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Research Article

Explicit Expression for Arbitrary Positive Integer Powers of Special Sparse Matrices

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Sparse matrices appear frequently in mathematical models. In this paper, we firstly present a general expression for the entries of the r th ($r \in \mathbb{N}$) power of a certain n-square sparse matrix, in terms of the Chebyshev polynomials of the second kind. Secondly, we present a method for integer positive powers of the skew matrix corresponding to these sparse matrices. This method will be inspiring to calculate the positive integer powers of the similar matrices. Finally, we present some examples to illustrate our results. Also, we give maple 18 procedures in order to verify our calculations.

1. Introduction

In order to solve some difference equations, differential equations, and delay differential equations, we meet the necessity to compute the arbitrary positive integer powers of the square matrix [1–5]. Many papers discuss this topic [6–19]. In this paper, we obtain the entries of positive integer powers of a n-square certain sparse matrix of the order $n = 4p + 3\,(p \in \mathbb{N})$ of the following form:

$$A = \begin{bmatrix}
0 & 0 & & & & & & & & 2\\
0 & 0 & -1 & & & & & & & \\
 & -1 & 0 & -1 & & & & & & \\
 & & -1 & 0 & 1 & & 0 & & & \\
 & & & 1 & 0 & 1 & & & & \\
 & & & & 1 & 0 & -1 & & & \\
 & & & & & -1 & 0 & -1 & & \\
 & & & & & & -1 & 0 & & \\
 & & & 0 & & & & & \ddots & \\
 & & & & & & & & 0 & 0\\
2 & & & & & & & & 0 & 0
\end{bmatrix}. \quad (1)$$

We compute the r th power ($r \in \mathbb{N}$) of the matrix and the skew matrix corresponding to it, by using the expression $A^r = TJ^rT^{-1}$ [14], where J is Jordan's form of the matrix A, and T is the transforming matrix A. Although these types of matrices may have applications, we investigate these matrices purely from a mathematical point of view. In this article, interesting formulas are presented that will be attractive for math enthusiasts. Now, we are beginning with the following definition.

Definition 1 (see [15]). The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \textit{when}\; x = \cos\theta,\ n = 0, 1, 2, \ldots. \quad (2)$$

With the definition above, we find that $U_n(x)$ satisfies the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), U_0(x) = 1, \\ U_1(x) = 2x, \quad n = 2, 3, \ldots. \quad (3)$$

2. Eigenvalues and Jordan's Form of Matrix A

In this section, we calculate the eigenvalues of matrix A which are given in (1). The following lemma helps us to calculate the eigenvalues of tridiagonal matrices.

Lemma 1 (see [16]). *Let* $\{H(n), \quad n = 1, 2, \ldots\}$ *be a sequence of tridiagonal matrices of the following form:*

$$H(n) = \begin{bmatrix} h_{11} & h_{12} & & & & & & \\ h_{21} & h_{22} & h_{23} & & & & & \\ & h_{32} & h_{33} & h_{34} & & & & \\ & & h_{43} & h_{44} & h_{45} & & 0 & \\ & & & h_{54} & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & h_{n-2n-1} & \\ & & & & & h_{n-1n-2} & h_{n-1n-1} & h_{n-1n} \\ & & & & & & h_{nn-1} & h_{nn} \end{bmatrix}. \tag{4}$$

Then, the successive determinants of $H(n)$ are given by the following recursive formula:

$$\begin{aligned} |H(1)| &= h_{11}, \\ |H(2)| &= h_{11}h_{22} - h_{12}h_{21}, \\ |H(n)| &= h_{nn}|H(n-1)| - h_{n-1n}h_{nn-1}|H(n-2)|. \end{aligned} \tag{5}$$

Let $\{\{H^*(n), \quad n = 1, 2, \ldots\}$ be a sequence of the tridiagonal matrix of the following form:

$$H^*(n) = \begin{bmatrix} h_{11} & -h_{12} & & & & & & \\ -h_{21} & h_{22} & -h_{23} & & & & & \\ & -h_{32} & h_{33} & h_{34} & & & & \\ & & h_{43} & h_{44} & h_{45} & & 0 & \\ & & & h_{54} & \ddots & & & \\ & & 0 & & & & (-1)^{p^2} h_{n-2n-1} & \\ & & & & & (-1)^{p^2} h_{n-1n-2} & h_{n-1n-1} & (-1)^{q^2} h_{n-1n} \\ & & & & & & (-1)^{q^2} h_{nn-1} & h_{nn} \end{bmatrix}, \tag{6}$$

where (p and q = 1, 2, 3, ...). Since the matrices $H(n)$ and $H^*(n)$ have the same recursive formula, it can be written as follows:

$$|H(n)| = |H^*(n)|. \tag{7}$$

Theorem 1. *The eigenvalues of A, as given in (4), are as follows:*

$$\lambda_k = -2\cos\frac{(k-1)\pi}{n-1}, \quad k = 1, 2, \ldots, n. \tag{8}$$

Proof. Let

$$D_n(x)=\begin{vmatrix} x & 0 & & & & & & & & & 2\\ 0 & x & -1 & & & & & & & & \\ & -1 & x & -1 & & & & & & & \\ & & -1 & x & 1 & & 0 & & & & \\ & & & 1 & x & 1 & & & & & \\ & & & & 1 & x & -1 & & & & \\ & & & & & -1 & x & -1 & & & \\ & & & & & & -1 & x & & & \\ & & & 0 & & & & & \ddots & & \\ & & & & & & & & & & 0\\ 2 & & & & & & & & & 0 & x \end{vmatrix},$$

$$\Delta_n(x)=\begin{vmatrix} x & 1 & & & & & & & \\ 1 & x & 1 & & & & & & \\ & 1 & x & 1 & & 0 & & & \\ & & 1 & x & 1 & & & & \\ & & & 1 & x & 1 & & & \\ & & & & 1 & x & 1 & & \\ & & & & & 1 & \ddots & & \\ & 0 & & & & & & x & 1\\ & & & & & & & 1 & x \end{vmatrix}. \tag{9}$$

If we calculate the determinant given in (9) by using the Laplace expansion, then we have

$$D_n(x)=\left(x^2-4\right)\Delta_{n-2}(x). \tag{10}$$

On the other hand, from Lemma 1, we get

$$\Delta_n(x)=x\Delta_{n-1}(x)-\Delta_{n-2}(x),n \geq 3,\Delta_1(x)=x,\Delta_2(x)=x^2-1. \tag{11}$$

Solving difference equation (11), we obtain

$$\Delta_n(x)=U_n\left(\frac{x}{2}\right), \tag{12}$$

where $U_n(x)$ is the n th degree Chebyshev polynomial of the second kind. From (10) and (12) follows.

$$\begin{aligned} D_n(x)&=\left(x^2-4\right)U_{n-2}\left(\frac{x}{2}\right),\\ D_n(-x)&=|A-xI|=\left(x^2-4\right)U_{n-2}\left(-\frac{x}{2}\right). \end{aligned} \tag{13}$$

Hence, the eigenvalues of A are

$$\lambda_k=-2\cos\frac{(k-1)\pi}{n-1}, \quad k=1,2,\ldots,n. \tag{14}$$

Since all the eigenvalues λ_k for $k=1,2,\ldots,n$ are simple, each eigenvalue λ_k corresponds single Jordan cell $J_1(\lambda_k)$ in the matrix J. Taking this into account, we write down Jordan's form of the matrix A.

$$J=\operatorname{diag}\left(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n\right). \tag{15}$$

And applying the relation $\lambda_k=-\lambda_{n-k+1}, k=1,2,\ldots,(n-1)/2$, we can write

$$J=\operatorname{diag}\left(\lambda_1,\lambda_2,\lambda_3,\ldots,0,\ldots,-\lambda_3,-\lambda_2,-\lambda_1\right). \tag{16}$$

□

3. General Expression for the Entries of A^r

In this section, we firstly find the transforming matrix T and its inverse for expression $A=TJT^{-1}$ by using Chebyshev polynomial second kind. Secondly, we derive a general expression for the entries of matrix A^r for $(r\in\mathbb{N})$. From relation, $A=TJT^{-1}$ follows

$$AT=TJ, \tag{17}$$

where $T=[T_1,T_2,\ldots,T_n]$ and $T_i=[T_{1i},T_{2i},\ldots,T_{ni}]^t$, for, $i=1,2,\ldots,n$; then, we can write

$$\begin{aligned} A\left[T_1,T_2,T_3,\ldots,T_n\right]&=\left[T_1,T_2,T_3,\ldots,T_n\right]\\ &\quad\cdot\operatorname{diag}\left(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n\right). \end{aligned} \tag{18}$$

Now, we have

$$AT_i=\lambda_iT_i, \quad i=1,2,3,\ldots,n. \tag{19}$$

From (19) follows.

$$\begin{bmatrix} 2T_{ni}\\ -T_{3i}\\ -T_{2i}-T_{4i}\\ -T_{3i}+T_{5i}\\ T_{4i}+T_{6i}\\ T_{5i}-T_{7i}\\ -T_{6i}-T_{8i}\\ -T_{7i}+T_{9i}\\ T_{8i}+T_{10i}\\ T_{9i}-T_{11i}\\ -T_{10i}-T_{12i}\\ \vdots\\ -T_{n-2i}\\ 2T_{1i} \end{bmatrix}=\begin{bmatrix} \lambda_iT_{1i}\\ \lambda_iT_{2i}\\ \lambda_iT_{3i}\\ \lambda_iT_{4i}\\ \lambda_iT_{5i}\\ \lambda_iT_{6i}\\ \lambda_iT_{7i}\\ \lambda_iT_{8i}\\ \lambda_iT_{9i}\\ \lambda_iT_{10i}\\ \lambda_iT_{11i}\\ \vdots\\ \lambda_iT_{n-1i}\\ \lambda_iT_{ni} \end{bmatrix}, \quad i=1,2,3,\ldots,n. \tag{20}$$

Solving the set of systems (20), we find the eigenvectors of the matrix A.

$$T_1=\begin{bmatrix}\alpha\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\\vdots\\-\alpha\end{bmatrix},T_n=\begin{bmatrix}\alpha\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\\vdots\\\alpha\end{bmatrix},T_i=\begin{bmatrix}0\\c_0\beta U_0\left(\frac{\lambda_i}{2}\right)\\c_1\beta U_1\left(\frac{\lambda_i}{2}\right)\\c_2\beta U_2\left(\frac{\lambda_i}{2}\right)\\c_3\beta U_3\left(\frac{\lambda_i}{2}\right)\\c_4\beta U_4\left(\frac{\lambda_i}{2}\right)\\c_5\beta U_5\left(\frac{\lambda_i}{2}\right)\\c_6\beta U_6\left(\frac{\lambda_i}{2}\right)\\c_7\beta U_7\left(\frac{\lambda_i}{2}\right)\\c_8\beta U_8\left(\frac{\lambda_i}{2}\right)\\c_9\beta U_9\left(\frac{\lambda_i}{2}\right)\\\vdots\\0\end{bmatrix}, \quad i=2,3,\ldots,n-1, \tag{21}$$

where $U_k(x)$ is the k th degree Chebyshev polynomial of the second kind, α and β are arbitrary numbers, and

$$c_i=\begin{cases}-1, & if, \quad i\in\mathbb{X}=\{4p-3|p=1,2,\ldots\},\\ 1, & if, \quad i\in\mathbb{N}\cup\{0\}-X.\end{cases} \tag{22}$$

Taking into account (21) and (22) and choosing $\alpha=\beta=1$ arbitrarily, we write down the transforming matrix T:

$$T=\begin{bmatrix}1&0&0&0&\ldots&1\\0&U_0\left(\frac{\lambda_2}{2}\right)&U_0\left(\frac{\lambda_3}{2}\right)&U_0\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&-U_1\left(\frac{\lambda_2}{2}\right)&-U_1\left(\frac{\lambda_3}{2}\right)&-U_1\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_2\left(\frac{\lambda_2}{2}\right)&U_2\left(\frac{\lambda_3}{2}\right)&U_2\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_3\left(\frac{\lambda_2}{2}\right)&U_3\left(\frac{\lambda_3}{2}\right)&U_3\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_4\left(\frac{\lambda_2}{2}\right)&U_4\left(\frac{\lambda_3}{2}\right)&U_4\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&-U_5\left(\frac{\lambda_2}{2}\right)&-U_5\left(\frac{\lambda_3}{2}\right)&-U_5\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_6\left(\frac{\lambda_2}{2}\right)&U_6\left(\frac{\lambda_3}{2}\right)&U_6\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_7\left(\frac{\lambda_2}{2}\right)&U_7\left(\frac{\lambda_3}{2}\right)&U_7\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&U_8\left(\frac{\lambda_2}{2}\right)&U_8\left(\frac{\lambda_3}{2}\right)&U_8\left(\frac{\lambda_4}{2}\right)&\ldots&0\\0&-U_9\left(\frac{\lambda_2}{2}\right)&-U_9\left(\frac{\lambda_3}{2}\right)&-U_9\left(\frac{\lambda_4}{2}\right)&\ldots&0\\\vdots&\vdots&\vdots&\vdots&\ldots&\vdots\\-1&0&0&0&\ldots&1\end{bmatrix}. \tag{23}$$

Denoting j th column of the inverse matrix T^{-1} by $\tau_j\,(T^{-1}=(\tau_1,\tau_2,\tau_3,\ldots,\tau_n))$, here,

$$\tau_j=\left[\tau_{1j},\tau_{2j},\tau_{3j},\ldots,\tau_{nj}\right]^t, \quad j=1,2,\ldots,n. \tag{24}$$

Then, we can write

$$[\tau_1,\tau_2,\tau_3,\ldots,\tau_{n-1},\tau_n]A=J[\tau_1,\tau_2,\tau_3,\ldots,\tau_{n-1},\tau_n]. \tag{25}$$

From (25), follows.

$$\begin{aligned}&[2\tau_n,-\tau_3,-\tau_2-\tau_4,-\tau_3+\tau_5,\tau_4+\tau_6,\ldots,\tau_{n-2},2\tau_1]\\&\quad=[J\tau_1,J\tau_2,\ldots,J\tau_{n-1},J\tau_n].\end{aligned} \tag{26}$$

Let $\tau_{i2} = m_i$ for $i = 1, 2, 3, \ldots, n$, then from solving the set of systems (26), we can conclude $\tau_{11} = -\tau_{1n} = \tau_{n1} = \tau_{nn} = \rho$ which ρ is an arbitrary number, $\tau_{i1} = \tau_{in} = 0$ for $i = 2, 3, \ldots, n-1$. Also, we can conclude $\tau_{i1} = \tau_{in} = 0$ for $i = 2, 3, \ldots n-1$, $\tau_{1j} = \tau_{nj} = 0$ for $j = 2, 3, \ldots n-1$, and other elements of the matrix T^{-1} are equal $m_i\times$ (Chebyshev polynomials second kind in eigenvalues matrix A). So we can write the columns matrix T^{-1} as follows:

$$\tau_1 = \begin{bmatrix} \rho \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \rho \end{bmatrix}, \tau_n = \begin{bmatrix} -\rho \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \rho \end{bmatrix}, \tau_j = d_{j-2} \begin{bmatrix} 0 \\ m_2 U_{j-2}\left(\frac{\lambda_2}{2}\right) \\ m_2 U_{j-2}\left(\frac{\lambda_3}{2}\right) \\ m_3 U_{j-2}\left(\frac{\lambda_4}{2}\right) \\ \vdots \\ m_{n-1} U_{j-2}\left(\frac{\lambda_{n-1}}{2}\right) \\ 0 \end{bmatrix}, \tag{27}$$

$$j = 2, 3, \ldots n-1,$$

where

$$d_j = \begin{cases} -1 & if, \quad j \in \mathbb{X} = \{4p-3 | p = 1, 2, \ldots\}, \\ 1 & if, \quad j \in \mathbb{N} \cup \{0\} - X. \end{cases} \tag{28}$$

Taking into account (27) and (28), we can write down the matrix T^{-1}:

$$T^{-1} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\rho \\ 0 & m_2U_0\left(\frac{\lambda_2}{2}\right) & -m_2U_1\left(\frac{\lambda_2}{2}\right) & m_2U_2\left(\frac{\lambda_2}{2}\right) & m_2U_3\left(\frac{\lambda_2}{2}\right) & m_2U_4\left(\frac{\lambda_2}{2}\right) & -m_2U_5\left(\frac{\lambda_2}{2}\right) & m_2U_6\left(\frac{\lambda_2}{2}\right) & m_2U_7\left(\frac{\lambda_2}{2}\right) & m_2U_8\left(\frac{\lambda_2}{2}\right) & -m_2U_9\left(\frac{\lambda_2}{2}\right) & \cdots & 0 \\ 0 & m_3U_0\left(\frac{\lambda_3}{2}\right) & -m_3U_1\left(\frac{\lambda_3}{2}\right) & m_3U_2\left(\frac{\lambda_3}{2}\right) & m_3U_3\left(\frac{\lambda_3}{2}\right) & m_3U_4\left(\frac{\lambda_3}{2}\right) & -m_3U_5\left(\frac{\lambda_3}{2}\right) & m_3U_6\left(\frac{\lambda_3}{2}\right) & m_3U_7\left(\frac{\lambda_3}{2}\right) & m_3U_8\left(\frac{\lambda_3}{2}\right) & -m_3U_9\left(\frac{\lambda_3}{2}\right) & \cdots & 0 \\ 0 & m_4U_0\left(\frac{\lambda_4}{2}\right) & -m_4U_1\left(\frac{\lambda_4}{2}\right) & m_4U_2\left(\frac{\lambda_4}{2}\right) & m_4U_3\left(\frac{\lambda_4}{2}\right) & m_4U_4\left(\frac{\lambda_4}{2}\right) & -m_4U_5\left(\frac{\lambda_4}{2}\right) & m_4U_6\left(\frac{\lambda_4}{2}\right) & m_4U_7\left(\frac{\lambda_4}{2}\right) & m_4U_8\left(\frac{\lambda_4}{2}\right) & -m_4U_9\left(\frac{\lambda_4}{2}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & m_{n-1}U_0\left(\frac{\lambda_{n-1}}{2}\right) & -m_{n-1}U_1\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_2\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_3\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_4\left(\frac{\lambda_{n-1}}{2}\right) & -m_{n-1}U_5\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_6\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_7\left(\frac{\lambda_{n-1}}{2}\right) & m_{n-1}U_8\left(\frac{\lambda_{n-1}}{2}\right) & -m_{n-1}U_9\left(\frac{\lambda_{n-1}}{2}\right) & \cdots & 0 \\ \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \rho \end{bmatrix}. \tag{29}$$

From $T^{-1}T = I$, we have

$$\rho + \rho = 1 \Rightarrow \rho = \frac{1}{2}. \tag{30}$$

And, for $i = 2, 3, \ldots, n-1$,

$$\begin{aligned}
& m_i \sum_{j=0}^{n-3} U_j^2\left(\frac{\lambda_i}{2}\right) = 1 \\
& \Rightarrow m_i \sum_{j=0}^{n-3} U_j^2\left(-\cos\frac{(i-1)\pi}{n-1}\right) = 1 \\
& \Rightarrow m_i \sum_{j=0}^{n-3} U_j^2\left(\cos\frac{(n-i)\pi}{n-1}\right) = 1 \\
& \Rightarrow m_i \sum_{j=0}^{n-3} \left[\frac{\sin(j+1)((n-i)/(n-1))\pi}{\sin((n-i)/(n-1))\pi}\right]^2 = 1 \\
& \Rightarrow m_i \sum_{j=1}^{n-2} \left[\frac{\sin j((i-1)/(n-1))\pi}{\sin((i-1)/(n-1))\pi}\right]^2 = 1 \\
& \Rightarrow \frac{m_i}{\sin^2((i-1)/(n-1))\pi} \sum_{j=1}^{n-2} \sin^2\frac{j(i-1)\pi}{n-1} = 1 \\
& \Rightarrow \frac{m_i}{2\sin^2((i-1)/(n-1))\pi} \sum_{j=1}^{n-2} \left(1-\cos\frac{2j(i-1)\pi}{n-1}\right) = 1 \\
& \Rightarrow \frac{m_i}{2\sin^2((i-1)/(n-1))\pi} \left[(n-2) - \sum_{j=1}^{n-2}\cos\frac{2j(i-1)\pi}{n-1}\right] = 1 \\
& \Rightarrow \frac{m_i}{2\sin^2((i-1)/(n-1))\pi} \left[\left(n-\frac{3}{2}\right) + \frac{\sin((i-1)/(n-1))\pi}{2\sin((i-1)/(n-1))\pi}\right] = 1 \left(\sum_{j=1}^{N}\cos j\theta = -\frac{1}{2} + \frac{\sin(N+1/2)\theta}{2\sin(\theta/2)}\right) \\
& \Rightarrow \frac{m_i(n-1)}{2\sin^2((i-1)/(n-1))\pi} = 1 \\
& \Rightarrow m_i = \frac{2\sin^2((i-1)/(n-1))\pi}{n-1}.
\end{aligned} \tag{31}$$

Therefore,

$$m_i = \frac{4-\lambda_i^2}{2(n-1)}, \quad i = 2,3,\ldots n-1. \tag{32}$$

From (30) and (32) follows.

$$T^{-1} = \frac{1}{2(n-1)}\begin{bmatrix} (n-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -(n-1) \\ 0 & (4-\lambda_2^2)U_0\left(\frac{\lambda_2}{2}\right) & -(4-\lambda_2^2)U_1\left(\frac{\lambda_2}{2}\right) & + & + & + & -(4-\lambda_2^2)U_5\left(\frac{\lambda_2}{2}\right) & (4-\lambda_2^2)U_6\left(\frac{\lambda_2}{2}\right) & + & + & - & \ldots & 0 \\ 0 & (4-\lambda_3^2)U_0\left(\frac{\lambda_3}{2}\right) & -(4-\lambda_3^2)U_1\left(\frac{\lambda_3}{2}\right) & + & + & + & -(4-\lambda_3^2)U_5\left(\frac{\lambda_3}{2}\right) & (4-\lambda_3^2)U_6\left(\frac{\lambda_3}{2}\right) & + & + & - & \ldots & 0 \\ 0 & (4-\lambda_4^2)U_0\left(\frac{\lambda_4}{2}\right) & -(4-\lambda_4^2)U_1\left(\frac{\lambda_4}{2}\right) & + & + & + & -(4-\lambda_4^2)U_5\left(\frac{\lambda_4}{2}\right) & (4-\lambda_4^2)U_6\left(\frac{\lambda_4}{2}\right) & + & + & - & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & (4-\lambda_{n-1}^2)U_0\left(\frac{\lambda_{n-1}}{2}\right) & -(4-\lambda_{n-1}^2)U_1\left(\frac{\lambda_{n-1}}{2}\right) & + & + & + & -(4-\lambda_{n-1}^2)U_5\left(\frac{\lambda_{n-1}}{2}\right) & (4-\lambda_{n-1}^2)U_6\left(\frac{\lambda_{n-1}}{2}\right) & + & + & - & \ldots & 0 \\ (n-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & (n-1) \end{bmatrix}. \tag{33}$$

By combining (15), (23), (32) and using the equality $A^r = TJ^rT^{-1}$, we compute the r th powers of the matrix A. Therefore, (i, j) th entry of matrix A^r can be given as follows:

$$[A^r]_{ij} = \left[TJ^rT^{-1}\right]_{ij} = \frac{1}{2(n-1)} \times \begin{cases} (n-1)\left[\frac{1}{j}\right](\lambda_1^r + \lambda_n^r) & if \quad i = 1, j = 1,2,\ldots,n-1 \\ (n-1)\left[\frac{i}{n}\right](-\lambda_1^r + \lambda_n^r) & if \quad j = 1, i = 2,3,\ldots n \\ \sum_{k=2}^{n-1} c_{i-2}d_{j-2}\lambda_k^r(4-\lambda_k^2)U_{i-2}\left(\frac{\lambda_k}{2}\right)U_{j-2}\left(\frac{\lambda_k}{2}\right) & if \quad i,j = 2,3,\ldots,n-1 \\ (n-1)\left[\frac{1}{i}\right](-\lambda_1^r + \lambda_n^r) & if \quad j = n, i = 1,2,\ldots n-1 \\ (n-1)\left[\frac{j}{n}\right](\lambda_1^r + \lambda_n^r) & if \quad i = n, j = 2,3,\ldots,n \end{cases}. \tag{34}$$

For any real number x, $[x]$ = the largest integer that is less than or equal to x.

In the continuation of this article, we calculate the positive integer powers of the skew matrices corresponding to matrix (1) as follows:

$$B=\begin{bmatrix} 2 & & & & & & & & & 0 & 0\\ & & & & & & & & -1 & 0 & 0\\ & & & & & & & -1 & 0 & -1 & \\ & & 0 & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 0 & 1 & & & \\ & & & & -1 & 0 & 1 & & & & \\ & & & -1 & 0 & -1 & & & & & \\ & & & 0 & -1 & & & & & & \\ & & \iddots & & & & & 0 & & & \\ 0 & 0 & & & & & & & & & \\ 0 & 0 & & & & & & & & & 2 \end{bmatrix}. \tag{35}$$

Let

$$J=\begin{bmatrix} & & & & 1\\ & 0 & & 1 & \\ & & \iddots & & \\ & 1 & & 0 & \\ 1 & & & & \end{bmatrix}. \tag{36}$$

And partition matrices A^r and JA^r are as follows:

(1) If r is an even number,

$$A^r=\begin{bmatrix} 2^r \\ [A^r] \\ 2^r \end{bmatrix}, JA^r=\begin{bmatrix} 2^r \\ [JA^r] \\ 2^r \end{bmatrix}. \tag{37}$$

(2) If r is an odd number,

$$A^r=\begin{bmatrix} 2^r \\ [A^r] \\ 2^r \end{bmatrix}, JA^r=\begin{bmatrix} 2^r \\ [JA^r] \\ 2^r \end{bmatrix}. \tag{38}$$

With simple calculations, we have

$$B=AJ. \tag{39}$$

Theorem 2. *If matrices A, B and J have the form (1), (34) and (35), respectively, then r th power ($r\in\mathbb{N}$) of matrix B_n can compute as follows.*

For $k=1,2,3,\ldots$.

State 1. If $n=4m+1$, $m=1,2,\ldots$, then

$$B^r=\begin{cases} A^r, & \text{for} \quad r=2k,\\ JA^r, & \text{for} \quad r=2k+1. \end{cases} \tag{40}$$

State 2. If $n=4m+3$, $m=1,2,\ldots$, then

$$B^r=\begin{cases} \begin{bmatrix} 2^r \\ -[A^r] \\ 2^r \end{bmatrix}, & for \quad r=4k-2,\\ JA^r, & for \quad r=4k-1,\\ A^r, & for \quad r=4k,\\ \begin{bmatrix} 2^r \\ -[JA^r] \\ 2^r \end{bmatrix}, & for \quad r=4k+1. \end{cases} \tag{41}$$

Proof. We prove this theorem by induction on k. □

State 3. The base case of $k=1$ is true, *because*, with simple calculations, we have

$$B^2=A^2, B^3=JA^3. \tag{42}$$

We suppose that the result is true for $k>1$ and consider case $k+1$.

By the induction hypothesis, we have

$$B^{2k}=A^{2k}, B^{2k+1}=JA^{2k+1}. \tag{43}$$

We also *show* that the case $k+1$ is true

$$\begin{aligned} B^{2k+2}&=B^{2k}B^2=A^{2k}A^2=A^{2k+2},\\ B^{2k+3}&=B^{2k+1}B^2=JA^{2k+1}A^2=JA^{2k+3}. \end{aligned} \tag{44}$$

State 4. The case $k=1$ is true, and after the *necessary* calculations, we have

$$B^2=\begin{bmatrix} 2^2 \\ -[A^2] \\ 2^2 \end{bmatrix}, B^3=JA^3, B^4=A^4, B^5=\begin{bmatrix} 2^5 \\ -[JA^5] \\ 2^5 \end{bmatrix}. \tag{45}$$

We suppose that the result is true for $k>1$ and *show* that case $k+1$ is true.

By the induction hypothesis, we have

$$\begin{aligned} B^{4k-2}&=\begin{bmatrix} 2^{4k-2} \\ -[A^{4k-2}] \\ 2^{4k-2} \end{bmatrix}, B^{4k-1}=JA^{4k-1}, B^{4k}\\ &=A^{4k}, B^{4k+1}=\begin{bmatrix} 2^{4k+1} \\ -[JA^{4k+1}] \\ 2^{4k+1} \end{bmatrix}. \end{aligned} \tag{46}$$

We also show that case $k+1$ is true

$$B^{4k+2} = B^{4k-2}B^4 = \begin{bmatrix} 2^{4k-2} \\ -\left[A^{4k-2}\right] \\ 2^{4k-2} \end{bmatrix} \begin{bmatrix} 2^2 \\ -\left[A^2\right] \\ 2^2 \end{bmatrix}^2$$
$$= \begin{bmatrix} 2^{4k-2} \\ -\left[A^{4k-2}\right] \\ 2^{4k-2} \end{bmatrix} \begin{bmatrix} 2^4 \\ \left[A^4\right] \\ 2^4 \end{bmatrix}$$
$$= \begin{bmatrix} 2^{4k+2} \\ -\left[A^{4k+2}\right] \\ 2^{4k+2} \end{bmatrix},$$
$$B^{4k+3} = B^{4k-1}B^4 = JA^{4k-1}A^4 = JA^{4k+3}, B^{4k+4} = B^{4k}B^4 = A^{4k}A^4 = A^{4k+4} \quad (47)$$
$$B^{4k+5} = B^{4k+1}B^4 = \begin{bmatrix} 2^{4k+1} \\ -\left[JA^{4k+1}\right] \\ 2^{4k+1} \end{bmatrix} \begin{bmatrix} 2^2 \\ -\left[A^2\right] \\ 2^2 \end{bmatrix}^2$$
$$= \begin{bmatrix} 2^{4k+1} \\ -\left[A^{4k+1}\right] \\ 2^{4k+1} \end{bmatrix} \begin{bmatrix} 2^4 \\ \left[A^4\right] \\ 2^4 \end{bmatrix}$$
$$= \begin{bmatrix} 2^{4k+5} \\ -\left[A^{4k+5}\right] \\ 2^{4k+5} \end{bmatrix}.$$

Thus, the formulas also hold for $k+1$, and the induction arguments are completed.

We can compute (i, j) th entry of matrix B^r in (35) by using formula (34) and Theorem 2.

4. Numerical Consideration

In this section, we give an example that calculates the powers matrix (1). Calculations given in this section can be verified by the Maple 18 procedure that is given in Appendix.

Example 1. If $n = 7$, then let A_1 be a 7×7 matrix, given in (1) as in the following:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (48)$$

$2rd$, $3rd$, and $4rd$ powers of the matrix A_1 are computed as in the following.

From (15), the eigenvalues of the matrix A_1 can be written for $k = 1, 2, \ldots, 7$ as follows:

$\lambda_k = -2\cos(k-1)\pi/n-1$, namely, $\lambda_1 = -2$, $\lambda_2 = -\sqrt{3}$,$\lambda_3 = -1$, $\lambda_4 = 0$, $\lambda_5 = 1$, $\lambda_6 = \sqrt{3}$, and $\lambda_7 = 2$.

From (23) and (32), we can write the transforming matrix T and its inverse as follows:

$$T=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & U_0\left(\frac{\lambda_2}{2}\right) & U_0\left(\frac{\lambda_3}{2}\right) & U_0\left(\frac{\lambda_4}{2}\right) & U_0\left(\frac{\lambda_5}{2}\right) & U_0\left(\frac{\lambda_6}{2}\right) & 0 \\ 0 & -U_1\left(\frac{\lambda_2}{2}\right) & -U_1\left(\frac{\lambda_3}{2}\right) & -U_1\left(\frac{\lambda_4}{2}\right) & -U_1\left(\frac{\lambda_5}{2}\right) & -U_1\left(\frac{\lambda_6}{2}\right) & 0 \\ 0 & U_2\left(\frac{\lambda_2}{2}\right) & U_2\left(\frac{\lambda_3}{2}\right) & U_2\left(\frac{\lambda_4}{2}\right) & U_2\left(\frac{\lambda_5}{2}\right) & U_2\left(\frac{\lambda_6}{2}\right) & 0 \\ 0 & U_3\left(\frac{\lambda_2}{2}\right) & U_3\left(\frac{\lambda_3}{2}\right) & U_3\left(\frac{\lambda_4}{2}\right) & U_3\left(\frac{\lambda_5}{2}\right) & U_3\left(\frac{\lambda_6}{2}\right) & 0 \\ 0 & U_4\left(\frac{\lambda_2}{2}\right) & U_4\left(\frac{\lambda_3}{2}\right) & U_4\left(\frac{\lambda_4}{2}\right) & U_4\left(\frac{\lambda_5}{2}\right) & U_4\left(\frac{\lambda_6}{2}\right) & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & \sqrt{3} & 1 & 0 & -1 & -\sqrt{3} & 0 \\ 0 & 2 & 0 & -1 & 0 & 2 & 0 \\ 0 & -\sqrt{3} & 1 & 0 & -1 & \sqrt{3} & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T^{-1}=\frac{1}{2(7-1)}\times\begin{bmatrix} (7-1) & 0 & 0 & 0 & 0 & 0 & -(7-1) \\ 0 & \left(4-\lambda_2^2\right)U_0\left(\frac{\lambda_2}{2}\right) & -\left(4-\lambda_2^2\right)U_1\left(\frac{\lambda_2}{2}\right) & \left(4-\lambda_2^2\right)U_2\left(\frac{\lambda_2}{2}\right) & \left(4-\lambda_2^2\right)U_3\left(\frac{\lambda_2}{2}\right) & \left(4-\lambda_2^2\right)U_4\left(\frac{\lambda_2}{2}\right) & 0 \\ 0 & \left(4-\lambda_3^2\right)U_0\left(\frac{\lambda_3}{2}\right) & -\left(4-\lambda_3^2\right)U_1\left(\frac{\lambda_3}{2}\right) & \left(4-\lambda_3^2\right)U_2\left(\frac{\lambda_3}{2}\right) & \left(4-\lambda_3^2\right)U_3\left(\frac{\lambda_3}{2}\right) & \left(4-\lambda_3^2\right)U_4\left(\frac{\lambda_3}{2}\right) & 0 \\ 0 & \left(4-\lambda_4^2\right)U_0\left(\frac{\lambda_4}{2}\right) & -\left(4-\lambda_4^2\right)U_1\left(\frac{\lambda_4}{2}\right) & \left(4-\lambda_4^2\right)U_2\left(\frac{\lambda_4}{2}\right) & \left(4-\lambda_4^2\right)U_3\left(\frac{\lambda_4}{2}\right) & \left(4-\lambda_4^2\right)U_4\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & \left(4-\lambda_5^2\right)U_0\left(\frac{\lambda_5}{2}\right) & -\left(4-\lambda_5^2\right)U_1\left(\frac{\lambda_5}{2}\right) & \left(4-\lambda_5^2\right)U_2\left(\frac{\lambda_5}{2}\right) & \left(4-\lambda_5^2\right)U_3\left(\frac{\lambda_5}{2}\right) & \left(4-\lambda_5^2\right)U_4\left(\frac{\lambda_5}{2}\right) & 0 \\ 0 & \left(4-\lambda_6^2\right)U_0\left(\frac{\lambda_6}{2}\right) & -\left(4-\lambda_6^2\right)U_1\left(\frac{\lambda_6}{2}\right) & \left(4-\lambda_6^2\right)U_2\left(\frac{\lambda_6}{2}\right) & \left(4-\lambda_6^2\right)U_3\left(\frac{\lambda_6}{2}\right) & \left(4-\lambda_6^2\right)U_4\left(\frac{\lambda_6}{2}\right) & 0 \\ (7-1) & 0 & 0 & 0 & 0 & 0 & (7-1) \end{bmatrix},$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{12} & \frac{\sqrt{3}}{12} & \frac{1}{6} & -\frac{\sqrt{3}}{12} & \frac{1}{12} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{12} & -\frac{\sqrt{3}}{12} & \frac{1}{6} & \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{49}$$

Then, we get

$$A_1^2 = TJ^2T^{-1} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$A_1^3 = TJ^3T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{50}$$

$$A_1^4 = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 6 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}.$$

Also, (i, j) th entry of the matrices A_1^2, A_1^3, and A_1^4 can be verified by the formula given in (34).

Appendix

Following Maple 18 procedure calculates the r th power of the $n \times n$ sparse matrix (1):

```
>restart:
with(ListTools):
power: = proc(n,r)
local k,lambda,i,j,c,d,u,x,A, Cell;
for kfrom 0 to n
do
c[k]: = 1:
lambda[k]: = −2.cos (k − 1) · Pi/n − 1:
if irem(k,4) = 1thenc[k]: = −1
end if:
d[k]: = c[k]:
u[k]: = sin((k + 1) · arccos(x))/ sin(arccos(x)).:
end do;
Cell: = [ ]:
for i from 1to n
do
for j from 1 to n
do
if i = 1 and member(j,seq(t,t = 1...(n − 1))) = true then
A[r,i,j]: = 1/2 · (n − 1).(n − 1).floor (1/j).
(lambda[1])^r + (lambda[n])^r)
end if;
if j = 1 and member(i,seq(t,t = 2...n)) = true then
A[r,i,j]: = 1/2 · (n − 1).(n − 1).floor (i/n).
(−(lambda[1])^r + (lambda[n])^r)
```

end if;

if member(i,seq(t,t = 2. . . $(n-1)$)) and member(j,seq(t,t = 2. . . $(n-1)$)) = true then;

A[r,i,j]: = $1/2\cdot(n-1)$ .sum$c[i-2].d[j-2]\cdot(\text{lambda}[\text{kappa}])^r$ .4 − (lambda ([kappa])^2).subs(x = lambda [kappa]/2, u [i−2]).subs(x = lambda [kappa]/2,u[j−2]), kappa = 2. . . $(n-1)$)

end if;

if j = n and member(i,seq(t,t = 1. . . $(n-1)$)) = true then;

A[r,i,j]: = $1/2\cdot(n-1).(n-1)$.floo1 $(1/i)$. $(-(\text{lambda}[1])^r+(\text{lambda}[n])^r)$

end if;

if i = n and member(j,seq(t,t = 2. . .n) = true then

A[r,i,j]: = $1/2\cdot(n-1).(n-1)$.floor (j/n). $((\text{lambda}[1])^r+(\text{lambda}[n])^r)$

end if;

Cell:= FlattenOnce([Cell,A[r,i,j]]);

od;

od;

interface(rtablesize = infinity): simplify(Matrix(n,n,Cell));

end proc;

>power(11,1)

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&2\\ 0&0&-1&0&0&0&0&0&0&0&0\\ 0&-1&0&-1&0&0&0&0&0&0&0\\ 0&0&-1&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&-1&0&0&0&0\\ 0&0&0&0&0&-1&0&-1&0&0&0\\ 0&0&0&0&0&0&-1&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 2&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}. \tag{51}$$

>power(11,2)

$$\begin{bmatrix} 4&0&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0&0\\ 0&0&2&0&-1&0&0&0&0&0&0\\ 0&1&0&2&0&1&0&0&0&0&0\\ 0&0&-1&0&2&0&-1&0&0&0&0\\ 0&0&0&1&0&2&0&1&0&0&0\\ 0&0&0&0&-1&0&2&0&-1&0&0\\ 0&0&0&0&0&1&0&2&0&1&0\\ 0&0&0&0&0&0&-1&0&2&0&0\\ 0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&4 \end{bmatrix}. \tag{52}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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