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Research Article

Integral Criteria for Weighted Bloch Functions

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The present manuscript gives analytic characterizations and interesting technique that involves the study of general $\mathfrak{w}$-Besov classes of analytic functions by the help of analytic $\mathfrak{w}$-Bloch functions. Certain special functions significant in both $\mathfrak{w}$-Besov-norms and $\mathfrak{w}$-Bloch norms framework and to introduce new important families of analytic classes. Interesting motivation of this concerned paper is to construct some new analytic function classes of general $\mathfrak{w}$-Besov-type spaces via integrals on concerned functions view points. The introduced analytic $\mathfrak{w}$-Bloch and $\mathfrak{w}$-Besov type of functions with some interesting properties for these classes of function spaces are established within the constructions of their norms. Using the defined analytic function spaces, various important relations are also derived.

1. Introduction

Operator theory and special classes of holomorphic function spaces have interesting significant roles in different branches of pure and applied mathematics as well as in recent studies of theoretical physics. Some concerned problems arising in operator and measure theories are framed in terms of certain classes of function spaces. Most of these classes can be studied and treated by using numerous kinds of measures and some types of operators which provide and evolve new means of recent studies in mathematical analysis. Besov and Bloch-type classes are the focus of such studies. These types of spaces are extremely applied in computational mathematical analysis and theoretical physics problems. In addition, these special function classes allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of function classes. Throughout this concerned paper, the following notations and definitions will be used.

Let $H(\mathbb{D})$ denote the class of all concerned holomorphic functions on the concerned unit disk $\mathbb{D}$. For $a \in \mathbb{D}$, the known concerned Möbius transformation $\varphi_a(w)$ is symbolized by

$$\varphi_a(w) = \frac{a-w}{1-\bar{a}w}, \text{ with } w \in \mathbb{D}. \tag{1}$$

For a specific point $a \in \mathbb{D}$ and $0 < R < 1$, the supposed pseudo-hyperbolic disk $D(a, R)$ with the supposed center a and supposed radius R is symbolized also by $D(a, R) = \varphi_a(\mathrm{RD})$.

The supposed concerned pseudo-hyperbolic disk $D(a, R)$ can be considered an Euclidean disk: with a specific center and radius being $(1-R^2)a/1-R^2|a|^2$ and $(1-|a|^2)R/1-R^2|a|^2$, respectively ([1]). Let ζ denote the concerned normalized specific area of Lebesgue-type measure on $\mathbb{D}$. We will need the following interesting known identity:

$$1-\left|\varphi_a(w)\right|^2 = \frac{\left(1-|a|^2\right)\left(1-|w|^2\right)}{|1-\bar{a}w|^2} = \left(1-|w|^2\right)\left|\varphi_a'(w)\right|. \tag{2}$$

For $a \in \mathbb{D}$, by concerned usual substitution $w = \varphi_a(w)$, we must consider the known Jacobian change in the concerned estimates given by $\mathrm{d}\zeta(w) = |\varphi_a'(w)|^2 \mathrm{d}\zeta(w)$. One should remark that $\varphi_a(\varphi_a(w)) = w$, and thus $\varphi_a^{-1}(w) = \varphi_a(w)$. For $a, w \in \mathbb{D}$ and $0 < r < 1$, the supposed pseudo-hyperbolic disc $D(a, R)$ is denoted by $D(a, R) = \{w \in \mathbb{D}: |\varphi_a(w)| < R\}$.

Let the concerned function of Green's type with concerned logarithmic singularity at the specific point $a \in \mathbb{D}$

$$g(w,a) = \log\left|\frac{1-\overline{a}w}{w-a}\right| = \log\frac{1}{|\varphi_a(w)|}. \tag{3}$$

Assume that C_h and C_h^* are any two concerned quantities which depend on a holomorphic-type function h on $\mathbb{D}$. These concerned quantities are said to be equivalent and symbolized by $C_h \approx C_h^*$; when we find a concerned finite positive constant Υ not depending on the holomorphic function h, for every holomorphic-type function h on $\mathbb{D}$, the following inequalities hold:

$$\frac{1}{\Upsilon}C_h^* \le C_h \le \Upsilon C_h^*. \tag{4}$$

When the concerned quantities C_h and C_h^* are equivalent, we deduce that

$$C_h < \infty \Leftrightarrow C_h^* < \infty. \tag{5}$$

Note that we say $\Theta_1 \lesssim \Theta_2$ (for two concerned functions Θ_1 and Θ_2) when we find a specific concerned constant $\gamma > 0$ for which $\Theta_1 \le \gamma\Theta_2$.

Next the following new general concerned function classes will be defined.

Definition 1. Let ϖ be a given concerned bounded nondecreasing and continuous function $\varpi\colon \mathbb{D} \longrightarrow \mathbb{R}^+$. The concerned function $h \in H(\mathbb{D})$ is said to belong to the weighted ϖ-Bloch-type space $\mathscr{B}_{(\varpi)}$, when

$$\|h\|_{\mathscr{B}_{(\varpi)}} = \sup_{w\in\mathbb{D}} \varpi(1-|w|)|h'(w)| < \infty. \tag{6}$$

Definition 2. Let ϖ be a given concerned bounded nondecreasing and continuous function $\varpi\colon \mathbb{D} \longrightarrow \mathbb{R}^+$. The concerned function $h \in H(\mathbb{D})$ is said to belong to the little weighted ϖ-Bloch-type space $\mathscr{B}_{(\varpi,0)}$, when

$$\|h\|_{\mathscr{B}_{(\varpi,0)}} = \lim_{|w|\longrightarrow 1^-} \varpi(1-|w|)|h'(w)| = 0. \tag{7}$$

Remark 1. One should obviously note that these new analytic-type classes are more general than the well-known Bloch and little Bloch analytic-type classes. If $\varpi(t) = t^\alpha$ with $\alpha \in (0,\infty)$, then the analytic α-Bloch classes which were introduced and studied in [2] will be followed. In addition, when $\alpha = 1$, we obtain the known analytic Bloch-type class that is given in [3].

The following new analytic weighted function classes will be introduced.

Definition 3. For a bounded nondecreasing continuous function $\varpi\colon \mathbb{D} \longrightarrow \mathbb{R}^+$, let $0 < p < \infty$, and a holomorphic function $h \in H(\mathbb{D})$ is said to belong to the ϖ-Besov space $B_{p,\varpi}(\varphi)$, when

$$\|h\|_{\varpi,p;\varphi}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|)^{2n-2} \varpi^p\left(1-|w|^2\right)\left(1-|\varphi_a(w)|\right)^p \mathrm{d}\zeta(w) < \infty. \tag{8}$$

Definition 4. For a bounded nondecreasing continuous function $\varpi\colon \mathbb{D} \longrightarrow (0,\infty)$, let $0 < p < \infty$, and an analytic function h in $\mathbb{D}$ is said to belong to the ϖ-Besov space $B_{p,\varpi}(g)$, when

$$\|h\|_{\varpi,p;g}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|)^{2n-2} \varpi^p\left(1-|w|^2\right) g(w,a)^p \mathrm{d}\zeta(w) < \infty. \tag{9}$$

Definition 5. (see [1]). Let $h \in H(\mathbb{D})$ and let $1 < p < \infty$. If

$$\|h\|_{B_p}^p = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p \left(1-|w|^2\right)^{p-2} \mathrm{d}\zeta(w) < \infty, \tag{10}$$

then h belongs to the Besov space B_p.

The aim of the current paper is to establish with concerned proofs various results on analytic-type function spaces with the help of holomorphic $\mathscr{B}_\varpi$-function type classes in some holomorphic concerned general Besov functions satisfying more extended general integral norm conditions portrayed by some general weights in the known concerned complex disk. The new results generalize and evolve some results in [1, 2, 4]. To substantiate the authenticity of the obtained results and to clear the importance of the defined function classes, some related concerned corollaries are furnished obviously too.

Remark 2. It is an interesting considerable remark to mention the generalizations of some holomorphic-type function spaces in $\mathbb{C}^n$ (see [1,5–10]) and others. In the same way, there are some extensions by the use of hypercomplex functions (see [11–15]) and others.

2. Some Integral Criteria for $\mathscr{B}_{\varpi}$ Functions

Some concerned interesting integral-type criteria for the analytic classes $\mathscr{B}_{(\varpi)}$ and the analytic general Besov classes are established in this concerned present section. The obtained analytic results are generalizing and evolving the corresponding results in [1, 2, 16] and others.

Now, suppose that $\varphi_a(W) = a - w/1 - aw$ is a Möbius transformation of $\mathbb{D}$, let $D(a,r) = \{w \in \mathbb{D}: |\varphi_a(w)| < r\}$, and let $g(w,a) = \log 1/|\varphi_a(w)|$ be the known Green's function on $\mathbb{D}$ with a specific concerned logarithmic singularity at the point $a \in \mathbb{D}$. Next, an interesting general result will be established.

Theorem 1. *Let $r \in (0,1), p \in (0,\infty), q \in [1,\infty)$. In addition, we let $h \in H(\mathbb{D})$. Therefore, the next concerned general quantities are equivalent:*

(a) $\|h\|^p_{\mathscr{B}_{(\varpi)}}$.

(b) For $0 < p < \infty$, we have

$$\sup_{a\in\mathbb{D}} \frac{1}{|D(a,r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p \varpi^p (1-|w|) \mathrm{d}\zeta(w) < \infty. \tag{11}$$

(c) For $0 < p < \infty$, we have

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \varpi^p (1-|w|) \mathrm{d}\zeta(w) < \infty. \tag{12}$$

(d) For $0 < p < \infty$ and $1 < q < \infty$, we have

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \varpi^p (1-|w|) \left(1-|\varphi_a(w)|^2\right)^q \mathrm{d}\zeta(w) < \infty. \tag{13}$$

(e) For $0 < p < \infty$, we have

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \left(\log \frac{1}{|w|}\right) \varpi^p (1-|w|) |\varphi_a'(w)|^2 \mathrm{d}\zeta(w) < \infty. \tag{14}$$

(f)

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (g(w,a))^p (1-|w|)^{p-2} \varpi^p (1-|w|) \mathrm{d}\zeta(w) < \infty. \tag{15}$$

Proof. For any concerned analytic-type function g on $\mathbb{D}$, $|g|^p$ is a concerned subharmonic function, and thus we deduce that

$$|g(0)|^p \le \frac{1}{\pi r^2} \iint_{D(0,r)} |g(w)|^p \mathrm{d}\zeta(w). \tag{16}$$

Employing g by $h' \circ \varphi_a$, the next inequality can be obtained:

$$|h'(a)|^p \le \frac{1}{\pi r^2} \iint_{D(0,r)} |h' \circ \varphi_a(w)|^p \mathrm{d}\zeta(w) = \frac{1}{\pi r^2} \iint_{D(0,r)} |h'(w)|^p \frac{\left(1-|\varphi_a(w)|^2\right)^2}{\left(1-|w|^2\right)^2} \mathrm{d}\zeta(w). \tag{17}$$

In view of the following concerned relations,

$$\frac{1-|\varphi_a(w)|^2}{1-|w|^2} = |\varphi_a'(w)|, \text{also } \frac{1-|\varphi_a(w)|^2}{1-|w|^2} \le \frac{4}{1-|a|^2} a, \quad w \in \mathbb{D} \,(\text{see}[14]). \tag{18}$$

Thus, the next inequality can be deduced:

$$|h'(a)|^p \le \frac{16}{\pi r^2 \left(1-|a|^2\right)^2} \iint_{D(a,r)} |h'(w)|^p \mathrm{d}\zeta(w). \tag{19}$$

Hence, using the known specific estimations $(1-|a|^2)^2 \sim (1-|w|^2)^2 \sim |D(a,r)|$, with $w \in D(a,r)$, the following concerned inequalities can be deduced:

$$|h'(a)|^p (1-|a|)^p \varpi^p (1-|a|) \le \frac{16(1-|a|)^p \varpi^p (1-|a|)}{\pi r^2 \left(1-|a|^2\right)^2} \iint_{D(a,r)} |h'(w)|^p \mathrm{d}\zeta(w). \tag{20}$$

Because $(1-|a|)^2 \sim (1-|a|^2)^2$,

$$\begin{aligned}\left|h'(a)\right|^p(1-|a|)^p\varpi^p(1-|a|) &\le \frac{16(1-|a|)^{p\alpha}\varpi^p(1-|a|)}{\pi r^2\left(1-|a|^2\right)^2}\iint_{D(a,r)}\left|h'(w)\right|^p d\zeta(w)\\ &\le \frac{16\eta}{\pi r^2|D(a,r)|^{1-p\alpha/2}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\varpi^p(1-|w|)d\zeta(w) = \frac{m(r)}{|D(a,r)|^{1-p/2}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\varpi^p(1-|w|)d\zeta(w),\end{aligned} \tag{21}$$

where η is a specific positive constant which is greater than zero and $m(r) = 16\eta/\pi r^2$ is a concerned constant which depends on r. Hence, for the two quantities (a) and (b), we have

$$(\mathrm{a}) \le k_1(\mathrm{b}), \tag{22}$$

where k_1 stands for a positive concerned constant.

Now, since $|D(a,r)| \sim (1-|w|^2)^2$, $\forall w \in D(a,r)$, so it is clear that $(b) \sim (c)$. From, the concerned inequality

$$1-\left|\varphi_a(w)\right|^2 > 1-r^2, \tag{23}$$

the following estimates can be simply obtained:

$$\begin{aligned}&\iint_{D(a,r)}\left|h'(w)\right|^p(1-|w|)^{p-2}\varpi^p(1-|w|)d\zeta(w)\\ &= \iint_{D(a,r)}\left|h'(w)\right|^p(1-|w|)^{p-2}\varpi^p(1-|w|)\frac{\left(1-\left|\varphi_a(w)\right|^2\right)^q}{\left(1-\left|\varphi_a(w)\right|^2\right)^q}d\zeta(w)\\ &\le \frac{1}{\left(1-r^2\right)^q}\iint_{D(a,r)}\left|h'(w)\right|^p(1-|w|)^{p-2}\varpi^p(1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^q d\zeta(w).\end{aligned} \tag{24}$$

Therefore, for the two quantities (c) and (d), we have

$$(\mathrm{c}) \le k_2(\mathrm{d}), \tag{25}$$

where k_2 stands for a positive concerned constant.

It is known that $1-|\varphi_a(w)|^2 \le 2g(w,a)$ for all $w, a \in \mathbb{D}$, and thus we can infer that the concerned quantity (d) is less than or equal to a concerned positive constant times the concerned quantity (f).

Using the next fundamental inequalities

$$\begin{aligned}&\iint_{D(a,r)}\left|h'(w)\right|^p(1-|w|)^{p-2}\varpi^p(1-|w|)(g(w,a))^p d\zeta(w)\\ &= \iint_{D(a,r)}\left|h'(\varphi_a(w))\right|^p(1-|\varphi_a(w)|)^p\varpi^p\left(1-\left|\varphi_a(w)\right|\right)\left(\log\frac{1}{|w|}\right)^q\frac{d\zeta(w)}{\left(1-|w|^2\right)^2} \le \|h\|^p_{\mathscr{B}^\alpha_\varpi}\iint_{D(a,r)}\left(\log\frac{1}{|w|}\right)^q\frac{d\zeta(w)}{\left(1-|w|^2\right)^2},\end{aligned} \tag{26}$$

where

$$\mathscr{T} = \iint_{D(a,r)}\left(\log\frac{1}{|w|}\right)^q\left(1-|w|^2\right)^{-2}d\zeta(w) < \infty, \tag{27}$$

we can easily infer that the concerned quantity (e) is less than or equal to a specific positive concerned constant times the concerned quantity (a).

With the help of the known inequality $1-|w|^2 \le 2\log 1/|w|$, where $w \in \mathbb{D}$, setting $q = 2$ in the concerned quantity

(d), we simply deduce that the concerned quantity (d) is also less than or equal to the concerned quantity (e). For the final step in the concerned proof, we consider the following estimates.

$$\begin{aligned} J &= \iint_{D(a,r)} |h'(w)|^p \left(\log\frac{1}{|w|}\right)^q \varpi^p(1-|w|)|\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \\ &= \left(\iint_{D_{1/4}} + \iint_{\mathbb{D}/D_{(1/4)}}\right)|h'(w)|^p \left(\log\frac{1}{|w|}\right)^q \varpi^p(1-|w|)|\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \\ &= J_1 + J_2, \end{aligned} \tag{28}$$

with $w \in D_{1/4} = \{w\colon |w| < 1/4\}$, $|\varphi_a'(w)|^2 = (1-|a|^2)/\ |1-\bar{a}w|^4 \le 1/|1-\ |w||^4 \le (4/3)^4$, and thus

$$\begin{aligned} J_1 &= \iint_{D_{1/4\in J_1}} |h'(w)|^p \left(\log\frac{1}{|w|}\right)^p \varpi^p(1-|w|)|\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \\ &\le \|h\|^p_{\mathscr{B}^\alpha_{(\varpi)}} \iint_{D_{1/4}} \left(\frac{(\log 1/|w|)}{(1-|w|)}\right)^p |\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \le \|h\|^p_{\mathscr{B}^\alpha_{(\varpi)}} \left(\frac{4}{3}\right)^{p+4} \iint_{D_{1/4}} \left(\log\frac{1}{|w|}\right)^p \mathrm{d}\zeta(w) = \left(\frac{4}{3}\right)^{p+4} T(p)\|h\|^p_{\mathscr{B}^\alpha_{(\varpi)}}, \end{aligned} \tag{29}$$

where the integral

$$T(p) = \iint_{D_{1/4}} \left(\log\frac{1}{|w|}\right)^p \mathrm{d}\zeta(w) < \infty. \tag{30}$$

For each specific point $w \in \mathbb{D}/(1/4)$, the following useful estimates hold:

$$\log\frac{1}{|w|} \le 4\left(1-|w|^2\right) \le 8(1-|w|), \tag{31}$$

which implies that

$$J_2 \le 8 \iint_{\mathbb{D}/D_{(1/4)}} |h'(w)|^p \left(\log\frac{1}{|w|}\right)^q \varpi^p(1-|w|)|\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \le 8^p \|h\|^p_{\mathscr{B}_{(\varpi)}} \iint_{\mathbb{D}/D_{(1/4)}} |\varphi_a'(w)|^2 \mathrm{d}\zeta(w) \le k_3 \|h\|^p_{\mathscr{B}_{(\varpi)}}, \tag{32}$$

where k_3 is a concerned positive constant. Thus, the concerned general quantity (e) is less than or equal to a concerned positive constant times the concerned general quantity (a). Thus, the interesting proof of Theorem 1 is completely established clearly. In Theorem 1, assuming that $\varpi(1-|w|) \equiv 1$, the following concerned corollary can be obtained clearly. □

Corollary 1. *Let $r \in (0,1) p \in (0,\infty), q \in [1,\infty)$ and let $h \in H(\mathbb{D})$. Therefore, the next concerned general quantities are equivalent:*

(a^*) $\|h\|^p_{\mathscr{B}_{(\varpi)}}$.

(b^*) *For $0 < p < \infty$, we have*

$$\sup_{a\in\mathbb{D}} \frac{1}{|D(a,r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p \mathrm{d}\zeta(w) < \infty. \tag{33}$$

(c^*) *For $0 < p < \infty$, we have*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \mathrm{d}\zeta(w) < \infty. \tag{34}$$

(d^*) *For $0 < p < \infty$ and $1 < q < \infty$, we have*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \left(1-|\varphi_a(w)|^2\right)^q \mathrm{d}\zeta(w) < \infty. \tag{35}$$

(e^*) *For* $0<p<\infty$, *we have*

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\left(\log\frac{1}{|w|}\right)^p\left|\varphi_a(w)\right|^2\mathrm{d}\zeta(w)<\infty. \tag{36}$$

(f^*)

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p(g(w,a))^p(1-|w|)^{p-2}\mathrm{d}\zeta(w)<\infty. \tag{37}$$

For the concerned holomorphic function classes $\mathscr{B}_{(\varpi,0)}$, *we give the following corresponding result induced from Theorem 1 on the specific boundary of the concerned unit disc* $\mathbb{D}$.

Theorem 2. *Let* $r\in(0,1)p\in(0,\infty),q\in[1,\infty)$ *and let* $h\in H(\mathbb{D})$. *Therefore, the next concerned general quantities are equivalent:*

(a_1) $\|h\|^p_{\mathscr{B}_{(\varpi,0)}}$.

(a_2) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\frac{1}{|D(a,r)|^{1-p/2}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\varpi^p(1-|w|)\mathrm{d}\zeta(w)<\infty. \tag{38}$$

(a_3) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\varpi^p(1-|w|)^{p-2}\varpi^p(1-|w|)\mathrm{d}\zeta(w)<\infty. \tag{39}$$

(a_4) *For* $0<p<\infty$ *and* $1<q<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\varpi^p(1-|w|)^{p-2}\varpi^p(1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)\mathrm{d}\zeta(w)<\infty. \tag{40}$$

(a_5) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\left(\log\frac{1}{|w|}\right)^p\varpi^p(1-|w|)\left|\varphi_a'(w)\right|^2\mathrm{d}\zeta(w)<\infty. \tag{41}$$

(a_6)

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p(g(w,a))^p(1-|w|)^{p-2}\varpi^p(1-|w|)\mathrm{d}\zeta(w)<\infty. \tag{42}$$

In Theorem 2, assuming that $\varpi(1-|w|)\equiv1$, the next interesting corollary can be established directly.

Corollary 2. *Let* $r\in(0,1)p\in(0,\infty),q\in[1,\infty)$ *and let* $h\in H(\mathbb{D})$. *Therefore, the next concerned general statements are equivalent:*

(b_1) $\|h\|^p_{\mathscr{B}_{(\varpi,0)}}$.

(b_2) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\frac{1}{|D(a,r)|^{1-p/2}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\mathrm{d}\zeta(w)<\infty. \tag{43}$$

(b_3) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p(1-|w|)^{p-2}\mathrm{d}\zeta(w)<\infty. \tag{44}$$

(b_4) *For* $0<p<\infty$ *and* $1<q<\infty$, *we have*

$$\lim_{|a|\longrightarrow 1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p(1-|w|)^{p-2}\left(1-\left|\varphi_a(w)\right|^2\right)^q d\zeta(w)<\infty. \tag{45}$$

(b_5) *For* $0<p<\infty$, *we have*

$$\lim_{|a|\longrightarrow 1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p\left(\log\frac{1}{|w|}\right)^p\left|\varphi_a'(w)\right|^2 d\zeta(w)<\infty. \tag{46}$$

(b_6)

$$\lim_{|a|\longrightarrow 1^-}\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}\left|h'(w)\right|^p(g(w,a))^p(1-|w|)^{p-2}d\zeta(w)<\infty. \tag{47}$$

3. More Analytic Characterizations

Extension of Kwon et al.'s result (see [4]) is also established in this current section with the help of holomorphic $\mathscr{B}_\varpi$-type functions. Holomorphic function significants in both weighted ϖ-Besov norms and $\mathscr{B}_\varpi$-Bloch norms framework and to introduce some new concerned families of holomorphic-type function classes.

Theorem 3. *Let* $0<p<\infty, 0\le s<\infty$, *with* $0<p<s+1$ *and let* $h\in H(\mathbb{D})$. *Thus,*

$$\|h\|_{\mathscr{B}_\varpi}^p\approx\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|h'(w)\right|^p\left(1-|w|^2\right)^{p-2}\varpi^p(1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p(g(w,a))^{-s}d\zeta(w). \tag{48}$$

Proof. In view of condition (b) in Theorem 1, we deduce that

$$\|h\|_{\mathscr{B}_\varpi}^p\approx w\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|h'(w)\right|^p\left(1-|w|^2\right)^{p-2}\varpi^p(1-|w|)d\zeta(w). \tag{49}$$

Set a concerned finite specific positive constant δ_R by

$$\delta_R\colon\left(1-R^2\right)^p\left(\log\frac{1}{1-R^2}\right)^{-s}. \tag{50}$$

Thus, we deduce that

$$\|h\|_{\mathscr{B}_\varpi}^p\approx\sup_{a\in\mathbb{D}}\int_{D(a,R)}\left|h'(w)\right|^p\left(1-|w|^2\right)^{p-2}\varpi^p(1-|w|)\left(\log\frac{1}{1-R^2}\right)^{-s}\left(1-R^2\right)^p d\zeta(w). \tag{51}$$

Using the inequality

$$\left|\varphi_a(w)\right|^2<R^2\Leftrightarrow 1-\left|\varphi_a(w)\right|^2>1-R^2, \tag{52}$$

we infer that

$$\|h\|_{\mathscr{B}_\varpi}^p\approx\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|h'(w)\right|^p\left(1-|w|^2\right)^{p-2}\varpi^p(1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p(g(w,a))^{-s}d\zeta(w). \tag{53}$$

Thus,

$$\|h\|_{\mathscr{B}_\omega}^p \le k^* \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|h'(w)\right|^p \left(1-|w|^2\right)^{p-2} \varpi^p (1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p (g(w,a))^{-s} \mathrm{d}\zeta(w). \tag{54}$$

where k^* is a specific positive constant that does not depend on a.

Now for the other direction, in view of changing variable base, that is replacing w by $\varphi_a(w)$, the following estimates can be deduced:

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|h'(w)\right|^p \left(1-|w|^2\right)^{p-2} \varpi^p (1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p (g(w,a))^{-s} \mathrm{d}\zeta(w) \\
&\le \|h\|_{\mathscr{B}_\omega}^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1-|w|^2\right)^{p-2} \left(1-\left|\varphi_a(w)\right|^2\right)^p (g(w,a))^{-s} \mathrm{d}\zeta(w) \\
&\left(=\|h\|_{\mathscr{B}_\omega}^p \sup_{a\in\mathbb{D}}\right) \int_{\mathbb{D}} \left(1-\left|\varphi_a(w)\right|^2\right)^p \left(1-|w|^2\right)^p \left(\log\frac{1}{|w|}\right)^{-s} \left(\frac{1-|a|^2}{|1-\overline{a}w|^2}\right)^2 \mathrm{d}\zeta(w).
\end{aligned} \tag{55}$$

With the help of the known equality (1), the following estimates can be obtained:

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|h'(w)\right|^p \left(1-|w|^2\right)^{p-2} \varpi^p (1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p (g(w,a))^{-s} \mathrm{d}\zeta(w) \\
&\le \|h\|_{\mathscr{B}_\omega}^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1-|w|^2\right)^{p-2} \left(\log\frac{1}{|w|}\right)^{-s} \mathrm{d}\zeta(w) \\
&\le (2)^{3p-2} \|h\|_{\mathscr{B}_\omega}^p \int_0^1 (1+R)^{-s} (1-R)^{p-s-2} \mathrm{d}r \\
&\le 4\pi (2)^{3p-s-2} \left|h_{\mathscr{B}_\omega}^p \int_0^1 (1-R)^{p-s-2} \mathrm{d}R < (2)^{3p-s} \pi \right| h_{\mathscr{B}_\omega}^p I(R,p,s),
\end{aligned} \tag{56}$$

where the specific integral $I(R,p,s) = \int_0^1 (1-R)^{p-s-2} \mathrm{d}R < +\infty$ when $p < s+1$. Therefore,

$$\|h\|_{\mathscr{B}_\omega}^p \ge k_1^* \sup a \in \mathbb{D} \int_{\mathbb{D}} \left|h'(w)\right|^p \left(1-|w|^2\right)^{p-2} \varpi^p (1-|w|)\left(1-\left|\varphi_a(w)\right|^2\right)^p (g(w,a))^{-s} \mathrm{d}\zeta(w). \tag{57}$$

where the specific constant

$$k_1^* = \frac{1}{(2)^{s-3p} (\pi) I(p,s)}. \tag{58}$$

Thus, the concerned proof of our desired result is completely obtained. □

Remark 3. Various research studies on advanced operator theory using numerous weighted classes of holomorphic functions have been actively appearing in some joyful important areas of mathematical sciences such as the theory of dynamical systems, the known probability theory, recent research in mathematical physics, and some branches of quantum mechanics. Such holomorphic

weighted classes of concerned function spaces are still under interest of considerations in the aforementioned applications. The defined holomorphic $\mathfrak{w}$-Bloch spaces as well as the holomorphic $\mathfrak{w}$-Besov spaces can be also applied in such interesting applications.

4. Conclusion

General analytic characterizations for some concerned extended classes of holomorphic Banach of function spaces of Bloch and Besov type are established and discussed in this article. The concerned proofs are obtained using two types of concerned holomorphic functions in $\mathbb{D}$. Both the considered functions and the proofs methods have concerned parameters which make deep help to the obtained results. As specific results, various new theorems and lemmas for the considered function classes are well derived. Further, the obtained concerned generalized results proved that the used function classes give better performance compared with existing results in the literature. The n-th partial derivative concerned quantities with general formulas including the extended type of functions are also well derived. The well-known analytic Bloch, the analytic Besov, and the Zygmund spaces are introduced as specific concerned special cases of the new defined classes. The concerned computations of the extended Bloch-type norm and the extended Besov-type norms related to these special cases can be established too.

The analytic characterizations of integral norms, the extended Besov space, and the specific analytic Bloch norms will give interesting general results and recent developments within this fascinating area of function spaces. Such concerned advanced aspects may be considered in some areas of further studies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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