Research Article

Integral-Type Fractional Equations with a Proportional Riemann–Liouville Derivative

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In this paper, we present the necessary conditions where integral-type fractional equations with a proportional Riemann–Liouville derivative have a unique solution. Also, we give an example to illustrate our work.

1. Introduction

Lately, many researchers have been focusing on the study of various types of fractional problems; we refer the reader to [1–17]. The fixed point and the monotone iterative techniques can be very useful tools to prove the existence and

uniqueness of a solution to this type of problems; see [1]. In this manuscript, inspired by the work of Jankowski in [1], we investigate the existence and uniqueness of a solution to the following problem:

$$\begin{aligned} D^{\alpha, \rho} \xi(t) &= g\left(t, \xi(t), \int_0^t \mathcal{K}(t, \tau) \xi(\tau) d\tau\right) \equiv \mathcal{F}\xi(t), \quad t \in J_0 = (0, a]; a > 0, \\ \tilde{\xi}(0) &= p, \end{aligned} \quad (1)$$

where $D^{\alpha, \rho} \xi(t)$ denotes a proportional Riemann–Liouville fractional derivative for $\rho \in [0, 1]$ and $0 < \alpha < 1$. Also, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $J = [0, a]$, and $\tilde{\xi}(0) = t^{1-\alpha} e^{(\rho-1/\rho)t} \xi(t)|_{t \rightarrow 0+}$. Now, we remind the reader of the definition of the proportional Riemann–Liouville fractional integral and derivative.

Definition 1 (see [18]). Let $\alpha \in \mathbb{C}; \operatorname{Re}(\alpha) \geq 0$, $0 < \rho \leq 1$, and $t > 0$.

- (i) The following integral is called the proportional Riemann–Liouville fractional integral:

$$I^{\alpha, \rho} f(t) := \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{[\rho-1/\rho](t-\tau)} (t-\tau)^{\alpha-1} f(\tau) d\tau. \quad (2)$$

- (ii) The following derivative is called the proportional Riemann–Liouville fractional derivative:

$$\begin{aligned} (D^{\alpha, \rho} f)(t) &= D_t^{n, \rho} I^{n-\alpha, \rho} f(t) \\ &= \frac{D_t^{n, \rho}}{\rho^{n-\alpha} \Gamma(n-\alpha)} \int_0^t e^{[\rho-1/\rho](t-\tau)} (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \end{aligned} \quad (3)$$

where $n = [\operatorname{Re}(\alpha)] + 1$ and $D_t^{1-\rho} = (1-\rho)f(t) + \rho f'(t)$.

Next, we present the following proposition.

Proposition 1 (see [18]). *If $\alpha, \gamma \in \mathbb{C}$, where $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\gamma) > 0$, then for any $0 < \rho \leq 1$, we have $I^{\alpha, \rho}(t^{\gamma-1} e^{(\rho-1/\rho)t})(x) = (\Gamma(\gamma)/\rho^\alpha \Gamma(\alpha + \gamma)) x^{\gamma+\alpha-1} e^{(\rho-1/\rho)x}$.*

In Section 2, we prove the existence and uniqueness of a solution to problem (1) using the fixed point technique. In Section 3, we prove the existence and uniqueness of a solution to problem (1) using the monotone iterative method. In the conclusion, we present an open question.

2. Fixed Point Approach

First of all, let $C_{1-\alpha}(J, \mathbb{R}) = \{f \in C([0, a], \mathbb{R}) | t^{1-\alpha} f \in C(J, \mathbb{R})\}$. Now, define the following two weighted norms:

$$\begin{aligned} \|f\|^* &= \max_{[0, a]} t^{1-\alpha} |f(t)|, \\ \|f\|_* &= \max_{[0, a]} t^{1-\alpha} e^{-\lambda t} |f(t)| \text{ for fixed } \lambda > 0. \end{aligned} \quad (4)$$

Theorem 1. *Let $0 < \alpha < 1, 0 < \rho \leq 1$, and $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathcal{K} \in C(J \times J, \times \mathbb{R})$. Let $\beta = (\rho - 1/\rho)$. Also, assume the following two hypotheses:*

(1) *There exist nonnegative constants H, V , and W such that $|\mathcal{K}(t, s)| < H$ and*

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq V|v_1 - u_1| + W|v_2 - u_2|. \quad (5)$$

(2) $b \equiv (a^\alpha/\Gamma(2\alpha)\rho^\alpha) [V + (HWa/2\alpha)] < 1$, for $\alpha \in (0, (1/2)]$.

Then, initial value problem (1) has a unique solution.

Proof. First, let $S\xi(t) = t^{1-\alpha} e^{(\rho-1/\rho)t} p + (1/\rho^\alpha \Gamma(\alpha)) \int_0^t e^{[(\rho-1/\rho)(t-\tau)]} (t-\tau)^{\alpha-1} \mathcal{F}\xi(\tau) d\tau$. Note that if S has a unique fixed point and that is $S\xi(t) = \xi(t)$, then initial value problem (1) has a unique solution, i. e., it will be enough to show that S is a contraction map. So, let $\xi, \mathcal{Y} \in C_{1-\alpha}(J, \mathbb{R})$; we have two cases:

Case 1: $\alpha \in (0, (1/2)]$.

$$\begin{aligned} S\xi(t) &= t^{1-\alpha} e^{\beta t} p + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \mathcal{F}\xi(\tau) d\tau, \\ \|S\xi - S\mathcal{Y}\|^* &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \max_{t \in J} t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} |\mathcal{F}\xi(\tau) - \mathcal{F}\mathcal{Y}(\tau)| d\tau \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \max_{t \in J} t^{1-\alpha} \left\{ \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V|\xi(\tau) - \mathcal{Y}(\tau)| + W \int_0^\tau |\mathcal{K}(t, s)\xi(s) - \mathcal{Y}(s)| ds \right] d\tau \right\} \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|^* \max_{t \in J} \left\{ t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V\tau^{\alpha-1} + HW \int_0^\tau s^{\alpha-1} ds \right] d\tau \right\} \\ &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|^* \max_{t \in J} t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V\tau^{\alpha-1} + HW \frac{\tau^\alpha}{\alpha} \right] d\tau \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|^* \max_{t \in J} t^{1-\alpha} e^{-\beta t} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V e^{\beta\tau} \tau^{\alpha-1} + HW e^{\beta\tau} \frac{\tau^\alpha}{\alpha} \right] d\tau \\ &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|^* \max_{t \in J} t^{1-\alpha} e^{-\beta t} \left[I^{\alpha, \rho}(V t^{\alpha-1} e^{\beta t}) + I^{\alpha, \rho}\left(\frac{HW}{\alpha} t^\alpha e^{\beta t}\right) \right] \\ &= \frac{\Gamma(\alpha) a^\alpha}{\Gamma(\alpha) \Gamma(2\alpha) \rho^\alpha} \left[V + \frac{HWa}{2\alpha} \right] \|\xi - \mathcal{Y}\|^* \\ &= b \|\xi - \mathcal{Y}\|^*. \end{aligned} \quad (6)$$

Hence, S is a contraction map. Therefore, S has a unique fixed point as desired.

Case 2: $\alpha \in ((1/2), 1)$; in this case, we use $\|\cdot\|_*$ with the positive constant $\lambda > 0$ such that

$$\sqrt{\lambda - \beta} > b_1 \equiv \frac{e^{-\beta a} (V\alpha + HWa) \Gamma(2\alpha - 1) \sqrt{a^{2\alpha-1}}}{\alpha \rho^\alpha \Gamma(\alpha) \sqrt{\Gamma(2(2\alpha - 1))}}. \quad (7)$$

It is not difficult to see the following:

$$(1) \int_0^t e^{2(\lambda-\beta)\tau} d\tau \leq (e^{2(\lambda-\beta)t}/2(\lambda-\beta))$$

$$(2) t^{1-\alpha} \sqrt{\int_0^t (t-\tau)^{2(\alpha-1)} \tau^{2(\alpha-1)} d\tau} = (\Gamma(2\alpha-1) \sqrt{a^{2\alpha-1}} / \sqrt{\Gamma(2(2\alpha-1))})$$

Also, recall the Schwarz inequality for integrals:

$$\begin{aligned} \int_0^t |f(\tau)g(\tau)| d\tau &\leq \sqrt{\int_0^t f^2(\tau) d\tau} \sqrt{\int_0^t g^2(\tau) d\tau}, \\ \|S\xi - S\mathcal{Y}\|_* &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \max_{t \in J} t^{1-\alpha} e^{-\lambda t} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} |\mathcal{F}\xi(\tau) - \mathcal{F}\mathcal{Y}(\tau)| d\tau \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} [V e^{\lambda\tau} \tau^{\alpha-1}] + HW e^{\lambda\tau} \int_0^\tau s^{\alpha-1} ds d\tau \right\} \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V \tau^{\alpha-1} e^{\lambda\tau} + HW \frac{\tau^\alpha}{\alpha} e^{\lambda\tau} \right] d\tau \right\} \\ &\leq \frac{V\alpha + HWa}{\alpha \rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \int_0^t e^{\beta(t-\tau)} e^{\lambda\tau} (t-\tau)^{(\alpha-1)} \tau^{\alpha-1} d\tau \right\} \\ &= \frac{V\alpha + HWa}{\alpha \rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \int_0^t e^{\beta(t-\tau)+\lambda\tau} (t-\tau)^{(\alpha-1)} \tau^{\alpha-1} d\tau \right\} \\ &\leq \frac{V\alpha + HWa}{\alpha \rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \times \sqrt{\int_0^t (t-\tau)^{2(\alpha-1)} \tau^{2(\alpha-1)} d\tau} \sqrt{\int_0^t e^{2\beta(t-\tau)+2\lambda\tau} d\tau} \right\} \\ &\leq \frac{V\alpha + HWa}{\alpha \rho^\alpha \Gamma(\alpha)} \|\xi - \mathcal{Y}\|_* \max_{t \in J} \left\{ t^{1-\alpha} e^{-\lambda t} \times \sqrt{\int_0^t (t-\tau)^{2(\alpha-1)} \tau^{2(\alpha-1)} d\tau} \sqrt{\int_0^t e^{2(\lambda-\beta)\tau} d\tau} \right\} \\ &\leq \frac{b_1}{\sqrt{\lambda-\beta}} \|\xi - \mathcal{Y}\|_*. \end{aligned} \quad (8)$$

Thus, S is a contraction map. Therefore, S has a unique fixed point as required. $\square$

As an application to Theorem 1, consider the following problem:

$$\begin{aligned} D^{\alpha,\rho} y(t) &= -L(t)y(t) + \xi(t), \\ \tilde{\xi}(0) &= r. \end{aligned} \quad (9)$$

If

$$\frac{a^\alpha}{\rho^\alpha \Gamma(2\alpha)} \max_{t \in J} |L(t)| < 1, \quad \text{for } 0 < \alpha \leq \frac{1}{2}, \quad (10)$$

then it is not difficult to see that, by using Theorem 1, problem (9) has a unique solution. In closing of this section, the following linear problem is considered:

$$\begin{aligned} D^{\alpha,\rho} \xi(t) &= -L(t)\xi(t) + z(t), \quad t \in J_0, \\ \tilde{\xi}(0) &= r. \end{aligned} \quad (11)$$

Now, we introduce the following hypothesis.

Hypothesis 1 (H_1)

- (1) $L(t) = L$, $t \in J$ or
- (2) The function L is nonconstant on J and

$$\frac{a^\alpha}{\rho^\alpha \Gamma(2\alpha)} \max_{t \in J} |L(t)| < 1 \quad \text{only if } \alpha \in \left(0, \frac{1}{2}\right). \quad (12)$$

The following lemma is a consequence of Theorem 1.

Lemma 1. *If $\alpha \in (0, 1)$, $L \in C(J, \mathbb{R})$, $z \in C_{1-\alpha}(J, \mathbb{R})$, and hypothesis (H_1) holds, then problem (11) has a unique solution.*

We would like to bring to the reader's attention that, in [1], in the hypothesis ρ should be as follows: $\rho \equiv (T^q/\Gamma(2q))[K + (WLT/2q)]$ which he used to prove the case where $q \in (0, (1/2)]$. This way, his result will be stronger or he can just change the last equality to the inequality.

3. Monotone Iterative Method

First of all, we start by introducing the following hypothesis.

Hypothesis 2 (H_2)

- (1) $L(t) = L$, $t \in J$ or
- (2) The function L is nonconstant, and if $L(t)$ is negative, then there exists $\bar{L}$ which is nondecreasing, where $-L(t) \leq \bar{L}(t)$ on J and for every $x \in J$, we have

$$\frac{e^{\beta x}}{\rho^\alpha \Gamma(\alpha)} \int_0^a (a-\tau)^{\alpha-1} e^{\beta(a-\tau)} \bar{L}(\tau) d\tau < 1. \quad (13)$$

Now, for our purpose, we prove the following useful lemma.

Lemma 2. Let $\alpha \in (0, 1)$ and $L \in C(J, [0, \infty))$ or $L \in C(J, (-\infty, 0])$. Also, denote by $\beta := (\rho - 1/\rho)$. Assume that $q \in C_{1-\alpha}(J, \mathbb{R})$ is a solution to the following problem:

$$\begin{aligned} D^{\alpha, \rho} q(t) &\leq -L(t)q(t), \quad t \in J_0, \\ \tilde{q}(0) &< 0. \end{aligned} \quad (14)$$

If (H_2) holds, then $q(t) \leq 0$ for all $t \in J$.

Proof. Assume that our lemma is false, that is, there exist $x, y \in [0, a]$ such that $q(x) = 0$, $q(y) > 0$, and $q(t) \leq 0$ for $t \in (0, x]$; $q(t) > 0$ for $t \in (x, y]$. Let x_0 be the first maximal point of q on $[x, y]$.

Case 1: assume that $L(t) \geq 0$ for all $t \in J$. Thus, $D^{\alpha, \rho} q(t) \leq 0$ for $t \in [x, y]$. Hence,

$$\int_x^{x_0} D^{\alpha, \rho} q(t) dt \leq 0. \quad (15)$$

Therefore, $B \equiv I^{\rho, 1-\alpha} q(x_0) - I^{\rho, 1-\alpha} q(x) \leq 0$, but

$$\begin{aligned} B &= \frac{1}{\rho^{1-\alpha} \Gamma(1-\alpha)} \left[\int_0^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau) d\tau - \int_0^x e^{\beta(x-\tau)} (x-\tau)^{-\alpha} q(\tau) d\tau \right] \\ &= \frac{1}{\rho^{1-\alpha} \Gamma(1-\alpha)} \left\{ \int_0^x \left[e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} - e^{\beta(x-\tau)} (x-\tau)^{-\alpha} \right] q(\tau) d\tau + \int_x^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau) d\tau \right\} \\ &> \frac{1}{\rho^{1-\alpha} \Gamma(1-\alpha)} \int_x^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau) d\tau > 0, \end{aligned} \quad (16)$$

which leads us to a contradiction given the fact that $B \leq 0$.

Case 2: assume that $L(t) \leq 0$ for all $t \in J$, and consider $\bar{L}$ to be nondecreasing on J . Now, if we apply $I^{\alpha, \rho}$ on problem (14), we obtain

$$q(t) - \tilde{q}(0) \frac{e^{\beta t} t^{\alpha-1}}{\rho^{\alpha-1} \Gamma(\alpha)} \leq -I^{\alpha, \rho} [L(t)q(t)], \quad \text{for } t \in [x, x_0]. \quad (17)$$

Notice that $\tilde{q}(0) (e^{\beta t} t^{\alpha-1} / \rho^{\alpha-1} \Gamma(\alpha)) \leq 0$ which is due to the fact that $\tilde{q}(0) \leq 0$. Thus,

$$\begin{aligned}
q(x_0) &\leq -\frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^{x_0} (x_0 - \tau)^{\alpha-1} e^{\beta(x_0-\tau)} L(\tau) q(\tau) d\tau \\
&= -\frac{1}{\rho^\alpha \Gamma(\alpha)} \left[\int_0^{x_0} (x_0 - \tau)^{\alpha-1} e^{\beta(x_0-\tau)} L(\tau) q(\tau) d\tau + \int_x^{x_0} (x_0 - \tau)^{\alpha-1} e^{\beta(x_0-\tau)} L(\tau) q(\tau) d\tau \right] \\
&< -\frac{q(x_0)}{\rho^\alpha \Gamma(\alpha)} \int_0^{x_0} (x_0 - \tau)^{\alpha-1} e^{\beta(x_0-\tau)} L(\tau) d\tau, \text{ let } \sigma = \frac{\tau}{x_0} \\
&= -\frac{q(x_0) e^{\beta x_0} x_0^\alpha}{\rho^\alpha \Gamma(\alpha)} \int_0^1 (1 - \sigma)^{\alpha-1} e^{\beta(1-\sigma)} L(\sigma x_0) d\sigma \\
&\leq \frac{q(x_0) e^{\beta x_0} x_0^\alpha}{\rho^\alpha \Gamma(\alpha)} \int_0^1 (1 - \sigma)^{\alpha-1} e^{\beta(1-\sigma)} \tilde{L}(\sigma a) d\sigma \\
&= \frac{q(x_0) e^{\beta(x_0+(1/a))} x_0^\alpha}{\rho^\alpha \Gamma(\alpha) a^\alpha} \int_0^a (a - \tau)^{\alpha-1} e^{\beta(a-\tau)} \tilde{L}(\tau) d\tau \\
&\leq \frac{q(x_0) e^{\beta(x_0+(1/a))}}{\rho^\alpha \Gamma(\alpha)} \int_0^a (a - \tau)^{\alpha-1} e^{\beta(a-\tau)} \tilde{L}(\tau) d\tau.
\end{aligned} \tag{18}$$

Hence, $q(x_0)[1 - (e^{\beta(x_0+(1/a))}/\rho^\alpha \Gamma(\alpha)) \int_0^a (a - \tau)^{\alpha-1} \tilde{L}(\tau) d\tau] \leq 0$. Using hypothesis (H_2) implies that $q(x_0) \leq 0$, which leads us to a contradiction, and this concludes our proof. $\square$

We say that y is a lower solution of problem (1) if

$$D^{\alpha,\rho} y(t) \leq \mathcal{F}y(t), \quad t \in J_0, \tilde{y}(0) \leq 0, \tag{19}$$

and we say that y is an upper solution of problem (1) if

$$D^{\alpha,\rho} y(t) \geq \mathcal{F}y(t), \quad t \in J_0, \tilde{y}(0) \geq 0. \tag{20}$$

Next, the following hypothesis is defined.

Hypothesis 3. (H_3) . There exists a function $L \in C(J, \mathbb{R})$ where

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq L(t)|v_1 - u_1| \quad \text{whenever } x_0 \leq u_1 \leq v_1 \leq y_0 \text{ and } u_2 \leq v_2. \tag{21}$$

Theorem 2. Assume that x_0 is a lower solution of problem (1) and y_0 is an upper solution of problem (1), where $x_0, y_0 \in C_{1-\alpha}(J, \mathbb{R})$. Moreover, assume that hypotheses H_1, H_2 , and H_3 hold; problem (1) has solutions in $[x_0, y_0] = \{y \in C_{1-\alpha}(J, \mathbb{R}) | x_0(t) \leq y(t) \leq y_0(t), t \in J_0, \tilde{x}_0(0) \leq \tilde{y}(0) \leq \tilde{y}_0(0)\}$.

Proof. Using Lemmas 1 and 2, the proof is similar to the proof of Theorem 2 in [1]. $\square$

Now, we present the following example.

Example 1. Let $0 < \alpha < 1$, $0 < \rho \leq 1$, $\beta = (\rho - 1/\rho)$, and $\mathbb{A}, \mathbb{B} \in C([0, 1], (0, \infty))$ such that $\mathbb{A}(t) \leq \mathbb{B}(t)$ for $t \in [0, 1]$. Now, consider the following problem:

$$\begin{aligned}
D^{\alpha,\rho} \xi(t) &\equiv \mathcal{F}\xi(t), \quad t \in J_0 = (0, 1], \\
\tilde{\xi}(0) &= 0,
\end{aligned} \tag{22}$$

where

$$\mathcal{F}\xi(t) = \frac{\rho^\alpha e^{\beta t} t^{1-\alpha}}{\Gamma(2-\alpha)} + \mathbb{A}(t) [te^{\beta t} - 1 - \xi(t)]^3 + \frac{\beta}{e^{\beta t} - 1} \mathbb{B}(t) \int_0^t [\sin(t\tau)]^4 \xi(\tau) d\tau. \tag{23}$$

Now, let $x_0(t) = 0$ and $y_0(t) = te^{\beta t}$; first, note that $x_0(t)$ is a lower solution of problem (22). Next, we show that $y_0(t)$ is an upper solution of problem (22):

$$\begin{aligned}\mathcal{F}y_0(t) &= \frac{\rho^\alpha e^{\beta t} t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t} - 1} \mathbb{B}(t) \int_0^t [\sin(t\tau)]^4 \tau e^{\beta \tau} d\tau \\ &\leq \frac{\rho^\alpha e^{\beta t} t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t} - 1} \mathbb{B}(t) \int_0^t e^{\beta \tau} d\tau \\ &= \frac{\rho^\alpha e^{\beta t} t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t} - 1} \mathbb{B}(t) \left[\frac{e^{\beta t}}{\beta} - \frac{1}{\beta} \right] < \frac{\rho^\alpha e^{\beta t} t^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= D^{\alpha, \rho} y_0(t).\end{aligned}\tag{24}$$

Thus, $y_0(t)$ is an upper solution of problem (22). Now, it is not difficult to see that all the hypotheses of Theorem 2 are satisfied. Therefore, problem (22) has solutions in $[x_0, y_0]$ if $\alpha \in ((1/2), 1)$, and for $\alpha \in (0, (1/2)]$, we need to assume that $(1/\rho^\alpha \Gamma(2\alpha)) \max_{t \in [0,1]} |\mathbb{A}(t)| < 1$.

4. Conclusion

In closing, note that the results of Jankowski [1] are a special case of our work which is by taking $\rho = 1$. Also, we would like to bring to the reader attention the following open question.

What are the necessary and sufficient conditions for problem (1) to have a unique solution if ρ is not constant, but it is a function of t say $g(t)$, so that the problem involves $D^{\alpha, g(t)}$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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