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Research Article

Best Proximity Points of Multivalued Hardy-Roger's Type (Cyclic) Contractive Mappings of *b*-Metric Spaces

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In this article, we introduce a new type of generalized multivalued Hardy and Roger's type proximal contractive and proximal cyclic contractive mappings of b-metric spaces and develop some results for the existence of best proximity point(s). Moreover, we obtain some results for the existence and uniqueness of best proximity points for single-valued mappings. Examples are given to explain the main results.

1. Introduction

The metric fixed point theory plays a very fundamental role in many fields of mathematics especially in nonlinear analysis and some related disciplines. The fundamental tool of this theory is the Banach contraction principle (shortly BCP) [1] which states that if a self-mapping $T: P \longrightarrow P$ of a complete metric space (P, ϱ) with metric ϱ satisfies

$$\varrho(Tp_1, Tp_2) \leq k\varrho(p_1, p_2), \tag{1}$$

for all $p_1, p_2 \in P$, and for some $k \in [0, 1)$, T has a unique fixed point, that is, there exists a point $p \in P$, such that

$$\varrho(p, Tp) = 0. \tag{2}$$

A mapping that satisfies (1) is known as Banach contraction. After this remarkable result, many mathematicians contributed for the development of fixed-point theory by producing many results with different generalized contractive mappings in complete metric spaces, for details one can see [2–8] and the references therein. One of the important generalizations of BCP was presented by Edelstein [9] in 1962. Later on, many mathematicians generalized Edelstein's result, for instance Meir and Keeler [10] in 1969 and Reich [11] in 1971. Reich's result has been further generalized by Hardy and Roger [12] in 1973 as follows.

Theorem 1. *Let (P, ϱ) be a metric space and $T: P \longrightarrow P$ a self-mapping satisfying the following conditions for all $p_1, p_2 \in P$:*

(1)

$$\varrho(Tp_1, Tp_2) \leq \alpha\varrho(p_1, Tp_1) + \beta\varrho(p_2, Tp_2) + \gamma\varrho(p_1, Tp_2) + \delta\varrho(p_2, Tp_1) + \tau\varrho(p_1, p_2), \tag{3}$$

where $\alpha, \beta, \gamma, \delta, \tau$ are nonnegative reals.
Set $\Omega = \alpha + \beta + \gamma + \delta + \tau$. Then,

(a) If P is complete and $\Omega < 1$, then T has a unique fixed point

(b) If (1) is modified as

(1′) for all $p_1 \neq p_2$ implies

$$\varrho(Tp_1, Tp_2) < \alpha\varrho(p_1, Tp_1) + \beta\varrho(p_2, Tp_2) + \gamma\varrho(p_1, Tp_2) + \delta\varrho(p_2, Tp_1) + \tau\varrho(p_1, p_2), \tag{4}$$

and P is compact, T is continuous, and $\Omega = 1$*; then, T has a unique fixed point.*

Nadler [13] in 1969 generalized the BCP in the context of multivalued mappings of complete metric spaces. Later on, Nadler's result has been generalized by Prolla [14] in 1983.

Meanwhile, the metric space has been generalized to b-metric space; by then, the fixed point theory has been further generalized for single-valued and multivalued mappings in the context of b-metric space, for instance, Bakhtin [15] in 1989 and Czerwik [16] in 1993.

For nonself mapping, $T: R \longrightarrow S$ (R and S are two nonempty sets), such that $R \cap T(R) = \varnothing$ (empty set); then, it is not possible to find the fixed point of T. The best way to deal with such situation is to explore a point r in R, such that

$$\varrho(r, \mathrm{Tr}) = \varrho(R, S), \tag{5}$$

where

$$\varrho(R, S) = \inf_{r\in R, s\in S} \varrho(r, s), \tag{6}$$

and if such a point in R exists, it is called the best proximity point of T. If $R = S$, then the best proximity point becomes a fixed point. So, best proximity point theory is the proper generalization of fixed-point theory. Fan's result [17] in 1969 was probably the first attempt in this direction.

Later on, many mathematicians extended Fan's result and developed some best proximity point results. For more details, one can see [18]. Best proximity point theory has been further developed by using different proximal contractions, for more details, one can see references [19–23].

Kirk [24] in 2003 introduced cyclic contraction and developed some fixed points results. Later on in 2006, Eldered and Veeramani [25] developed some best proximity point results for cyclic contractions.

Basha in 2019 [21] introduced proximal contractive and proximal cyclic contractive mappings and developed some results for the existence and uniqueness of best proximity point.

Recently, in 2021, Hiranmoy et al. [26] introduced proximal Kannan-type and proximal cyclic Kannan-type contractive mappings in metric spaces (compare with [21]) and developed some best proximity point results.

Motivated by the contractive mappings of Hiranmoy, we introduce the notion of multivalued Hardy and Roger's type proximal and cyclic proximal contractive mappings and develop some results for the existence of best proximity points in b-metric space. Furthermore, we give some examples to explain the results.

2. Preliminaries

Throughout this article, $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_1$, and $\wp(P)$ denote the set of reals, nonnegative reals, positive integers, nonnegative integers, and collection of nonempty subsets of P, respectively.

Definition 2. Let P be a nonempty set and $b \geq 1$ a real number. The mapping $\varrho_b: P \times P \longrightarrow [0, \infty)$ is a b-metric and (P, ϱ_b) is called b-metric space if ϱ_b satisfies the following axioms:

(b_1) $\varrho_b(p_1, p_2) = 0$ if and only if $p_1 = p_2$

(b_2) $\varrho_b(p_1, p_2) = \varrho_b(p_2, p_1)$

(b_3) $\varrho_b(p_1, p_2) \leq b[\varrho_b(p_1, p_3) + \varrho_b(p_3, p_2)]$, for all $p_1, p_3, p_2 \in P$.

Throughout this paper, ϱ and ϱ_b denote metric and b-metric, respectively. Now, suppose that R and S are two nonempty subsets of (P, ϱ_b). Define

$$\begin{aligned} \varrho_b(R, S) &= \inf\{\varrho_b(r, s): r \in R, s \in S\}, \\ R_0 &= \{r \in R: \varrho_b(r, s) = \varrho_b(R, S) \text{ for some } s \in S\}, \\ S_0 &= \{s \in S: \varrho_b(r, s) = \varrho_b(R, S) \text{ for some } r \in R\}. \end{aligned} \tag{7}$$

Definition 3. A b-metric space (P, ϱ_b) is boundedly compact if every bounded sequence in P has a convergent subsequence (compare with [27]).

Definition 4 (see [26]). Let R and S be two nonempty subsets of (P, ϱ). A mapping $T: R \longrightarrow S$ is said to be a proximal Kannan-type contractive mapping if

$$\begin{aligned} &\left.\begin{aligned} &\varrho(r_1, \mathrm{Tr}_3) = \varrho(R, S) \\ &\varrho(r_2, \mathrm{Tr}_4) = \varrho(R, S) \\ &r_3 \neq r_4 \end{aligned}\right\} \text{implies } \varrho(r_1, r_2) < \frac{1}{2}(\varrho(r_1, r_3) + \varrho(r_2, r_4)), \\ &\left.\begin{aligned} &\varrho(r_1, \mathrm{Tr}_3) = \varrho(R, S) \\ &\varrho(r_2, \mathrm{Tr}_4) = \varrho(R, S) \\ &r_3 = r_4 \end{aligned}\right\} \text{implies } \varrho(r_1, r_2) \leq \frac{1}{2}(\varrho(r_1, r_3) + \varrho(r_2, r_4)), \end{aligned} \tag{8}$$

hold for all $r_1, r_2, r_3, r_4 \in R$.

Definition 5. Let R and S be two nonempty subsets of (P, ϱ_b). Then, a mapping $T\colon R\cup S \longrightarrow R\cup S$ is said to be cyclic if $T(R)\subset S$ and $T(S)\subset R$ (compare with [26]).

Definition 6 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ). A cyclic mapping $T\colon R\cup S \longrightarrow R\cup S$ is said to be a proximal cyclic Kannan-type contractive mapping if*

$$\left.\begin{array}{l}\varrho(r_1, \mathrm{Tr}_3) = \varrho(R,S)\\ \varrho(r_2, \mathrm{Tr}_4) = \varrho(R,S)\\ \varrho(r_3, r_4) > \varrho(R,S)\end{array}\right\}\text{implies } \varrho(r_1, r_2) < \frac{1}{2}\left(\varrho(r_1, r_3) + \varrho(r_2, r_4)\right),$$
$$\left.\begin{array}{l}\varrho(r_1, \mathrm{Tr}_3) = \varrho(R,S)\\ \varrho(r_2, \mathrm{Tr}_4) = \varrho(R,S)\\ \varrho(r_3, r_4) = \varrho(R,S)\end{array}\right\}\text{implies } \varrho(r_1, r_2) = \varrho(R,S), \tag{9}$$

that hold for all $r_1, r_2, r_3, r_4 \in R$.

In the following, we introduce a compact weak proximal pair in b-metric space.

Definition 7. Let R and S be two nonempty subsets of (P, ϱ_b). The pair (R,S) is said to be a compact weak proximal pair if for bounded sequences (r_n) in R and (s_n) in S with $\varrho_b(r_n, s_n) \longrightarrow \varrho_b(R,S)$ as $n \longrightarrow \infty$, the sequences (r_n) and (s_n) have convergent subsequences in R and S, respectively (compare with [26]).

Remark 8. Note that if $R = S$ in above definition, then (R,R) is a compact weak proximal pair if and only if R is boundedly compact.

Now, we present a lemma in the context of b-metric space (analogous to [[26], Lemma 2.2]) that will be used in the sequel to prove our main results.

Lemma 9. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is bounded, and (R,S) is a compact weak proximal pair. Then, $R_0 \neq \varnothing$, and hence, so is S_0.*

Proof. As

$$\varrho_b(R,S) = \inf\{\varrho_b(r,s)\colon r\in R, s\in S\}, \tag{10}$$

so for each $n \in N$, there exists $r_n \in R$ and $s_n \in S$, such that

$$\varrho_b(R,S) \leq \varrho_b(r_n, s_n) < \varrho_b(R,S) + \frac{1}{n}. \tag{11}$$

Therefore, the sequence $(\varrho_b(r_n, s_n))$ converges to $\varrho_b(R,S)$. Now, we assume that R is bounded. So, there exists a positive real number K, such that $\varrho_b(r_n, r_m) \leq K$ for all $n, m \in \mathbb{N}$, so we have

$$\begin{aligned}\varrho_b(s_n, s_m) &\leq b\left(\varrho_b(s_n, r_n) + \varrho_b(r_n, s_m)\right)\\ &\leq b\left[\varrho_b(s_n, r_n) + b\left(\varrho_b(r_n, r_m) + \varrho_b(r_m, s_m)\right)\right],\end{aligned} \tag{12}$$

which implies

$$\varrho_b(s_n, s_m) < b\left[\varrho_b(R,S) + 1 + b\left(K + \varrho_b(R,S) + 1\right)\right). \tag{13}$$

Therefore, (r_n) and (s_n) are bounded sequences. So by compact weak proximality of the pair (R,S), there exist subsequences (r_{n_k}) of (r_n) and (s_{n_k}) of (s_n), such that (r_{n_k}) converges to $r_\star \in R$ and (s_{n_k}) converges to $s_\star \in S$. Therefore,

$$\varrho_b\left(r_{n_k}, s_{n_k}\right) \longrightarrow \varrho_b(r_\star, s_\star) \text{ as } k \longrightarrow \infty. \tag{14}$$

Thus, we have

$$\varrho_b(r_\star, s_\star) = \varrho_b(R,S). \tag{15}$$

So, $r_\star \in R_0$ and $s_\star \in S_0$. Hence, $R_0 \neq \varnothing$ and $S_0 \neq \varnothing$. Similarly, if S is bounded, then $R_0 \neq \varnothing$ and $S_0 \neq \varnothing$. □

Theorem 10 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ), such that at least one of R and S is bounded, and (R,S) is a compact weak proximal pair. Let $T\colon R \longrightarrow S$ be a proximal Kannan-type contractive mapping and assume that*

- *(i) $T(R_0) \subset S_0$*
- *(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho(r_n, s_n))$ converges to $\varrho(R,S)$, then $\lim_{n\longrightarrow\infty} \varrho(r_n, r_{n+1}) = 0$.*

Then, T has a unique best proximity point in R.

Theorem 11 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ), such that at least one of R and S is bounded, and (R,S) is a compact weak proximal pair. Let $T\colon R\cup S \longrightarrow R\cup S$ be a proximal cyclic Kannan-type contractive mapping and assume that the following conditions hold:*

- *(i) $T(R_0) \subset S_0$ and $T(S_0) \subset R_0$*
- *(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho(r_n, s_n))$ converges to $\varrho(R,S)$, then $\lim_{n\longrightarrow\infty} \varrho(r_n, r_{n+1}) = 0$.*

Then, the following conditions hold:

(a) *There exist* $r \in R$ *and* $s \in S$*, such that* $\varrho(r, \mathrm{Tr}) = \varrho(R, S)$ *and* $\varrho(s, Ts) = \varrho(R, S)$

(b) *If* $r \in R$ *and* $s \in S$*, such that* $\varrho(r, \mathrm{Tr}) = \varrho(R, S)$ *and* $\varrho(s, Ts) = \varrho(R, S)$*, then* $\varrho(r, s) = \varrho(R, S)$*.*

Now, we introduce the notions of a new type of generalized multivalued Hardy and Roger's proximal contractive and proximal cyclic contractive mappings.

Definition 12. Let R and S be two nonempty subsets of (P, ϱ_b). A mapping $T: R \longrightarrow \wp(S)$ is said to be a new type of generalized multivalued Hardy and Roger's proximal contractive mapping if

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ r_3 \neq r_4\end{array}\right\},$$

implies,

$$\left.\begin{array}{l}\varrho_b(r_1, r_2) < \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \dfrac{\gamma}{b^2}\varrho_b(r_3, r_4)\\ + \dfrac{\delta}{b}\varrho_b(r_1, r_4) + \dfrac{\tau}{b}\varrho_b(r_2, r_3)\end{array}\right\},$$

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ r_3 = r_4\end{array}\right\},$$

implies

$$\begin{aligned}\varrho_b(r_1, r_2) &\leq \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_3)\\ &\quad + \frac{\delta}{b}\varrho_b(r_1, r_4) + \frac{\tau}{b}\varrho_b(r_2, r_3),\end{aligned} \tag{16}$$

which hold for all $r_1, r_2, r_3, r_4 \in R$, where

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1 \tag{17}$$

$,\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

Remark 13. If in the Definition 12, we replace $T: R \longrightarrow \wp(S)$ by $T: R \longrightarrow S$, then T is said to be a new type of generalized Hardy and Roger's proximal contractive mapping.

Definition 14. Let R and S be two nonempty subsets of (P, ϱ_b). A multivalued cyclic mapping $T: R \cup S \longrightarrow \wp(R) \cup \wp(S)$ is said to be a new type of generalized multivalued Hardy and Roger's proximal cyclic contractive mapping if

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ \varrho_b(r_3, r_4) > \varrho_b(R, S)\end{array}\right\},$$

implies,

$$\left.\begin{array}{l}\varrho_b(r_1, r_2) < \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \dfrac{\gamma}{b^2}\varrho_b(r_3, r_4)\\ + \dfrac{\delta}{b}\varrho_b(r_1, r_4) + \dfrac{\tau}{b}\varrho_b(r_2, r_3)\end{array}\right\},$$

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ \varrho_b(r_3, r_4) = \varrho_b(R, S)\end{array}\right\} \text{implies} \varrho_b(r_1, r_2) = \varrho_b(R, S), \tag{18}$$

which hold for all $r_1, r_2, r_3, r_4 \in R$, where

$$\alpha + \beta + \gamma + 2\tau = 1, \alpha \neq 1, \beta \neq 1, \gamma + \delta + \tau < 1, \alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+. \tag{19}$$

Remark 15. If in the Definition 14, we replace $T: R \cup S \longrightarrow \wp(R) \cup \wp(S)$ by $T: R \cup S \longrightarrow R \cup S$, then T is said to be a new type of generalized Hardy and Roger's proximal cyclic contractive mapping.

3. Best Proximity Points Results for a New Type of Multivalued Hardy and Roger's Proximal Contractive Mappings in *b*-metric Space

The following is our main result of this section.

Theorem 16. *Let* R *and* S *be two nonempty subsets of* (P, ϱ_b)*, such that at least one of* R *and* S *is bounded and* (R, S) *is a compact weak proximal pair. Let* $T: R \longrightarrow \wp(S)$ *be a new type of generalized multivalued Hardy and Roger's proximal contractive mapping. Further assume that*

(i) *For each* $r \in R_0$, $\mathrm{Tr} \subset S_0$

(ii) *If* (r_n) *and* (s_n) *are two bounded sequences in* R *and* S*, respectively, such that* $(\varrho_b(r_n, s_n))$ *converges to* $\varrho_b(R, S)$*, then* $\lim_{n \longrightarrow \infty} \varrho_b(r_n, r_{n+1}) = 0$.

Then, T *has a best proximity point.*

Proof. Lemma 9 implies $R_0 \neq \varnothing$. Let $r_0 \in R_0$; then, $\mathrm{Tr}_0 \subset S_0$. We can pick an element $s_1 \in \mathrm{Tr}_0 \subset S_0$, so that there exists $r_1 \in R$, such that

$$\varrho_b(r_1, s_1) = \varrho_b(R, S). \tag{20}$$

Continuing this way, we can construct sequences (r_n) in R and (s_n) in Tr_{n-1}, such that

$$\varrho_b(r_n, s_n) = \varrho_b(R, S), \tag{21}$$

for all $n \in \mathbb{N}$. Therefore,

$$\varrho_b(R, S) \le \varrho_b(r_n, Tr_{n-1}) \le \varrho_b(r_n, s_n) = \varrho_b(R, S), \tag{22}$$

that is,

$$\varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S). \tag{23}$$

If $r_n = r_{n-1}$ for some $n \in \mathbb{N}$, then r_{n-1} is the best proximity point of T, and the proof is completed. So, we may assume that $r_n \ne r_{n-1}$ for all $n \in \mathbb{N}$. Now, we show that (r_n) and (s_n) are bounded sequences. As we have

$$\begin{gathered} \varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S), \\ \varrho_b(r_{n+1}, \mathrm{Tr}_n) = \varrho_b(R, S), \\ r_n \ne r_{n-1}, \end{gathered} \tag{24}$$

so by the given condition, we obtain

$$\begin{aligned} \varrho_b(r_n, r_{n+1}) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_n), \\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_n) + \frac{\tau}{b}\varrho_b(r_{n+1}, r_{n-1}) \\ &\le \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) \\ &\quad + \gamma\varrho_b(r_{n-1}, r_n) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1})), \end{aligned} \tag{25}$$

which implies

$$(1 - (\beta + \tau))\varrho_b(r_n, r_{n+1}) < (\alpha + \gamma + \tau)\varrho_b(r_{n-1}, r_n). \tag{26}$$

If $1 - (\beta + \tau) = 0$, then $\beta + \tau = 1$, so (17) implies $\alpha = \gamma = \tau = 0$, and so $\beta = 1$, a contradiction. Therefore, $1 - (\beta + \tau) \ne 0$ and $1 - (\beta + \tau) = \alpha + \gamma + \tau$. Hence, we get

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n), \tag{27}$$

that is,

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n) < \cdots < \varrho_b(r_0, r_1) = K\,(\text{say}). \tag{28}$$

As

$$\begin{gathered} \varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S), \\ \varrho_b(r_m, Tr_{m-1}) = \varrho_b(R, S), \end{gathered} \tag{29}$$

so, if $r_{n-1} = r_{m-1}$, then we have

$$\begin{aligned} \varrho_b(r_n, r_m) &\le b(\varrho_b(r_n, r_{m-1}) + \varrho_b(r_{m-1}, r_m)), \\ &\le b[b(\varrho_b(r_n, r_{n-1}) + \varrho_b(r_{n-1}, r_{m-1})) \\ &\quad + \varrho_b(r_{m-1}, r_m)], \\ &\le (b^2 + b)K. \end{aligned} \tag{30}$$

If $r_{n-1} \ne r_{m-1}$, then

$$\begin{aligned} \varrho_b(r_n, r_m) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_m, r_{m-1}) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_{m-1}), \\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_{m-1}) + \frac{\tau}{b}\varrho_b(r_m, r_{n-1}) \\ &\le (\alpha + \beta)K + \frac{\gamma}{b^2}[b(\varrho_b(r_{n-1}, r_n) + b(\varrho_b(r_n, r_m) + \varrho_b(r_m, r_{m-1})))] \\ &\quad + \delta(\varrho_b(r_n, r_m) + \varrho_b(r_m, r_{m-1})) + \tau(\varrho_b(r_m, r_n) + \varrho_b(r_n, r_{n-1})), \end{aligned} \tag{31}$$

which implies

$$(1 - (\gamma + \delta + \tau)\varrho_b(r_n, r_m) < (\alpha + \beta + 2\gamma + \delta + \tau)K), \tag{32}$$

and by (2), $\gamma + \delta + \tau < 1$. Therefore,

$$\varrho_b(r_n, r_m) < \frac{(\alpha + \beta + \gamma + \delta + \tau)}{1 - (\gamma + \delta + \tau)}K. \tag{33}$$

Hence, (r_n) is a bounded sequence. Furthermore, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \varrho_b(s_n, s_m) &\le b(\varrho_b(s_n, r_n) + \varrho_b(r_n, s_m)) \\ &\le b\varrho_b(R, S) + b^2(\varrho_b(r_n, r_m) + \varrho_b(r_m, s_m)) \\ &\le (b + b^2)\varrho_b(R, S) + b^2\frac{(\alpha + \beta + \gamma + \delta + \tau)}{1 - (\gamma + \delta + \tau)}K. \end{aligned} \tag{34}$$

Therefore, (s_n) is also a bounded sequence. From (69), it is clear that $(\varrho_b(r_n, r_{n+1}))$ is a decreasing sequence of nonnegative real numbers and hence convergent. Using hypothesis (ii), $(\varrho_b(r_n, r_{n+1}))$ converges to 0. Now, by compact weak proximality of the pair (R, S), there exist two subsequences (r_{n_k}) of (r_n) and (s_{n_k}) of (s_n), such that (r_{n_k}) converges to some $r_\star \in R$ and (s_{n_k}) converges to some $s_\star \in S$. Consequently,

$$\varrho_b(r_{n_k}, s_{n_k}) \longrightarrow \varrho_b(r_\star, s_\star) \text{as } k \longrightarrow \infty. \tag{35}$$

Consequently,

$$\varrho_b(r_\star, s_\star) = \varrho_b(R, S). \tag{36}$$

Thus, $r_\star \in R_0$, which implies $Tr_\star \subset S_0$. For each $\nu \in Tr_\star$, there exists $v \in R$, such that $\varrho_b(v, \nu) = \varrho_b(R, S)$. Now,

$$\varrho_b(R, S) \leq \varrho_b(v, Tr_\star) \leq \varrho_b(v, \nu) = \varrho_b(R, S), \tag{37}$$

which implies

$$\varrho_b(v, Tr_\star) = \varrho_b(R, S). \tag{38}$$

Moreover, we have

$$\varrho_b(r_{n_k+1}, r_\star) \leq b\left(\varrho_b(r_{n_k+1}, r_{n_k}) + \varrho_b(r_{n_k}, r_\star)\right). \tag{39}$$

Letting $k \longrightarrow \infty$, we get

$$\lim_{k \longrightarrow \infty} r_{n_k+1} = r_\star, \tag{40}$$

then using the facts

$$\begin{gathered} \varrho_b(r_{n_k+1}, Tr_{n_k}) = \varrho_b(R, S), \\ \varrho_b(v, Tr_\star) = \varrho_b(R, S), \end{gathered} \tag{41}$$

we get

$$\begin{aligned} \varrho_b(r_{n_k+1}, v) &\leq \alpha\varrho_b(r_{n_k+1}, r_{n_k}) + \beta\varrho_b(v, r_\star) + \frac{\gamma}{b^2}\varrho_b(r_{n_k}, r_\star) \\ &\quad + \frac{\delta}{b}\varrho_b(r_{n_k+1}, r_\star) + \frac{\tau}{b}\varrho_b(v, r_{n_k}) \\ &\leq \alpha\varrho_b(r_{n_k+1}, r_{n_k}) + \beta\varrho_b(v, r_\star) + \gamma\varrho_b(r_{n_k}, r_\star) \\ &\quad + \delta\varrho_b(r_{n_k+1}, r_\star) + \tau\varrho_b(v, r_{n_k}). \end{aligned} \tag{42}$$

Letting $k \longrightarrow \infty$, we get

$$[1 - (\beta + \tau)]\varrho_b(v, r_\star) \leq 0. \tag{43}$$

It implies $v = r_\star$. Thus, we have $\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S)$, that is, $r_\star$ is a best proximity point of T. This completes the proof.

Now, we give an example to explain our claim. □

Example 17. Let $P = \mathbb{R}$, $R = [1, 2]$, and $S = [1/4, 1/2]$. Consider

$$\varrho_b(p_1, p_2) = |p_1 - p_2|^2, \tag{44}$$

for all $p_1, p_2 \in \mathbb{R}$. Then, ϱ_b is a b-metric on P with $b = 2$. It implies $\varrho_b(R, S) = \{1/4\}$, $R_0 = \{1\}$, and $S_0 = \{1/2\}$; now, define a mapping $T: R \longrightarrow \wp(S)$ as follows:

$$\text{Tr} = \left\{\frac{1}{2^n}: 1 \leq n \leq r\right\}. \tag{45}$$

It implies for each $r \in R_0$, $\text{Tr} \subset S_0$. Now, we check T is a new type of generalized multivalued Hardy and Roger's proximal contractive mapping. Let

$$r_1, r_2, r_3, r_4 \in R. \tag{46}$$

Then, we discuss two possible cases.

Case 1: if

$$r_1, r_2, r_3, r_4 > 1, \tag{47}$$

then

$$\begin{gathered} \varrho_b(r_1, \text{Tr}_3) \neq d(R, S), \\ \varrho_b(r_2, \text{Tr}_4) \neq d(R, S). \end{gathered} \tag{48}$$

Case 2: if

$$r_1 = r_2 = r_3 = r_4 = 1, \tag{49}$$

then

$$\begin{gathered} \varrho_b(r_1, \text{Tr}_3) = d(R, S), \\ \varrho_b(r_2, \text{Tr}_4) = d(R, S), \\ r_3 = r_4. \end{gathered} \tag{50}$$

It implies

$$\begin{aligned} \varrho_b(r_1, r_2) = 0 &\leq \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \frac{\delta}{b}\varrho_b(r_1, r_4) \\ &\quad + \frac{\tau}{b}\varrho_b(r_2, r_3), \end{aligned} \tag{51}$$

where

$$\alpha + \beta + \gamma + 2\tau = 1, \alpha \neq 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{52}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$. So, all axioms of Theorem 16 are satisfied. Hence, T has the best proximity points set $\{1\}$.

Theorem 18. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is bounded, and (R, S) is a compact weak proximal pair. Let $T: R \longrightarrow S$ be a new type of generalized Hardy and Roger's proximal contractive mapping and assume that*

(i) For each $r \in R_0$, $\text{Tr} \in S_0$

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho_b(r_n, s_n))$ converges to $\varrho_b(R, S)$, then $\lim_{n \longrightarrow \infty}\varrho_b(r_n, r_{n+1}) = 0$

Then, T has a unique best proximity point.

Proof. Existence of best proximity point follows from Theorem 16. Now, to prove the uniqueness, consider r_1 and r_2 be two distinct best proximity points of T. Then, we have

$$\begin{aligned} \varrho_b(r_1, \mathrm{Tr}_1) &= \varrho_b(R, S), \\ \varrho_b(r_2, \mathrm{Tr}_2) &= \varrho_b(R, S), \\ r_1 &\neq r_2. \end{aligned} \tag{53}$$

It implies

$$\begin{aligned} \varrho_b(r_1, r_2) &< \alpha\varrho_b(r_1, r_1) + \beta\varrho_b(r_2, r_2) + \frac{\gamma}{b^2}\varrho_b(r_1, r_2) + \frac{\delta}{b}\varrho_b(r_1, r_2) + \frac{\tau}{b}\varrho_b(r_1, r_2), \\ &\leq \alpha\varrho_b(r_1, r_1) + \beta\varrho_b(r_2, r_2) + \gamma\varrho_b(r_1, r_2) + \delta\varrho_b(r_1, r_2) + \tau\varrho_b(r_1, r_2), \end{aligned} \tag{54}$$

so

$$\varrho_b(r_1, r_2) < (\gamma + \delta + \tau)\varrho_b(r_1, r_2) < \varrho_b(r_1, r_2), \tag{55}$$

a contradiction as $\gamma + \delta + \tau < 1$. Hence, T has a unique best proximity point. □

Corollary 19. *If we take in Theorem 18* $b = 1$ *and* $\alpha = \beta = 1/2, \gamma = \delta = \tau = 0$, *then we get Theorem 10.*

4. Best Proximity Points Results for a New Type of Multivalued Hardy and Roger's Proximal Cyclic Contractive Mappings in *b*-metric Space

In this section, we consider new type of multivalued Hardy and Roger's proximal cyclic contractive mapping for the existence of best proximity points.

Theorem 20. *Let R and S be two nonempty subsets of* (P, ϱ_b), *such that at least one of R and S is a bounded subset of P and* (R, S) *is a compact weak proximal pair. Let* $T: R \cup S \longrightarrow \wp(R) \cup \wp(S)$ *be a new type of generalized multivalued Hardy and Roger's proximal cyclic contractive mapping and assume that*

(i) *For each* $r \in R_0$, $Ts \subset S_0$, *and for each* $s \in S_0$, $Ts \subset R_0$, *and*

(ii) *If* (r_n) *and* (s_n) *are two bounded sequences in R and S, respectively, such that* $(\varrho_b(r_n, s_n))$ *converges to* $\varrho_b(R, S)$, *then* $\lim_{n\longrightarrow\infty}\varrho_b(r_n, r_{n+1}) = 0$.

Then, there exist $r \in R$ *and* $s \in S$, *such that* $\varrho_b(r, \mathrm{Tr}) = \varrho_b(R, S)$ *and* $\varrho_b(s, Ts) = \varrho_b(R, S)$, *and furthermore,* $\varrho_b(r, s) = \varrho_b(R, S)$.

Proof. Since (R, S) is a compact weakly proximal pair and at least one of R and S is bounded, so by Lemma 9, it follows that $R_0 \neq 0$ and $S_0 \neq 0$. Let $r_0 \in R_0$ and $s_0 \in S_0$ imply $\mathrm{Tr}_0 \subset S_0$ and $Ts_0 \subset R_0$, so there exists $r_1 \in R$ and $s_1 \in S$, such that

$$\varrho_b(r_1, \nu_1) = \varrho_b(R, S), \tag{56}$$

$\nu_1 \in Tr_0$. Continuing this way, we construct sequences (r_n) in R, (s_n) in S, (ν_n) in Tr_{n-1}, and (υ_n) in Ts_{n-1}, such that

$$\varrho_b(r_n, \nu_n) = \varrho_b(R, S), \tag{57}$$

and

$$\varrho_b(s_n, \upsilon_n) = \varrho_b(R, S). \tag{58}$$

It implies

$$\varrho_b(R, S) \leq \varrho_b(r_n, Tr_{n-1}) \leq \varrho_b(r_n, \nu_n) = \varrho_b(R, S), \tag{59}$$

and

$$\varrho_b(R, S) \leq \varrho_b(s_n, Ts_{n-1}) \leq \varrho_b(s_n, \upsilon_n) = \varrho_b(R, S), \tag{60}$$

for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \varrho_b(r_n, Tr_{n-1}) &= \varrho_b(R, S), \\ \varrho_b(s_n, Ts_{n-1}) &= \varrho_b(R, S). \end{aligned} \tag{61}$$

First, we assume that R is bounded. Then, there exists a positive real number K, such that $\varrho_b(r_n, r_m) \leq K$ for all $n, m \in \mathbb{N}$. Therefore, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \varrho_b(\nu_n, \nu_m) &\leq b(\varrho_b(\nu_n, r_n) + \varrho_b(r_n, \nu_m)), \\ &\leq b(\varrho_b(R, S) + b(\varrho_b(r_n, r_m) + \varrho_b(r_m, \nu_m))), \\ &\leq b(\varrho_b(R, S) + bK + b\varrho_b(R, S)), \end{aligned} \tag{62}$$

implies

$$\varrho_b(\nu_n, \nu_m) \leq (b + b^2)\varrho_b(R, S) + b^2K. \tag{63}$$

Therefore, (ν_n) is bounded. Also, T is cyclic, so for each $s \in S$, $Ts \subset R$, and so $\upsilon_n \in R$, for all $n \in \mathbb{N}$. Therefore, there exists a positive real number K_1, such that $\varrho_b(\upsilon_n, \upsilon_m) \leq K_1$. It implies (υ_n) is bounded, so

$$\begin{aligned} \varrho_b(s_n, s_m) &\leq b(\varrho_b(s_n, \upsilon_n) + \varrho_b(\upsilon_n, s_m)), \\ &\leq b(\varrho_b(R, S) + b(\varrho_b(\upsilon_n, \upsilon_m) + \varrho_b(\upsilon_m, s_m))), \\ &\leq (b + b^2)\varrho_b(R, S) + b^2K_1. \end{aligned} \tag{64}$$

It implies (s_n) is bounded. Thus, (r_n), (s_n), (ν_n), and (υ_n) all are bounded sequences. On a similar line, we can prove (r_n), (s_n), (ν_n), and (υ_n) are bounded whenever S is bounded. Since (R,S) is a compact weak pair, therefore, there exist subsequences (r_{n_k}), (s_{n_k}), (ν_{n_k}), and (υ_{n_k}) of (r_n), (s_n), (ν_n), and (υ_n), respectively, such that $r_{n_k} \longrightarrow r_\star \in R$, $s_{n_k} \longrightarrow s_\star \in S$, $\nu_{n_k} \longrightarrow \nu_\star \in S$, and $\upsilon_{n_k} \longrightarrow \upsilon_\star \in R$, as $k \longrightarrow \infty$. First, we show that $\varrho_b(r_\star, Tr_\star) = \varrho_b(R,S)$. As we have

$$\varrho_b(r_n, Tr_{n-1}) = \varrho_b(R,S), \tag{65}$$

and

$$\varrho_b(r_{n+1}, \mathrm{Tr}_n) = \varrho_b(R,S), \tag{66}$$

so if $\varrho_b(r_{n-1}, r_n) > \varrho_b(R,S)$, then we have

$$\begin{aligned}\varrho_b(r_n, r_{n+1}) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_n)\\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_n) + \frac{\tau}{b}\varrho_b(r_{n+1}, r_{n-1}),\\ &\le \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \gamma\varrho_b(r_{n-1}, r_n) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1}))\\ &\le (\alpha+\gamma)\varrho_b(r_{n-1}, r_n) + \beta\varrho_b(r_n, r_{n+1}) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1})).\end{aligned} \tag{67}$$

So,

$$(1-(\beta+\tau))\varrho_b(r_n, r_{n+1}) < (\alpha+\gamma+\tau)\varrho_b(r_{n-1}, r_n). \tag{68}$$

If $1-(\beta+\tau)=0$, then $\beta+\tau=1$, so (19) implies $\alpha=\gamma=\delta=\tau=0$, and $\beta=1$ is a contradiction. Therefore,

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n). \tag{69}$$

If

$$\varrho_b(r_{n-1}, r_n) = \varrho_b(R,S), \tag{70}$$

then

$$\varrho_b(r_n, r_{n+1}) = \varrho_b(r_{n-1}, r_n). \tag{71}$$

Therefore,

$$\varrho_b(r_n, r_{n+1}) \le \varrho_b(r_{n-1}, r_n), \tag{72}$$

for all $n \in \mathbb{N}$, and hence, the sequence $(\varrho_b(r_n, r_{n+1}))$ is a convergent sequence of real numbers. By hypothesis (ii), it follows that $(\varrho_b(r_n, r_{n+1}))$ converges to 0. Now,

$$\begin{aligned}\varrho_b\left(r_{n_k+1}, r_\star\right) &\le b\left(\varrho_b\left(r_{n_k+1}, r_{n_k}\right) + \varrho_b\left(r_{n_k}, r_\star\right)\right)\\ &\longrightarrow 0 \text{ as } k \longrightarrow \infty\end{aligned} \tag{73}$$

.

Therefore, $\lim_{k\longrightarrow\infty} r_{n_k+1} = r_\star$. Again, we have $\varrho_b(r_{n_k}, \nu_{n_k}) \longrightarrow \varrho_b(r_\star, \nu_\star)$ as $k \longrightarrow \infty$, and hence, we get

$$\varrho_b(r_\star, \nu_\star) = \varrho_b(R,S). \tag{74}$$

So, $r_\star \in R_0$ implies $Tr_\star \subset S_0$. Thus, there exists $\upsilon \in R$, such that

$$\varrho_b(\upsilon, \nu) = \varrho_b(R,S), \tag{75}$$

where $\nu \in Tr_\star$. It implies

$$\varrho_b(\upsilon, Tr_\star) = \varrho_b(R,S). \tag{76}$$

If

$$\varrho_b\left(r_{n_k}, r_\star\right) = \varrho_b(R,S), \tag{77}$$

only for finitely many k, then we can exclude those r_{n_k} from (r_{n_k}) and then assume

$$\varrho_b\left(r_{n_k}, r_\star\right) > \varrho_b(R,S) \tag{78}$$

for all k. If

$$\varrho_b\left(r_{n_k}, r_\star\right) = \varrho_b(R,S), \tag{79}$$

for infinitely many k, then we can extract a subsequence $(r_{n_{k_l}})$ from (r_{n_k}), such that

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S), \tag{80}$$

for all l. This gives

$$\lim_{l\longrightarrow\infty} \varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S) \text{ implies } \varrho_b(R,S) = 0. \tag{81}$$

From the relations

$$\left.\begin{aligned}\varrho_b(\upsilon, Tr_\star) &= \varrho_b(R,S),\\ \varrho_b\left(r_{n_{k_l}+1}, Tr_{n_{k_l}}\right) &= \varrho_b(R,S),\end{aligned}\right\} \tag{82}$$

and

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S), \tag{83}$$

we get

$$\varrho_b\left(r_{n_{k_l+1}}, \upsilon\right) = \varrho_b(R,S) = 0, \tag{84}$$

for all l. Taking limit as $l \longrightarrow \infty$, we get

$$\varrho_b(r_\star, v) = 0, \tag{85}$$

so $r_\star = v$. Therefore, we have

$$\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S). \tag{86}$$

Next, we assume that

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) > \varrho_b(R, S), \tag{87}$$

for all k; then, from relations

$$\left.\begin{array}{l}\varrho_b(v, Tr_\star) = \varrho_b(R, S),\\ \varrho_b\left(r_{n_{k_l}+1}, Tr_{n_{k_l}}\right) = \varrho_b(R, S),\end{array}\right\} \tag{88}$$

we get

$$\begin{aligned}\varrho_b\left(r_{n_{k_l}+1}, v\right) &< \alpha\varrho_b(v, r_\star) + \beta\varrho_b\left(r_{n_{k_l}+1}, r_{n_{k_l}}\right)\\ &+ \frac{\gamma}{b^2}\varrho_b\left(r_\star, r_{n_{k_l}}\right) + \frac{\delta}{b}\varrho_b\left(r_{n_{k_l}+1}, r_\star\right) + \frac{\tau}{b}\varrho_b\left(v, r_{n_{k_l}}\right).\end{aligned} \tag{89}$$

Taking limit as $k \longrightarrow \infty$ in above, we get

$$\begin{aligned}\varrho_b(v, r_\star) &\leq \alpha\varrho_b(v, r_\star) + \frac{\tau}{b}\varrho_b(v, r_\star),\\ \varrho_b(v, r_\star) &\leq \alpha\varrho_b(v, r_\star) + \tau\varrho_b(v, r_\star).\end{aligned} \tag{90}$$

It implies

$$(1 - (\alpha + \tau))\varrho_b(v, r_\star) \leq 0. \tag{91}$$

If $\alpha + \tau = 1$, then (19) implies $\beta = \gamma = \tau = 0$ which implies $\alpha = 1$, a contradiction, so

$$\varrho_b(v, r_\star) = 0 \text{ implies } v = r_\star. \tag{92}$$

Hence,

$$\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S). \tag{93}$$

Similarly, we can prove

$$\varrho_b(s_\star, Ts_\star) = \varrho_b(R, S). \tag{94}$$

Now, let $r \in R, s \in S$, such that

$$\varrho_b(r, \mathrm{Tr}) = \varrho_b(R, S), \tag{95}$$

and

$$\varrho_b(s, Ts) = \varrho_b(R, S). \tag{96}$$

If $\varrho_b(r, s) > \varrho_b(R, S)$, then

$$\begin{aligned}\varrho_b(r, s) &< \alpha\varrho_b(r, r) + \beta\varrho_b(s, s) + \frac{\gamma}{b^2}\varrho_b(r, s) + \frac{\delta}{b}\varrho_b(r, s)\\ &+ \frac{\tau}{b}\varrho_b(r, s),\end{aligned} \tag{97}$$

so

$$\varrho_b(r, s) < \gamma\varrho_b(r, s) + \delta\varrho_b(r, s) + \tau\varrho_b(r, s). \tag{98}$$

It implies

$$(1 - (\gamma + \delta + \tau))\varrho_b(r, s) < 0, \tag{99}$$

which further implies

$$\varrho_b(r, s) < 0, \tag{100}$$

a contradiction. So, $\varrho_b(r, s) = \varrho_b(R, S)$. This completes the proof. □

Theorem 21. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is a bounded subset of P and (R, S) is a compact weak proximal pair. Let $T\colon R \cup S \longrightarrow R \cup S$ be a new type of generalized Hardy and Roger's proximal cyclic contractive mapping and assume that*

(i) For each $r \in R_0$, $\mathrm{Tr} \in S_0$, and for each $s \in S_0$, $Ts \in R_0$,

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho_b(x_n, y_n))$ converges to $\varrho_b(R, S)$, then $\lim_{n\longrightarrow\infty}\varrho_b(r_n, r_{n+1}) = 0$.

Then, there exist $r \in R$ and $s \in S$, such that $\varrho_b(r, \mathrm{Tr}) = \varrho_b(R, S)$ and $\varrho_b(s, Ts) = \varrho_b(R, S)$. Furthermore, $\varrho_b(r, s) = \varrho_b(R, S)$.

Proof. Following Theorem 20, we can get the required result. □

Corollary 22. *If we take $b = 1$, $\alpha = \beta = 1/2$, and $\gamma = \delta = \tau = 0$ in Theorem 21, we get Theorem 11.*

5. Some Fixed Points Results

In this section, we derive some fixed points results from our main results.

Theorem 23. *Let (P, ϱ_b) be a boundedly compact b-metric space; then,*

(i) A mapping $T\colon P \longrightarrow \wp(P)$, such that for $q_1, q_2 \in P$, there exist $p_1 \in Tq_1$ and $p_2 \in Tq_2$, such that

$$\begin{aligned}\varrho_b(p_1, p_2) &< \alpha\varrho_b(p_1, q_1) + \beta\varrho_b(p_2, q_2) + \frac{\gamma}{b^2}\varrho_b(q_1, q_2)\\ &+ \frac{\delta}{b}\varrho_b(p_1, q_2) + \frac{\tau}{b}\varrho_b(p_2, q_1), \text{s.}\end{aligned} \tag{101}$$

For $q_1 \neq q_2$
And

$$\left.\begin{array}{l}\varrho_b(p_1, p_2) \leq \alpha\varrho_b(p_1, q_1) + \beta\varrho_b(p_2, q_2) +\\ \dfrac{\delta}{b}\varrho_b(p_1, q_2) + \dfrac{\tau}{b}\varrho_b(p_2, q_1)\end{array}\right\}, \tag{102}$$

For $q_1 = q_2$, *where*

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{103}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

(ii) If (p_n) *is a bounded sequence in* P, *then* $\lim_{n\longrightarrow\infty}(p_n, p_{n+1}) = 0$.

Then, $Fix(T)$ *(set of fixed points of* T*) is nonempty*

Theorem 24. *Let* (P, ϱ_b) *be a boundedly compact b-metric space; then,*

(i) A mapping $T: P \longrightarrow P$, *such that for all* $q_1, q_2 \in P$ *with* $q_1 \neq q_2$,

$$\varrho_b(Tq_1, Tq_2) < \alpha\varrho_b(Tq_1, q_1) + \beta\varrho_b(Tq_2, q_2) + \frac{\gamma}{b^2}\varrho_b(q_1, q_2) + \frac{\delta}{b}\varrho_b(Tq_1, q_2) + \frac{\tau}{b}\varrho_b(Tq_2, q_1), \tag{104}$$

where

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{105}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

(ii) If (p_n) *is a bounded sequence in* P, *then* $\lim_{n\longrightarrow\infty}(p_n, p_{n+1}) = 0$

Then, $Fix(T)$ *is singleton.*

6. Conclusion

We presented a new type of generalized multivalued Hardy and Roger's proximal contractive and proximal cyclic contractive mappings in b-metric spaces and developed results for the existence of best proximity points. Furthermore, we have derived results for the existence and uniqueness of best proximity points for new type of generalized Hardy and Roger's proximal contractive and proximal cyclic contractive mappings. Our results are the generalization of the results already existing in literature.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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