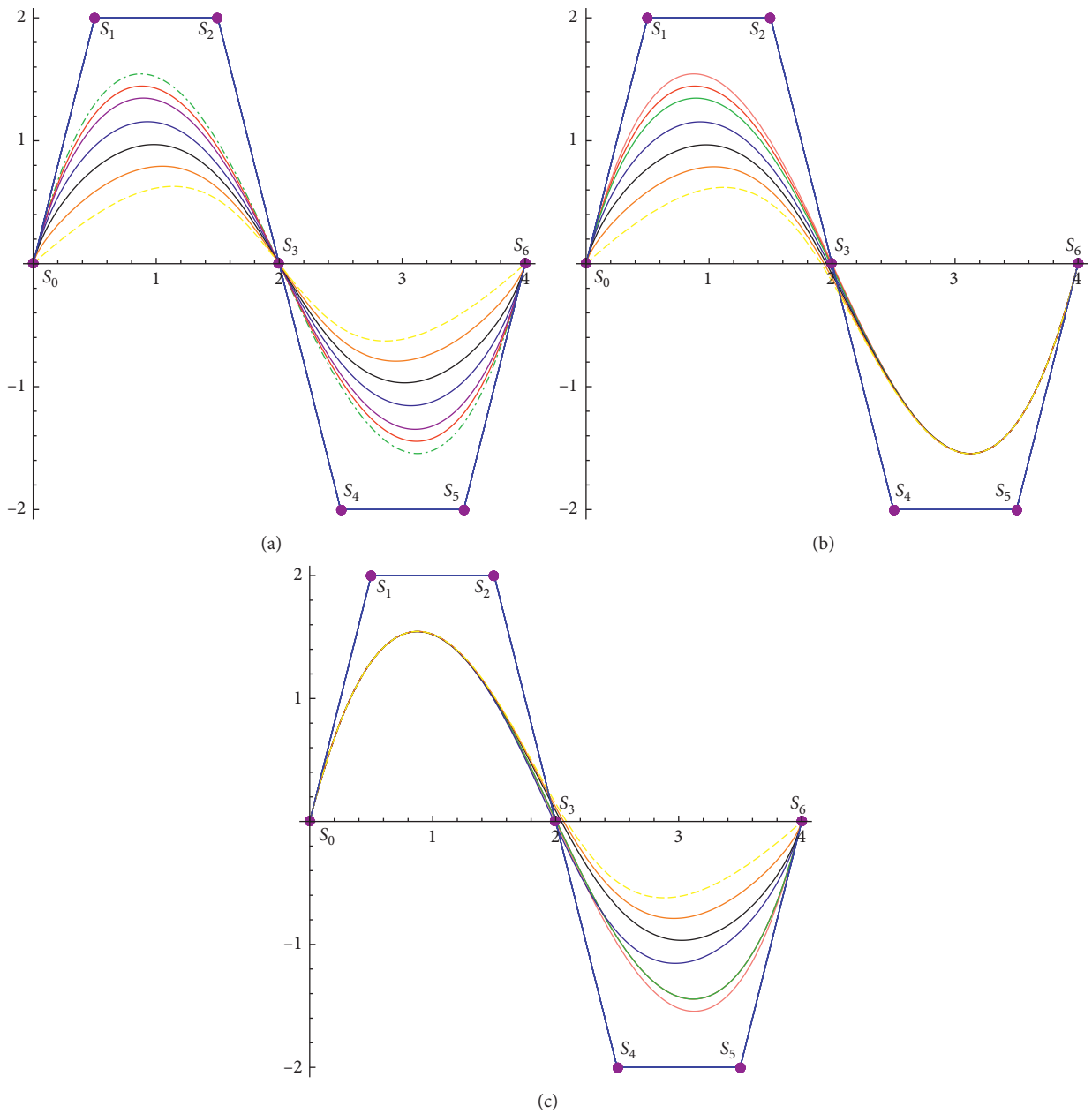


Figure 5: Effect on shape of the curve by changing the shape parameters. (a) $\omega = \mu = 1, 0.5, 0, -1, -2, -3, -4$. (b) $\mu = 1$ and $\omega = 1, 0.5, 0, -1, -2, -3, -4$. (c) $\omega = 1$ and $\mu = 1, 0.5, 0, -1, -2, -3, -4$.

$$
\begin{aligned}
S_6 &= T_0, \\
T_1 &= \frac{-4S_5 - \omega S_5 + 8S_6 + \mu 1 S_6 + \omega S_6}{4 + \mu 1}, \\
T_2 &= \frac{6(4+\mu 1)S_3 - 4(24 + 7\omega + \mu 1(7 + 2\omega))(S_5 - S_6) + \mu 1(4+\mu 1)T_3}{(4+\mu 1)(6+\mu 1)}.
\end{aligned}
\tag{15}
$$

Proof

(1) Using $\alpha(\pi/2) = \alpha_1(0)$, we get $S_6 = T_0$. C^0 continuity is achieved.

(2) For C^1 continuity, we must have C^0 continuity first to get $S_6 = T_0$, and $\alpha'(\pi/2) = \alpha_1'(0)$ can be used to get $T_1 = -4S_5 - \omega S_5 + 8S_6 + \mu 1 S_6 + \omega S_6/4 + \mu 1$.

(3) For C^2 continuity, we must have C^0 continuity and C^1 continuity conditions; additionally the condition $\alpha''(\pi/2) = \alpha_1(0)$ is used to get

$$\begin{aligned} S_6 &= T_0, \\ T_1 &= \frac{-4S_5 - \omega S_5 + 8S_6 + \mu 1 S_6 + \omega S_6}{4 + \mu 1}, \\ T_2 &= \frac{6(4+\mu 1)S_3 - 4(24 + 7\omega + \mu 1(7 + 2\omega))(S_5 - S_6) + \mu 1(4+\mu 1)T_3}{(4+\mu 1)(6+\mu 1)}. \end{aligned} \tag{16}$$

□

5.1. C^0 Continuity of ST-Bézier Curve. For C^0 parametric continuity, we consider any two ST-Bézier curve segments having the same joint point. In this case, the last control point of initial curve and the first control point of second curve are the same. By changing the values of control parameters, we can see the variation in figures. The values of different shape control parameters are used to modify the shape of values; see Figure 6.

5.2. C^1 Continuity of ST-Bézier Curve. For parametric continuity of degree 1, consider two adjacent ST-Bézier curve segments with shape control parameters. In this case, we should have common tangents of the two curve segments at joint point. The first two control points of second curve can be achieved as given in Theorem 3. Figures 7(a)–7(c) can be obtained by varying the shape control parameters via C^1 continuity constraints.

5.3. C^2 Continuity of ST-Bézier Curve. The C^2 continuity can be achieved by connecting two adjacent ST-Bézier curve segments if they fulfill the C^0 and C^1 continuity conditions. The second derivative of these segments at joint point must be the same as that of C^2 continuity. The control points of the first curve can be chosen according to the designer's requirement, while the control points for the second curve can be obtained from Theorem 3. Different values of shape control parameters can be used to obtain different curves as given in Figure 8.

6. Geometric Continuity of ST-Bézier Curve

To get more smoothness between any two adjacent ST-Bézier curves, geometric continuity conditions have been derived. In geometric continuity conditions, one extra parameter is involved, which is used for the modification of the curve.

Theorem 4. *Suppose that two adjacent segments of ST-Bézier curve $\alpha(\eta) = \sum_{i=0}^{6} S_i \Phi_i(\eta)$ and $\alpha(\eta) = \sum_{i=0}^{6} T_i \Phi_i(\eta)$ with control points $S_0, \ldots, S_6$ and $T_0, \ldots, T_6$, respectively, can be connected via $G^k, k = 0, 1, 2$ smooth continuities at joint point if they match the following necessary and sufficient conditions:*

(1) $S_6 = T_0$, for G^0 continuity.

(2) $S_6 = T_0, T_1 = -4S_5 - \omega S_5 + 4S_6 + \omega S_6 + 4\lambda S_6 + \mu 1 \lambda S_6/(4+\mu 1)\lambda$, for G^1 continuity.

(3) $S_6 = T_0, T_1 = -4S_5 - \omega S_5 + 4S_6 + \omega S_6 + 4\lambda S_6 + \mu 1 \lambda S_6/(4+\mu 1)\lambda$,

$$\begin{aligned} T_2 = \Bigg\{ &\frac{1}{2(4+\mu 1)(6+\mu 1)\lambda^3} \Big(12(4+\mu 1)\lambda S_3 + \big((4+\mu 1)(4+\omega)\beta - 8(4+\mu 1)(3+\omega)\lambda - 8(3+\mu 1)(4+\omega)\lambda^2\big) S_5 \\ &+ \left(-(4+\mu 1)(4+\omega)\beta + 4\lambda \begin{pmatrix} (4+\mu 1)(3+2\omega) \\ +2(3+\mu 1)(4+\omega)\lambda + 3(4+\mu 1)\lambda^2 \end{pmatrix} \right) S_6 + 2\mu 1(4+\mu 1)\lambda^3 T_3 \Bigg), \end{aligned} \tag{17}$$

for G^2 continuity.

Proof

1. For G^0 continuity, $\alpha(\pi/2) = \alpha_1(0)$ yields $S_6 = T_0$.

(2) For G^1 continuity, the expressions $\alpha(\pi/2) = \alpha_1(0)$ and $\alpha(\pi/2) = \lambda\alpha'(0), \lambda > 0$, yield $S_6 = T_0$, $T_1 = -4S_5 - \omega S_5 + 4S_6 + \omega S_6 + 4\lambda S_6 + \mu 1 \lambda S_6/(4+\mu 1)\lambda$.

(3) For G^2 continuity, as we know that $\alpha(\pi/2) = S_6 = T_0 = \alpha_1(0), \alpha'(\pi/2) = \lambda\alpha_1'(0), \lambda > 0$, and the reverse normal vector $D = \alpha'(\pi/2) \times \alpha''(\pi/2)$ of $\alpha(\eta)$ and vice normal vector $D_1 = \alpha_1'(0) \times \alpha_1''(0)$ of $\alpha_1(\eta)$ in $\eta = \pi/2$ have the same

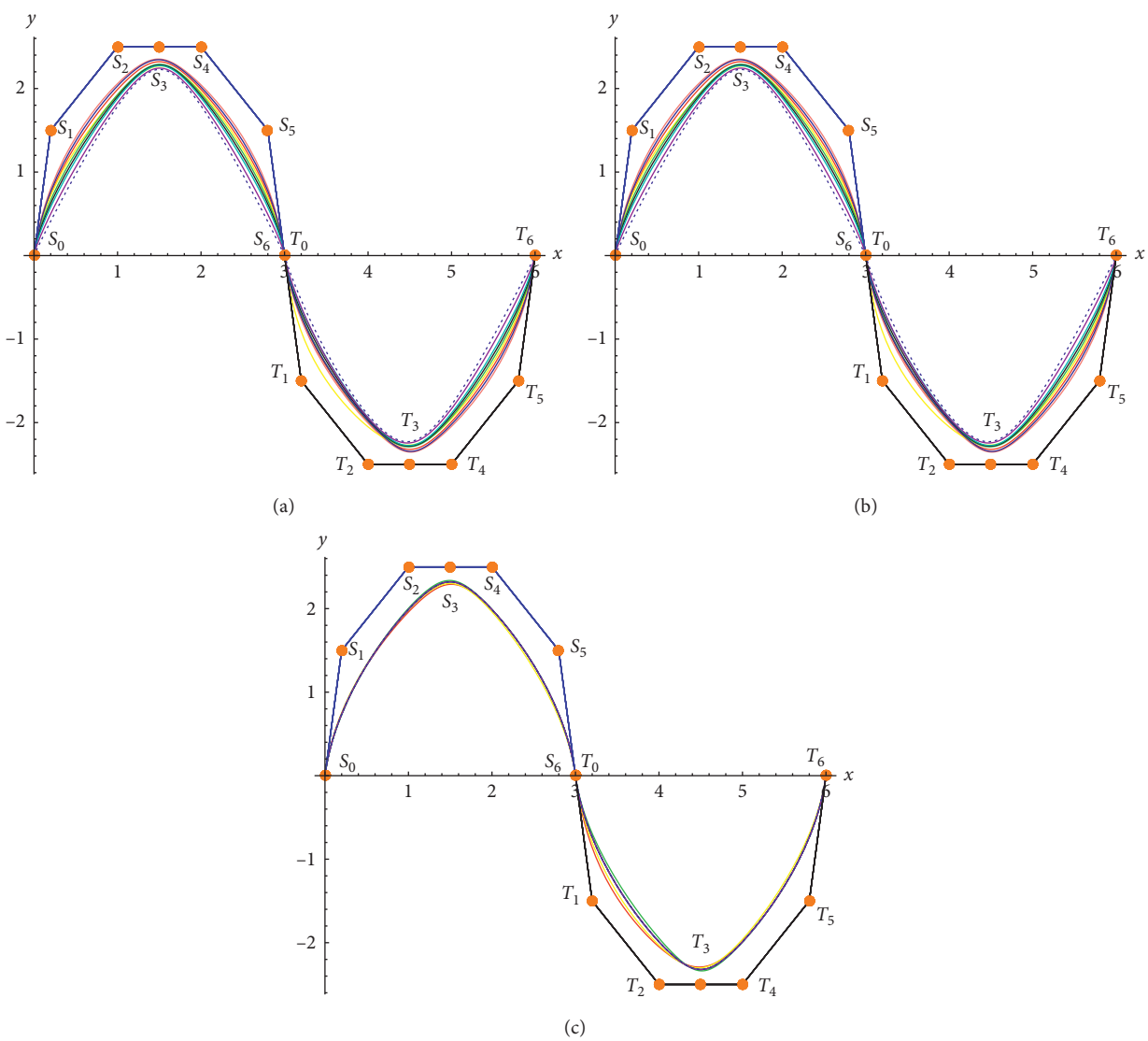


Figure 6: C^0 continuity of ST-Bézier curve segments with various shape parameters.

direction, the four vectors $\alpha'(\pi/2), \alpha'(0), \alpha''(\pi/2), \alpha''(0)$ are coplanar. So, we consider $\alpha''(\pi/2) = \mu\alpha_1''(0) + \beta\alpha_1'(0)$, where $\mu, \beta > 0$ are arbitrarily constants. Since the curvatures at the final point of the first curve and at the initial point of the second curve are the same,

$$\kappa\left(\frac{\pi}{2}\right) = \frac{\left|\alpha'(\pi/2) \times \alpha''(\pi/2)\right|}{\left|\alpha'(\pi/2)\right|^3} = \frac{\lambda\mu\left|\alpha_1'(0) \times \alpha_1''(0)\right|}{\lambda^3\left|\alpha_1'(0)\right|^3} = \frac{\left|\alpha_1'(0) \times \alpha_1''(0)\right|}{\left|\alpha_1'(0)\right|^3} = \kappa_1(0), \tag{18}$$

where $\mu = \lambda^2$, to meet the G^2 continuity conditions. □

6.1. G^1 Continuity of ST-Bézier Curve. C^0 and G^0 both have the same significance. For G^1 continuity, the tangents of the first curve and the second curve at joint point are the same. The parameter λ is any positive scale factor which is used for modification of curves. In Figure 9, by having different values of shape control parameters, the variation in figures can be seen.

6.2. G^2 Continuity of ST-Bézier Curve. The geometric continuity of degree 2 between any two adjacent ST-Bézier curves is given here. Different figures display the behavior of various shape parameters and scale factors. In Figures 9(a)–9(c), the shape parameters for both curves vary, while the scale factors remain fixed. In Figures 10(a)–10(d), the shape

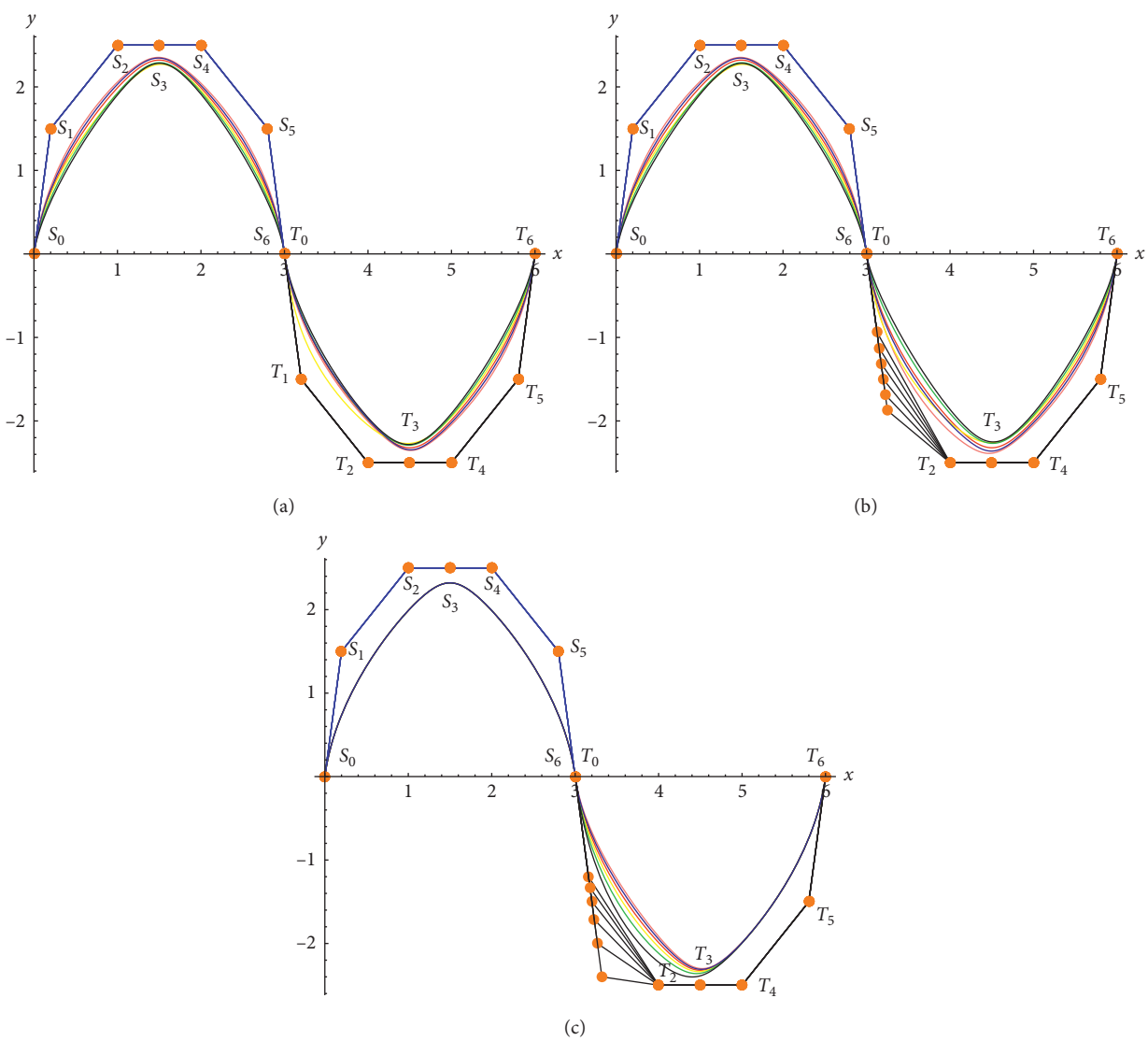


FIGURE 7: C^1 continuity of ST-Bézier curve segments with various shape parameters. (a) $\mu = \omega = \mu 1 = \omega 1, 0, 0.5, 1, -0.5, -1, -1.5$. (b) $\mu = \omega = 0, 0.5, 1, 0.5, -1, -1.5, \mu 1 = \omega 1 = 0$. (c) $\mu 1 = \omega 1 = 0, 0.5, 1, -0.5, -1, -1.5, \mu 1 = \omega 1 = 0$.

parameters for both curves remain the same, while by changing the scale factors of the second curve the variation in the figures is obvious.

7. Construction of Some Closed and Open Curves

Modeling and shape designing play a very important role in CAGD/CAD. ST-Bézier curves are also very useful for modeling and construction. So, by using ST-Bézier curves with various shape control parameters, we can construct some closed curves as shown in Figure 11. In Figure 11(a), closed curves are generated by using ST-Bézier curve when $\mu = \omega = 1$ (dot dashed, green), 0.5 (thick, red), 0 (thick, purple), −1 (thick, blue), −2 (thick, black), −3 (thick, orange), and −4 (dashed, yellow). Similarly, in Figures 11(b) and 11(c), some closed curves are obtained by continuity conditions when $\mu = \omega = 1$ (dot dashed, green), $\omega = 1$, $\mu = 0.5$ (thick, red), $\omega = 1$, $\mu = 0$ (thick, purple), $\omega = 1, \mu = 1$ (thick, blue), $\omega = 1$, $\mu = -2$ (thick, black), $\omega = 1$, $\mu = -3$ (thick, orange), and $\omega = 1, \mu = -4$ (dashed, yellow). By having different values of shape parameters, the influence can be seen as in Figures 11(a)–11(c). By using ST-Bézier curves with various shape control parameters, we can construct some open curves as shown in Figure 12. In Figure 12(a), the open curve is generated by using ST-Bézier curve when $\mu = \omega = 1$ (dot dashed, green), 0.5 (thick, red), 0 (thick, purple), −1 (thick, blue), −2 (thick, black), −3 (thick, orange), and −4 (dashed, yellow). Similarly, in Figures 12(b) and 12(c), some open curves are obtained by continuity conditions when $\mu = \omega = 1$ (dot dashed, green), $\omega = 1$, $\mu = 0.5$ (thick, red), $\omega = 1$, $\mu = 0$ (thick, purple), $\omega = 1, \mu = 1$ (thick, blue), $\omega = 1$, $\mu = -2$ (thick, black), $\omega = 1$, $\mu = -3$ (thick, orange), and $\omega = 1, \mu = -4$ (dashed, yellow). By having different values of shape parameters, the influence can be seen as in Figures 12(a)–12(c).

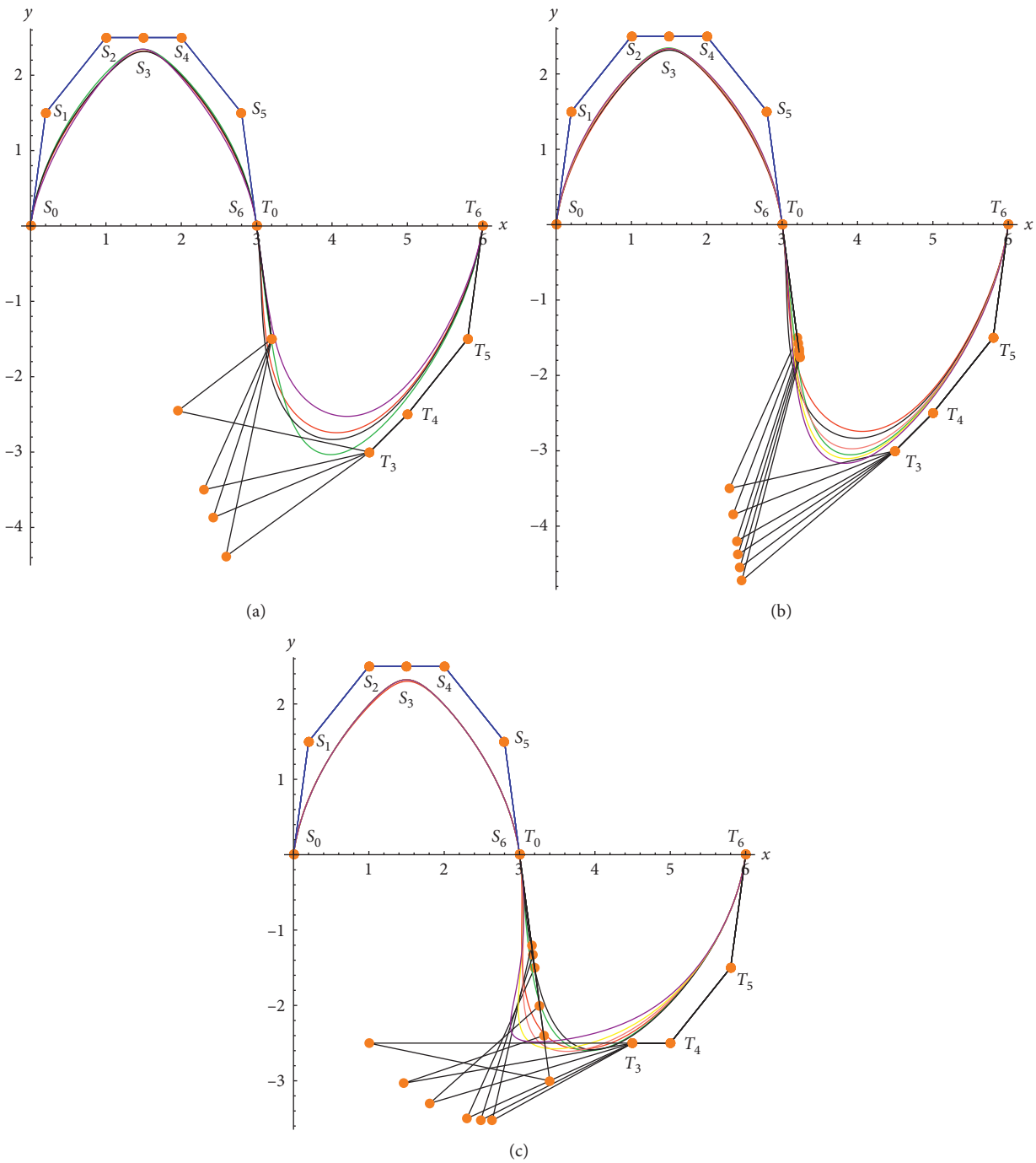


FIGURE 8: C^2 continuity of ST-Bézier curve segments with various shape parameters. (a) $\mu = \omega = \mu 1 = \omega 1 = 0, 0.5, 0.2, -0.5$. (b) $\mu = \omega = 0, 0.2, 0.4, 0.5, 0.6, 0.7, \mu 1 = \omega 1 = 1$. (c) $\mu = \omega = 0, \mu 1 = \omega 1 = 0, 0.5, 1, -1, -1.5, -2$.

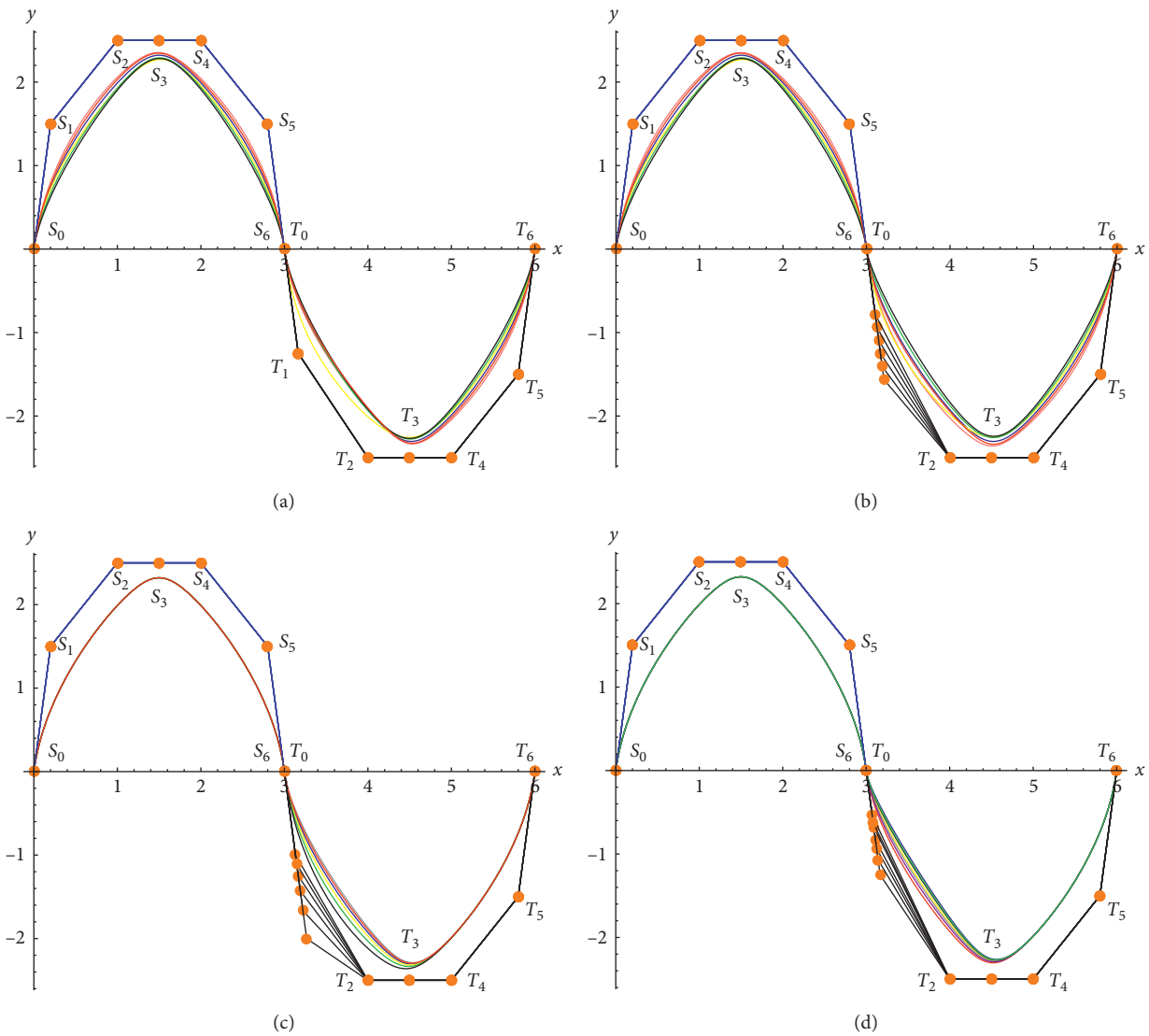


Figure 9: G^1 continuity of ST-Bézier curve segments with various shape parameters. (a) $\mu = \omega = \mu 1 = \omega 1 = 0, 0.5, 1, -1, -1.5, -2, \lambda = 1.2$. (b) $\mu = \omega = 0, 0.5, 1, -0.5, -1, -1.5, \mu 1 = \omega 1 = 0, \lambda = 1.2$. (c) $\mu 1 = \omega 1 = 0, 0.5, 1, -0.5, -1, 1.5, \mu = \omega = 0, \lambda = 1.2$. (d) $\mu 1 = \omega 1 = \mu = \omega = 0, \lambda = 1.2, 1.4, 1.6, 1.8, 2.2, 2.4, 2.8$.

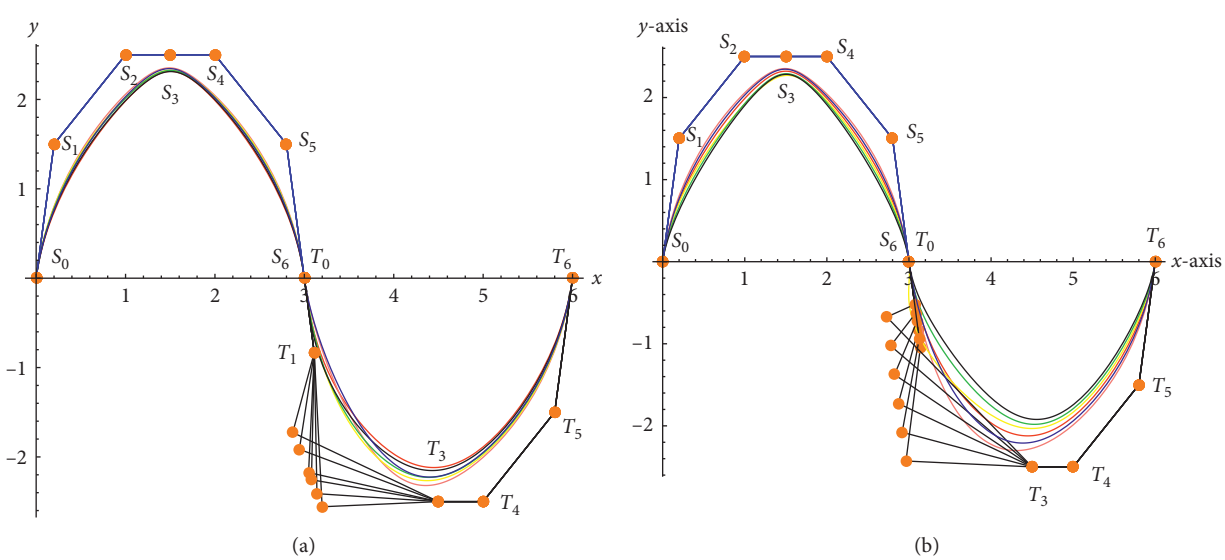


Figure 10: Continued.

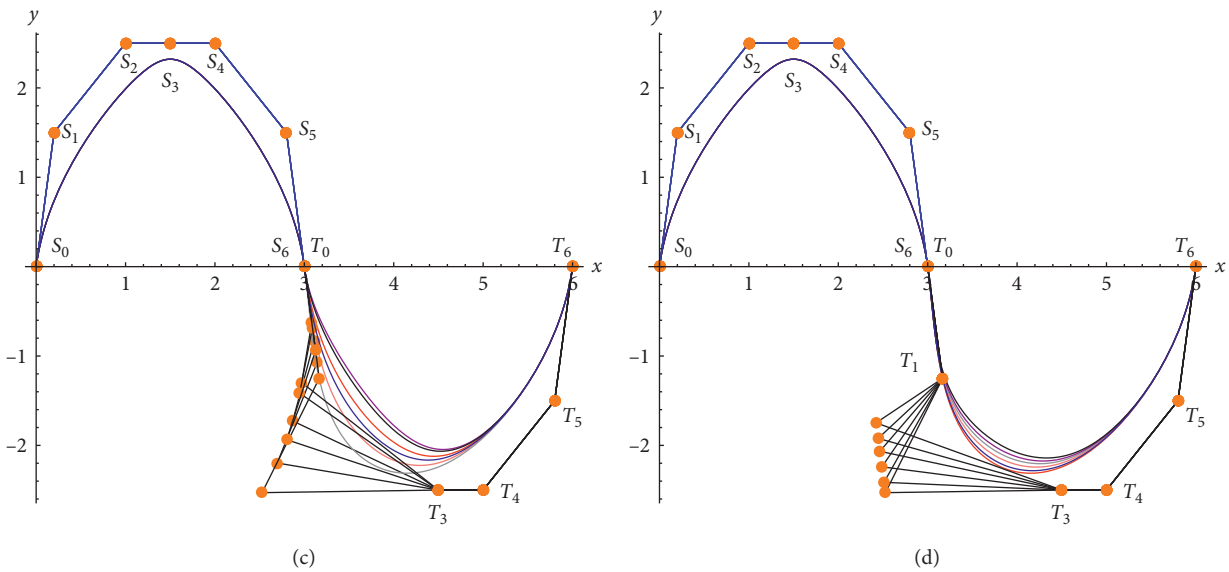


Figure 10: G^2 continuity of ST-Bézier curve segments with various shape parameters. (a) $\mu = \omega = \mu 1 = \omega 1 = 0, 0.2, 0.5, 0.6, 0.8, \lambda = 1.8, \beta = 1.1$. (b) $\mu 1 = \omega 1 = 0, \lambda = 1.8, \beta = 1.1, \mu = \omega = 0, 0.2, 0.5, 0.6, 0.8$. (c) $\mu = \omega = \mu 1 = \omega 1 = 0, \beta = 1.1, \lambda = 1.2, 1.4, 1.6, 1.8, 2.2, 2.4$. (d) $\mu = \omega = \mu 1 = \omega 1 = 0, \beta = 1.1, 1.5, 2.1, 2.7, 3.2, 3.8, \lambda = 1.2$.

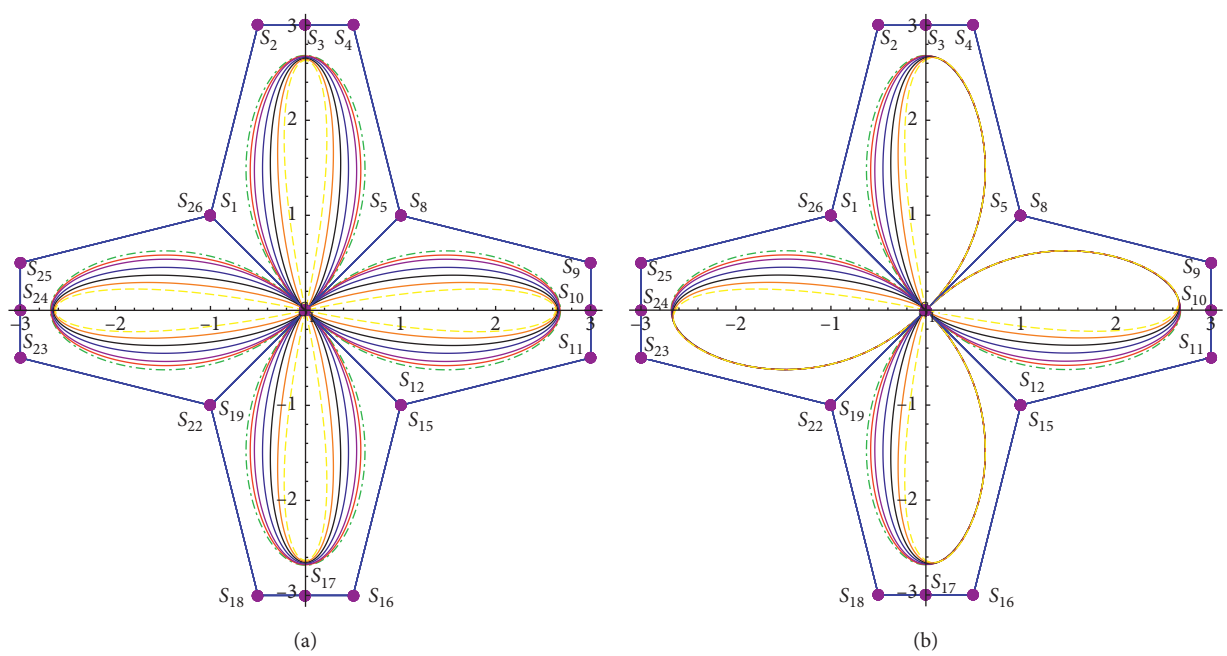
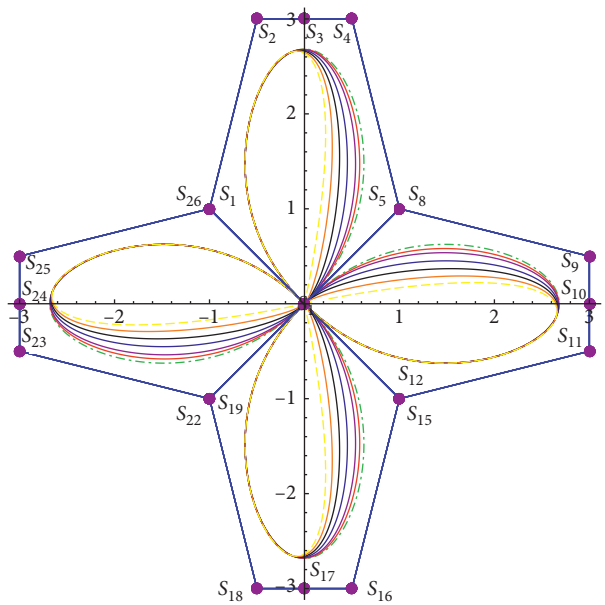
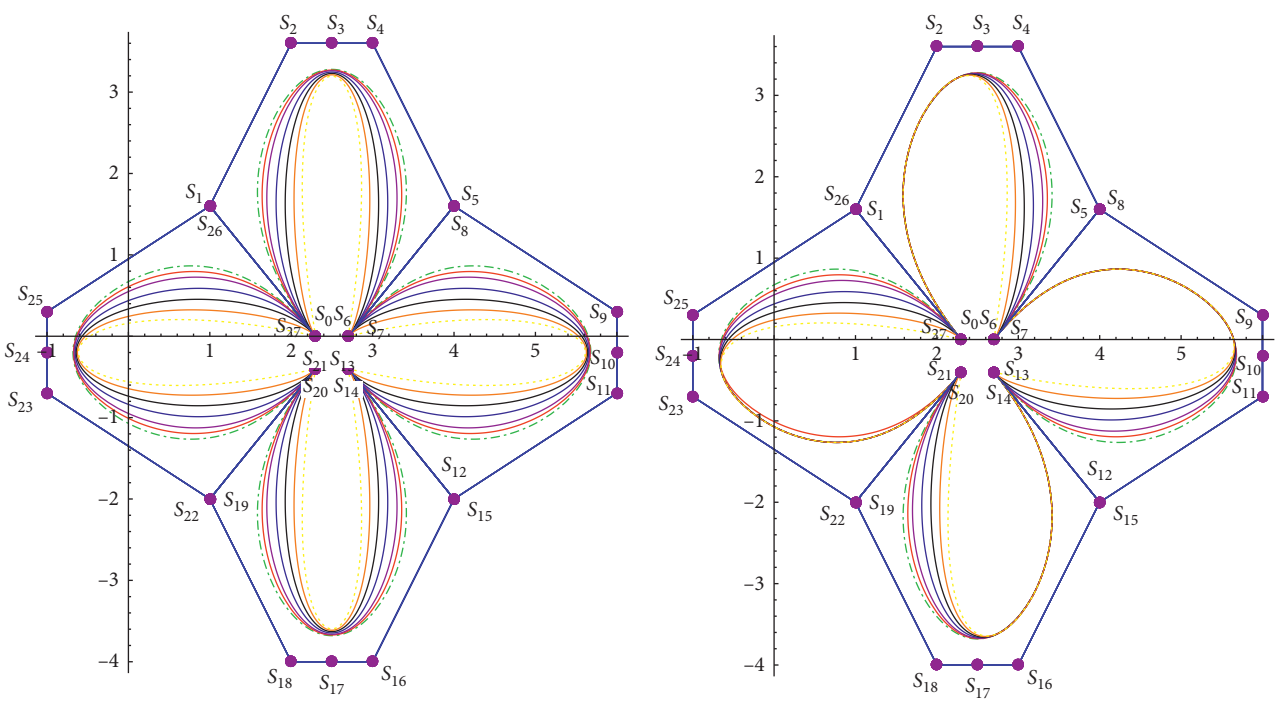


Figure 11: Continued.



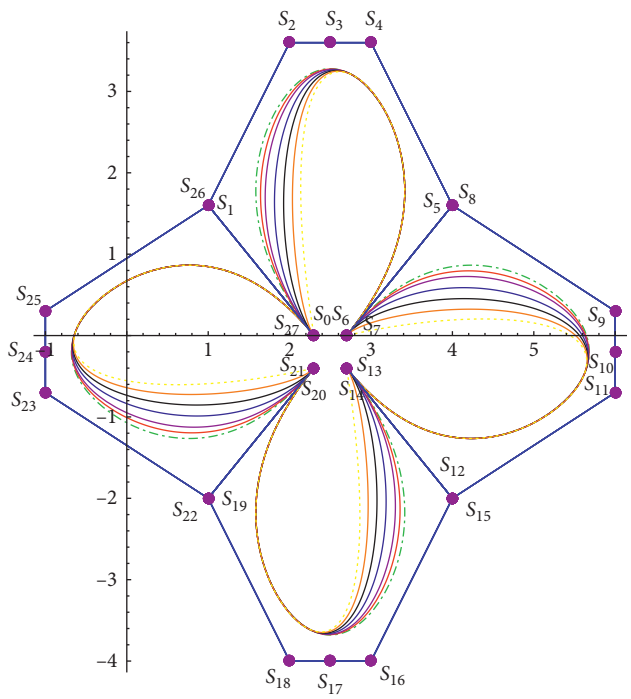
(c)

Figure 11: Construction of some closed curves by different values of shape parameters. (a) $\mu = \omega = 1, 0.5, 0, -1, -2, -3, -4$. (b) $\omega = 1, 0.5, 0, -1, -2, -3, -4, \mu = 1$. (c) $\mu = 1, 0.5, 0, -1, -2, -3, -4, \omega = 1$.



(a) (b)

Figure 12: Continued.



(c)

Figure 12: Construction of some open curves by different values of shape parameters. (a) $\mu = \omega = 1, 0.5, 0, -1, -2, -3, -4$. (b) $\mu = 1, 0.5, 0, -1, -2, -3, -4, \omega = 1$. (c) $\omega = 1, 0.5, 0, -1, -2, -3, -4, \mu = 1$.

8. Conclusions

In this research, a newly constructed ST-Bernstein basis and Bézier curve with two shape parameters has been proposed. It can be concluded that its geometric properties are similar to those of the classical Bézier curve. The shape of the curve can be regulated by changing the values of shape parameters. The suggested curve can be used to create open and closed curves with different values of shape parameter. The parametric and geometric continuities for two adjacent ST-Bézier curves are also presented, which demonstrate the efficiency of adjoining the ST-Bézier curves.

Data Availability

The experimental data used to support the findings of this study are included within the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this study.

Authors' Contributions

All authors equally contributed to this work and read and approved the final manuscript.

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