

Retraction

Retracted: Evidence Theory in Picture Fuzzy Set Environment

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] H. Garg, R. Sujatha, D. Nagarajan, J. Kavikumar, and J. Gwak, "Evidence Theory in Picture Fuzzy Set Environment," *Journal of Mathematics*, vol. 2021, Article ID 9996281, 8 pages, 2021.

Research Article

Evidence Theory in Picture Fuzzy Set Environment

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Picture fuzzy set is the most widely used tool to handle the uncertainty with the account of three membership degrees, namely, positive, negative, and neutral such that their sum is bound up to 1. It is the generalization of the existing intuitionistic fuzzy and fuzzy sets. This paper studies the interval probability problems of the picture fuzzy sets and their belief structure. The belief function is a vital tool to represent the uncertain information in a more effective manner. On the other hand, the Dempster–Shafer theory (DST) is used to combine the independent sources of evidence with the low conflict. Keeping the advantages of these, in the present paper, we present the concept of the evidence theory for the picture fuzzy set environment using DST. Under this, we define the concept of interval probability distribution and discuss its properties. Finally, an illustrative example related to the decision-making process is employed to illustrate the application of the presented work.

1. Introduction

Decision making is based on experts opinion, and often experts have to take decisions based on limited data or knowledge. Thus, in any decision-making process, two types of uncertainties arise. Epistemic uncertainties occur due to lack of knowledge, insufficient data, and ambiguity, whereas aleatory uncertainty is due to the randomness of the physical system under study [1–6]. Probability theory is proposed to deal with randomness and is not effective in dealing with epistemic uncertainties. Theory based on evidence to handle uncertainty is Dempster–Shafer theory [7, 8]. It has vast applications [9]. In a finite discrete space, Dempster–Shafer theory can be interpreted as a generalization of probability theory where probabilities are assigned to sets as opposed to

mutually exclusive singletons. In traditional probability theory, evidence is associated with only one possible event. In Dempster–Shafer theory, evidence can be associated with multiple possible events, in contrast to one single event. DS theory was extended to fuzzy sets by Zadeh [10, 11]. DS theory in fuzzy sets is presented in [12–16]. DS theory in intuitionistic domain was introduced by Grzegorzewski and Mrowka in [17]. Representation theorem was proved by Riecan [18, 19]. Gerstenkorn and Manko defined intuitionistic probability in two ways [20]. Also, Feng et al. [21] used intuitionistic fuzzy lower and upper approximation operators. Fuzzy clustering based on DS theory was applied for breast cancer cell detection in [22]. Pavement condition distress index was calculated using DS theory in [23], and the combination of quantum theory and DS theory has been

investigated in [24]. Belief degrees and belief structures are required for Dempster–Shafer theory.

Cong and Kreinovich [25] introduced picture fuzzy sets and developed some operations and relations on them. Temporal picture fuzzy soft set and related concepts were developed in [26]. Picture fuzzy geometric operators were proposed, and using it, the multiple attribute decision-making problem was addressed in [27]. P -order and R -order union and the intersection of internal (external) cubic picture fuzzy sets were discussed in [28]. In the interval-valued picture fuzzy domain, the similarity measures are developed in [29]. Einstein information-based aggregation operators applied in group decision-making problem were dealt in [30]. Distance measure and dissimilarity measure are defined in [31]. Generalized weighted distance measure, the generalized weighted Hausdorff distance measure, and the generalized hybrid weighted distance measure between LPFSs and their properties are discussed and applied to TOPSIS [32]. Decomposition theorems for PFS are proved in [33].

The estimation of the probability of belief function in the environment of fuzzy events and intuitionistic fuzzy events exist in literature. Dempster–Shafer theory is based on belief degrees and structures with precision, but in decision-making situations, the data are incomplete, and there is a lack of information. In face recognition, when two persons have highly similar features, a classifier may be unable to give a precise decision. In such situations, its belief degree may be imprecise. Thus, in the decision-making process involving elucidation of multiple experts' opinion, interval-valued belief degree is appropriate. Interval probability distribution based on Dempster–Shafer evidence theory in fuzzy and intuitionistic fuzzy environment is discussed by the authors. Fuzzy theory takes into account membership function, in contrast to intuitionistic fuzzy which accommodates non-membership. In some decision-making situations, some experts prefer neutral membership. Picture fuzzy assents positive, negative, and neutral memberships, thereby providing refusal degree. Thus, picture fuzzy sets are more apposite in decision making, indicating the need for extension of evidence theory for picture fuzzy sets. The main motivation of this paper is to frame probability distribution based on evidence theory for picture fuzzy sets and illustrate it through a suitable example.

2. Background

Dempster–Shafer theory of evidence is based on a finite set of mutually exclusive elements, called the frame of discernment denoted by Ω . 2^Ω is the power set of Ω , and it contains all possible unions of the sets in Ω . Atomic sets are the singleton sets in a frame of discernment.

DS theory can express and deal with uncertainty in crisp sets. However, it fails to handle vague information and linguistic terms. Thus, fuzzy evidence theory was developed, and it was extended to intuitionistic fuzzy sets. In this paper, we define evidence theory using picture fuzzy sets. The

probability distribution is expressed as an interval. The following definitions are provided as background for this paper.

- (1) *Definition* [7, 8]: let $\Omega = A_1, A_2, \dots, A_n$ be the frame of discernment. A basic belief assignment or basic probability assignment (BPA) is a function $m: 2^\Omega \rightarrow [0, 1]$ satisfying the conditions: $m(\emptyset) = 0$; $\sum_{A \subseteq \Omega} m(A) = 1$. For each subset $A \subseteq \Omega$, the value taken by the BPA at A is called the basic probability assigned to A and denoted by $m(A)$. A subset A of Ω is called the focal element of a belief function m if $m(A) > 0$.
- (2) *Dempster's rule of combination* [7, 8]: let m_1 and m_2 be two basic probability assignments (BPAs) on the frame of discernment Ω , where the BPAs m_1 and m_2 are independent. The orthogonal sum based on Dempster's rule of combination defined by $m = m_1 \oplus m_2$ is $m(A) = (1/1 - K) \sum_{B \cap C = A} m_1(B) m_2(C)$. The conflict between the BPAs m_1 and m_2 is $K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)$.
- (3) *Definition* [25]: a picture fuzzy set A on a universe X is of the form $A = x, P_A(x), Nu_A(x), Ng_A(x) | x \in X$ where $P_A(x) \in [0, 1]$ is the degree of positive membership of x in A , $Nu_A(x) \in [0, 1]$ is the degree of neutral membership of x in A , and $Ng_A(x) \in [0, 1]$ is the degree of negative membership of x in A . These memberships satisfy the condition $P_A(x) + Nu_A(x) + Ng_A(x) = 1$. Further, the refusal degree of x in A is $R_A(x) = 1 - (P_A(x) + Nu_A(x) + Ng_A(x))$.
- (4) *Arithmetic operations on intervals*: consider $[a_1, a_2]$ and $[b_1, b_2]$, with $a_1, b_1 > 0$. Then, the arithmetic operations on these intervals are given by
 - (a) Addition: $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$
 - (b) Subtraction: $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$
 - (c) Multiplication: $[a_1, a_2] \cdot [b_1, b_2] = [a_1 \cdot b_1, a_2 \cdot b_2]$
 - (d) Division: $([a_1, a_2]/[b_1, b_2]) = [(a_1/b_2), (a_2/b_1)]$

3. Picture Fuzzy Interval Probability (PFIP)

Probability distribution in the framework of picture fuzzy sets is introduced in this section. The probability distribution is in the form of an interval. The validation of this interval probability distribution is examined. Further, this definition coincides with fuzzy and intuitionistic interval probabilities when the negative, neutral, and refusal degrees are zero for the former and neutral membership is zero for the latter. Let $X = x_1, x_2, \dots, x_n$ be a universe of discourse and F be the set of all focal elements. A picture fuzzy belief function m is given as $\{\langle A_i^{\text{PF}}, m(A_i^{\text{PF}}), P_{A_i}^{\text{PF}}(x_j), Ng_{A_i}^{\text{PF}}(x_j), Nu_{A_i}^{\text{PF}}(x_j) \rangle\}$ where $A_i^{\text{PF}} \in F, x_j \in X$. Then, the probability of $x_j, j = 1, 2, \dots, n$ is defined as $\bar{P}(x_j) = [\bar{a}_j, \bar{b}_j]$, where \bar{a}_j and \bar{b}_j are given by

$$\bar{a}_j = \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{[\sum_{j=1}^n (-Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))] - R_{A_i}^{PF}(x_j)}, \quad \left(\sum_{j=1}^n \bar{a}_j\right) + (\bar{b}_q - \bar{a}_q) \leq 1, \quad (3)$$

$$\bar{b}_j = \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}{[\sum_{j=1}^n (P_{A_i}^{PF}(x_j))] + R_{A_i}^{PF}(x_j)}. \quad (2)$$

$$\left(\sum_{j=1}^n \bar{b}_j\right) - (\bar{b}_q - \bar{a}_q) \geq 1, \quad \forall q \in 1, 2, \dots, n.$$

Theorem 1. The picture fuzzy set interval probability estimation $\bar{P}(x_j) = [\bar{a}_j, \bar{b}_j]$, $j = 1, 2, \dots, n$ forms a valid interval-valued probability distribution in $X = x_1, x_2, \dots, x_n$.

The picture fuzzy set interval probability estimation is defined on picture fuzzy set. In picture fuzzy set, the rejection membership is given by $R_{A_i}^{PF}(x_j) = 1 - P_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j) - Nu_{A_i}^{PF}(x_j)$. Consider equation (1):

Proof. To prove that $\bar{P}(x_j) = [\bar{a}_j, \bar{b}_j]$, $j = 1, 2, \dots, n$ is a valid interval-valued probability distribution, we need to prove that

$$\bar{a}_j = \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{[\sum_{j=1}^n (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))] - R_{A_i}^{PF}(x_j)}, \quad (4)$$

where \bar{a}_j can be rewritten as

$$\bar{a}_j = \sum_{A_j^{PF} \in F} \frac{m(A_j^{PF})P_{A_j}^{PF}(x_j)}{\left[\sum_{\substack{k=1 \\ k \neq j}}^n (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) - (1 - Nu_{A_i}^{PF}(x_k) - Ng_{A_i}^{PF}(x_j)) - R_{A_i}^{PF}(x_j) \right]}, \quad (5)$$

$$\bar{a}_j = \sum_{A_j^{PF} \in F} \frac{m(A_j^{PF})P_{A_j}^{PF}(x_j)}{\left[\sum_{\substack{k=1 \\ k \neq j}}^n (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) + P_{A_j}^{PF}(x_j) \right]}.$$

Now,

$$\begin{aligned} \bar{b}_j &= \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}{[\sum_{j=1}^n (P_{A_i}^{PF}(x_j))] + R_{A_i}^{PF}(x_j)} \\ &= \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}{[\sum_{j=1}^n (P_{A_i}^{PF}(x_j))] + (1 - P_{A_i}^{PF}(x_j) - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))} \\ &= \sum_{A_i^{PF} \in F} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}{\left[\sum_{\substack{k=1 \\ k \neq j}}^n (P_{A_i}^{PF}(x_k)) + (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) \right]} \end{aligned} \quad (6)$$

TABLE 1: Comparison of the proposed approach with DST, fuzzy, and intuitionistic approaches.

S. no.	Dempster–Shafer theory	Evidence theory for fuzzy sets	Evidence theory for intuitionistic fuzzy sets	Evidence theory for picture fuzzy sets
1	Based on probability distribution	Based on fuzzy probability distribution	Based on intuitionistic probability distribution	Based on picture fuzzy probability distribution
2	Deals with aleatory uncertainties	Deals with epistemic uncertainties	Deals with epistemic uncertainties along with hesitancy in experts' opinion	Deals with epistemic uncertainties along with neutral in experts' opinion
3	Can be provided as interval probability distribution [34]	Can be provided as interval probability distribution [35]	Can be provided as interval probability distribution [9]	Can be provided as interval probability distribution (<i>present study</i>)
4	—	Supports degree of membership for belief functions	Supports degrees of membership and nonmembership for belief functions	Supports degrees of positive, negative, and neutral for belief functions
5	—	—	Accommodates hesitancy degree	Accommodates refusal degree
6	—	Reduces to crisp case interval probability distribution when degree	Reduces to fuzzy interval probability distribution when degree of nonmembership	Reduces to fuzzy interval probability distribution when degree negative, neutral, and refusal memberships are zero; reduces to intuitionistic interval probability distribution when neutral membership is zero

Again, for $\forall q \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
& \sum_{j=1}^n \frac{P_{A_i}^{\text{PF}}(x_j)}{\sum_{\substack{k=1 \\ k \neq j}}^n (1 - Nu_{A_i}^{\text{PF}}(x_k) - Ng_{A_i}^{\text{PF}}(x_k)) + P_{A_i}^{\text{PF}}(x_j)} \\
& \leq \sum_{j=1}^n \frac{P_{A_i}^{\text{PF}}(x_j)}{1 - P_{A_i}^{\text{PF}}(x_q) + \left[\sum_{\substack{k=1 \\ k \neq q}}^n (P_{A_i}^{\text{PF}}(x_k)) \right]} = \frac{\sum_{\substack{j=1 \\ j \neq q}}^n P_{A_i}^{\text{PF}}(x_j)}{1 - P_{A_i}^{\text{PF}}(x_q) + \sum_{\substack{k=1 \\ k \neq q}}^n P_{A_i}^{\text{PF}}(x_k)} \quad (7) \\
& = 1 - \frac{1 - P_{A_i}^{\text{PF}}(x_q)}{1 - P_{A_i}^{\text{PF}}(x_q) + \left[\sum_{\substack{k=1 \\ k \neq q}}^n (P_{A_i}^{\text{PF}}(x_k)) \right]}
\end{aligned}$$

Consider

$$\sum_{j=1}^{j \neq q} \frac{1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)}{\left[\sum_{\substack{k=1 \\ k \neq q}}^n (P_{A_i}^{PF}(x_k)) + (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) \right]} \geq \sum_{j=1}^{j \neq q} \frac{1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)}{\left[\sum_{\substack{k=1 \\ k \neq q}}^n (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) + P_{A_i}^{PF}(x_q) \right]} \tag{8}$$

$$= 1 - \frac{P_{A_i}^{PF}(x_q)}{\left[\sum_{\substack{k=1 \\ k \neq q}}^n (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) + P_{A_i}^{PF}(x_q) \right]}$$

$$\sum_{\substack{j=1 \\ j \neq q}}^n \bar{a}_q = \sum_{A_i} \sum_{\substack{j=1 \\ j \neq q}}^n \frac{m(A_i^{PF}) P_{A_i}^{PF}(x_j)}{\sum_{\substack{k=1 \\ k \neq j}}^n [1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)] + P_{A_i}^{PF}(x_j)}$$

$$\leq \sum m(A_i^{PF}) \left(1 - \frac{1 - Nu_{A_i}^{PF}(x_k) - Ng_{A_i}^{PF}(x_k)}{1 - Nu_{A_i}^{PF}(x_q) - Ng_{A_i}^{PF}(x_q) + \sum_{\substack{j=1 \\ j \neq q}}^n (P_{A_i}^{PF}(x_j))} \right) \tag{9}$$

$$= \sum m(A_i^{PF}) - \sum_{k \neq q} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_k) - Ng_{A_i}^{PF}(x_k))}{1 - Nu_{A_i}^{PF}(x_q) - Ng_{A_i}^{PF}(x_q) + \sum_{j=1}^n (P_{A_i}^{PF}(x_j))} = 1 - \bar{b}_q,$$

$$\sum_{\substack{j=1 \\ j \neq q}}^n \bar{b}_q = \sum_{j=1}^n \sum_{\substack{j \neq q \\ k \neq j}} \frac{m(A_i^{PF})(1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}{\left[P_{A_i}^{PF}(x_j) + (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)) \right]}$$

$$= \sum_{A_i} m(A_i^{PF}) \sum_{\substack{j=1 \\ j \neq q}}^n \frac{1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)}{\sum_{\substack{k=1 \\ k \neq q}}^n (P_{A_i}^{PF}(x_j)) + (1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j))}$$

$$\geq \sum_{A_i} m(A_i^{PF}) \left(1 - \frac{P_{A_i}^{PF}(x_q)}{P_{A_i}^{PF}(x_q) + \sum_{k=1, k \neq q}^n [1 - Nu_{A_i}^{PF}(x_k) - Ng_{A_i}^{PF}(x_k)]} \right) \tag{10}$$

$$= \sum_{A_i} m(A_i^{PF}) - \sum_{k \neq q} \frac{m(A_i^{PF}) P_{A_i}^{PF}(x_q)}{P_{A_i}^{PF}(x_q) + \sum_{k=1}^n [1 - Nu_{A_i}^{PF}(x_k) - Ng_{A_i}^{PF}(x_k)]} = 1 - \bar{a}_q.$$

Hence, finally by equations (9) and (10),

$$\sum_{j=1}^n \overline{a}_q \leq 1 - \overline{b}_q \quad \sum_{j=1}^n \overline{b}_q \geq 1 - \overline{a}_q \quad (11)$$

Thus, $\sum_{j=1}^b \overline{a}_j + (\overline{b}_q - \overline{a}_q) \leq 1 \quad \sum_{j=1}^n \overline{a}_j + (\overline{b}_q - \overline{a}_q) \geq 1, \quad \forall q \in \{1, 2, \dots, n\}.$

Therefore, $\overline{P_{A_i}^{PF}}(x_j) = [\overline{a}_j, \overline{b}_j]$ is an interval-valued probability distribution in X . \square

Theorem 2. *The interval picture fuzzy probability estimation $A_i^{PF} = \overline{P_{A_i}^{PF}}(x_j) = [\overline{a}_j, \overline{b}_j]$ if it reduces to interval intuitionistic probability estimation A_i^{IF} . Further, it also reduces to interval fuzzy probability estimation.*

Proof. In intuitionistic fuzzy set, the neutral membership $Nu_{A_i}^{PF}(x_j) = 0$:

$$\begin{aligned} \overline{a}_j &= \sum \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{\sum_{j=1}^n [1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)] - R_{A_i}^{PF}(x_j)} \\ &= \sum \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{\sum_{j=1}^n [1 - Ng_{A_i}^{PF}(x_j)] - R_{A_i}^{PF}(x_j)}, \\ \overline{b}_j &= \sum \frac{m(A_i^{PF})1 - Nu_{A_i}^{PF}(x_j) - Ng_{A_i}^{PF}(x_j)}{\sum_{j=1}^n [P_{A_i}^{PF}(x_j)] + R_{A_i}^{PF}(x_j)}. \end{aligned} \quad (12)$$

In fuzzy sets, $Nu_{A_i}^{PF}(x_j) = 0 = R_{A_i}^{PF}(x_j)$:

$$\begin{aligned} \overline{a}_j &= \sum \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{\sum_{j=1}^n P_{A_i}^{PF}(x_j)}, \\ \overline{b}_j &= \sum \frac{m(A_i^{PF})P_{A_i}^{PF}(x_j)}{\sum_{j=1}^n P_{A_i}^{PF}(x_j)}. \end{aligned} \quad (13)$$

The interval probability estimation based on evidence theory for picture fuzzy sets is established. The similarities and contrasts against the fuzzy and intuitionistic fuzzy sets are provided in Table 1. \square

3.1. Example 1. Submarines form a significant and crucial part of the navy of any country. The overall cost of the submarine includes the cost of paint used to coat it. Properly formulated protective coatings are important to the durability and performance of the submarine. Coating systems for the underwater parts of a ship should be corrosion-inhibiting, antifouling, abrasion-resistant, smooth, and compatible with cathodic protection. To minimise bunker (fuel) costs, the underwater hull should remain smooth during service. Consequently, a coating system should be applied as evenly as possible, and it should provide long-term protection against corrosion and fouling. Increased

hull friction due to fouling can result in up to 40% more fuel consumption compared to a clean hull and greater air pollution because of the extra fuel burned to maintain a ship’s speed. Systems for the underwater hull/boottop areas consist of anticorrosive paint and antifouling paint on top of it. Thus, choosing a cost-effective paint with anticorrosive and antifouling properties is important. Often the estimated cost is based on experts’ opinion as it decides the cost of the submarine. Let the universe of discourse be $X = \{80, 90, 100\}$. The assessment result from some experts is that the probability of “assigning about 90 thousand rupees for paint” is 0.5, the probability of “assigning a small amount of money for paint” is 0.3, and the probability of “assigning a large amount of money for paint” is 0.2. The linguistic terms “about 90 thousand,” “small amount of money,” and “large amount of money” can be captured using picture fuzzy sets, as the opinion of experts often differs and uncertainty is involved. Let these linguistic terms be expressed by three picture fuzzy events A, B, and C, respectively. These focal elements can be expressed as

$$\begin{aligned} A &= \{(80, 0.7, 0.1, 0.1), (90, 1, 0, 0), (100, 0.7, 0.1, 0.1)\}, \\ B &= \{(80, 0.7, 0.2, 0.1), (90, 0.5, 0.1, 0.3), (100, 0.3, 0.3, 0.3)\}, \\ C &= \{(80, 0.5, 0.2, 0.2), (90, 0.6, 0.1, 0.1), (100, 1, 0, 0)\}. \end{aligned} \quad (14)$$

The picture fuzzy interval probability distribution is calculated using the proposed approach as

$$\begin{aligned} \overline{P}(80) &= [0.307, 0.3545], \\ \overline{P}(90) &= [0.34055, 0.40915], \\ \overline{P}(100) &= [0.27955, 0.3302]. \end{aligned} \quad (15)$$

To make a decision on the assigning money to paint is obtained by comparing the picture fuzzy interval probability distribution. Based on comparison of intervals using their centers, $\overline{P}(100) \leq \overline{P}(80) \leq \overline{P}(90)$. Thus, the decision of assigning 90 thousand for paint can be inferred.

3.2. Example 2. Let us continue with the previous case study of choosing suitable paint for the submarine. Decision is often based on more than one variety of paints. For each paint variety, more than one expert opinion is obtained to take a decision as the cost of paint for the submarine is higher. Suppose the independent opinion of two experts for a variety of paint is obtained based on four main factors of corrosion-inhibiting, antifouling, abrasion-resistant, and smoothness. These factors can be taken as the frame of discernment. Let this frame of discernment be

TABLE 2: Basic probability assignments by two experts.

S. no.	BPA m_1	BPA m_2
1	$\{x_1, x_3\} = [0.045, 0.514]$	$\{x_1, x_3\} = [0.181, 0.25]$
2	$\{x_3\} = [0.081, 0.172]$	$\{x_1, x_2, x_4\} = [0.016, 0.28]$
3	$\{x_1, x_2, x_3\} = [0.46, 0.57]$	$\{x_1, x_4\} = [0.2, 0.25]$
4	$\{x_2, x_4\} = [0.32, 0.48]$	$\{x_1, x_2, x_3, x_4\} = [0.21, 0.38]$

TABLE 3: Midpoint of combined IPFP.

S. no.	Combined IPFP m_{12}	Midpoints
1	$\{x_1, x_2\} = [0.0736, 0.159605]$	0.1166025
2	$\{x_1, x_3\} = [0.08326, 0.14250456]$	0.11288228
3	$\{x_2, x_4\} = [0.0512, 0.1344043]$	0.09280215
4	$\{x_1, x_2, x_3\} = [0.0966, 0.216606]$	0.156603

$\Omega = \{x_1, x_2, x_3, x_4\}$. Let m_1 and m_2 be two BPAs defined on Ω , given by Table 2.

By Dempster rule of combination, the combined opinion of the two experts can be estimated. The combined opinion of interval picture fuzzy probability is given by

$$\begin{aligned}
 m_{12}(x_1) &= [0.092, 0.1425], \\
 m_{12}(x_2) &= [7.2000072E - 3, 0.1439246], \\
 m_{12}(x_3) &= [0.22806, 0.171505], \\
 m_{12}(x_4) &= [0.064, 0.12], \\
 m_{12}(x_1, x_2) &= [0.0736, 0.159605], \\
 m_{12}(x_1, x_3) &= [0.08326, 0.14250456], \\
 m_{12}(x_2, x_4) &= [0.0512, 0.1344043], \\
 m_{12}(x_1, x_2, x_3) &= [0.0966, 0.216606].
 \end{aligned} \tag{16}$$

The decision of selecting this variety of paint is suitable for submarine is decided based on the interval picture fuzzy probability distribution by comparison of these interval values with more than one characteristic. The midpoints of the intervals are given in Table 3.

Based on Table 3, the paint has three attributes, namely, corrosion-inhibiting, antifouling, and abrasion-resistant, but it lacks smoothness. Thus, the paint variety can be used with less smoothness in the finish.

4. Conclusion

In this paper, we have utilized the picture fuzzy set to address the uncertainty and vagueness in the data. The picture fuzzy set captures the uncertainty of the element with respect to the three membership degrees such that their sum is bounded by 1. In this paper, we reviewed the definition and properties of the interval probability distribution for the picture fuzzy information using the belief function and DST. The proof of their validation is also given in the work. By employing the belief functions on picture fuzzy information systems, the interval probability can be estimated and hence ranking of the number can be accessed. The functionality of the structure is also explained with two numerical examples.

Also, the combined IPFP is used to compare two experts' opinions of the choice of paint.

In the future, we will utilize the belief function to address the decision-making problems arising under the different environmental issues such as greenhouse gas emissions, healthcare, green supplier selection, and so on. Also, we have established some generalized measures to combine the different preference values, and therefore, in the future work, we will try to develop different information measures for determining the nature of the decision-making process [36–38].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] R. Ferdous, F. Khan, R. Sadiq, P. Amyotte, and B. Veitch, "Analyzing system safety and risks under uncertainty using a bow-tie diagram: an innovative approach," *Process Safety and Environmental Protection*, vol. 91, pp. 1–18, 2013.
- [2] R. Ferdous, F. Khan, R. Sadiq, P. Amyotte, and B. Veitch, "Handling and updating uncertain information in bowtie analysis," *Journal of Loss Prevention in the Process Industries*, vol. 25, p. 819, 2012.
- [3] A. S. Markowski and M. S. Mannan, "Fuzzy risk matrix," *Journal of Hazardous Materials*, vol. 159, p. 1527, 2008.
- [4] M. Yazdi, "The application of bow-tie method in hydrogen sulfide risk management using layer of protection analysis (LOPA)," *Journal of Failure Analysis and Prevention*, vol. 17, pp. 291–303, 2017.
- [5] Y. Hong, H. J. Pasman, S. Sachdeva, A. S. Markowski, and M. S. Mannan, "A fuzzy logic and probabilistic hybrid approach to quantify the uncertainty in layer of protection analysis," *Journal of Loss Prevention in the Process Industries*, vol. 43, p. 1017, 2016.
- [6] A. S. Markowski and A. Kotynia, "Bow-tie" model in layer of protection analysis," *Process Safety and Environmental Protection*, vol. 89, pp. 205–213, 2011.
- [7] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," *The Annals of Mathematical Statistics*, vol. 38, no. 2, pp. 325–339, 1967.
- [8] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ, USA, 1976.
- [9] Y. Song and X. Wang, "Probability estimation in the framework of intuitionistic fuzzy evidence theory,"

- Mathematical Problems in Engineering*, vol. 2015, Article ID 412045, 10 pages, 2015.
- [10] L. A. Zadeh, "Probability measures of fuzzy events," *Journal of Mathematical Analysis and Applications*, vol. 23, no. 2, pp. 421–427, 1968.
- [11] L. A. Zadeh, "Fuzzy sets and information granularity," *Advances in Fuzzy Set Theory and Applications*, p. 318, North-Holland, Amsterdam, The Netherlands, 1979.
- [12] M. Ishizuka, K. S. Fu, and J. T. Yao, "Inference procedures under uncertainty for the problem-reduction method," *Information Sciences*, vol. 28, no. 3, pp. 179–206, 1982.
- [13] R. R. Yager, "Generalized probabilities of fuzzy events from fuzzy belief structures," *Information Sciences*, vol. 28, no. 1, pp. 45–62, 1982.
- [14] H. Ogawa and K. S. Fu, "An inexact inference for damage assessment of existing structures," *International Journal of Man-Machine Studies*, vol. 22, no. 3, pp. 295–306, 1985.
- [15] J. Yen, "Generalizing the Dempster-Shafer theory to fuzzy sets," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 20, no. 3, pp. 559–570, 1990.
- [16] H. Zhu and O. Basir, "An adaptive fuzzy evidential nearest neighbor formulation for classifying remote sensing images," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 43, no. 8, pp. 1874–1889, 2005.
- [17] P. Grzegorzewski and E. Mrówka, "Probability of intuitionistic fuzzy events," *Soft Methods in Probability, Statistics and Data Analysis*, Springer, Heidelberg, Germany, pp. 105–115, 2002.
- [18] B. Riecan, *Representation of Probabilities on IFS Events, Soft Methodology and Random Information Systems*, pp. 243–248, Springer, Heidelberg, Germany, 2004.
- [19] B. Riecan, "On a problem of Radko Mesiar: general form of IF probabilities," *Fuzzy Sets and Systems*, vol. 157, no. 11, pp. 1485–1490, 2006.
- [20] T. Gerstenkorn and J. Manko, "On a hesitancy margin and a probability of intuitionistic fuzzy events," *Notes Intuitionistic Fuzzy Sets*, vol. 7, no. 1, pp. 4–9, 2001.
- [21] T. Feng, J.-S. Mi, and S.-P. Zhang, "Belief functions on general intuitionistic fuzzy information systems," *Information Sciences*, vol. 271, pp. 143–158, 2014.
- [22] S. Ben Chaabane, "A new method based on evidence theory and fuzzy clustering for the breast cancer cells images segmentation," *International Journal of Scientific & Engineering Research*, vol. 8, no. 11, pp. 540–566, 2017.
- [23] S. Wadalkar, R. K. Lad, and R. K. Jain, "Performance assessment of flexible pavements: fuzzy evidence theory approach," *Civil Engineering Journal*, vol. 6, no. 8, pp. 1492–1502, 2020.
- [24] X. Gao, L. Pan, and Y. Deng, "The quantum pythagorean fuzzy evidence theory based on negation in quantum of mass function," *International Journal of Intelligent Systems*, 2020.
- [25] B. Cong and V. Kreinovich, "Picture fuzzy sets—a new concept for computational intelligence problems," in *Proceedings of the 2013 Third World Congress on Information and Communication Technologies (WICT 2013)*, Hanoi, Vietnam, December 2013.
- [26] S. V. Manemaran and R. Nagarajan, "Temporal generated N -picture fuzzy soft dimensions via algebraic structures," *International Journal of Research and Analytical Reviews*, vol. 5, no. 4, pp. i660–i666, 2018.
- [27] C. Wang, X. Zhou, H. Tu, and S. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their applications in multiple attribute decision-making," *Italian Journal of Pure and Applied Mathematics*, vol. 37, pp. 477–492, 2017.
- [28] S. Ashraf, S. Abdullah, and A. Qadir, "Novel concept of cubic picture fuzzy sets," *Journal of New Theory*, vol. 24, pp. 59–72, 2018.
- [29] P. Liu, M. Munir, T. Mahmood, and K. Ullah, "Some similarity measures for interval-valued picture fuzzy sets and their applications in decision-making," *Information (Switzerland)*, vol. 10, no. 12, pp. 1–23, 2019.
- [30] S. Khan, S. Abdullah, and S. Ashraf, "Picture fuzzy aggregation information based on Einstein operations and their application in decision making," *Mathematical Sciences*, vol. 13, no. 3, pp. 213–229, 2019.
- [31] N. Van Dinh and N. Xuan Thao, "Some measures of picture fuzzy sets and their application in multi-attribute decision making," *International Journal of Mathematical Sciences and Computing*, vol. 4, no. 3, pp. 23–41, 2018.
- [32] D. Liu, Y. Luo, and Z. Liu, "The linguistic picture fuzzy set and its application in multi-criteria decision-making: an illustration to the topsis and todim methods based on entropy weight," *Symmetry*, vol. 12, no. 1170, pp. 1–27, 2020.
- [33] P. Dutta and S. Ganju, "Some aspects of picture fuzzy set," *Transactions of A. Razmadze Mathematical Institute*, vol. 172, pp. 164–175, 2018.
- [34] E. S. Lee, "An interval dempster-shafer approach," *Computers & Mathematics with Applications*, vol. 24, no. 7, pp. 89–95, 1992.
- [35] C. Lucas and B. Nadjar Araabi, "Generalization of the dempstershafer theory: a fuzzy-valued measure," *IEEE Transactions On Fuzzy Systems*, vol. 7, no. 3, pp. 255–270, 1999.
- [36] L. Wang, H. Garg, and N. Li, "Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight," *Soft Computing*, vol. 25, no. 2, pp. 973–993, 2021.
- [37] A. Devraj and P. Chellamani, "Picture fuzzy labelling graphs with an application," *Annals of Optimization Theory & Practices*, vol. 3, no. 3, pp. 117–132, 2020.
- [38] P. A. Ejegwa, I. C. Onyeke, and V. Adah, "An algorithm for an improved intuitionistic fuzzy correlation measure with medical diagnostic application," *Annals of Optimization Theory & Practices*, vol. 3, no. 3, pp. 51–68, 2020.