

## Research Article

# Cubic Pythagorean Fuzzy Graphs

G. Muhiuddin <sup>1</sup>, Saira Hameed,<sup>2</sup> Afeefa Maryam,<sup>2</sup> and Uzma Ahmad <sup>2</sup>

<sup>1</sup>Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia

<sup>2</sup>Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan

Correspondence should be addressed to G. Muhiuddin; [chishtygm@gmail.com](mailto:chishtygm@gmail.com)

Received 9 March 2022; Revised 14 April 2022; Accepted 4 May 2022; Published 28 June 2022

Academic Editor: Gohar Ali

Copyright © 2022 G. Muhiuddin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of the study is to explore graph theory based on cubic Pythagorean fuzzy sets. The concept of cubic Pythagorean fuzzy graphs (CuPFGs) is introduced in this research work. In addition, we define certain fundamental operations on CuPFGs including semistrong product, lexicographical product, and symmetric difference of two CuPFGs and demonstrate some of their key characteristics. Meanwhile, to investigate the preference of decision makers, cubic Pythagorean fuzzy preference relation is defined. Moreover, through a practical example, the applicability of our proposed work in multicriteria decision-making is described, and to clarify the organization of the proposed method, a frame diagram is presented.

## 1. Introduction

To deal with uncertain and vague information, Zadeh [1] initiated the concept of fuzzy set theory. Fuzzy sets (FSs) have many applications in the field of science and technology. Atanassov [2] put forward the idea of intuitionistic fuzzy set (IFS), a new tool of presenting ambiguous and uncertain information by associating degree of belongingness ( $\lambda$ ) and nonbelongingness ( $\mu$ ) with condition  $0 \leq \lambda + \mu \leq 1$ . After successful implementation of IFS in numerous fields, Gorzalczy [3] introduced interval-valued fuzzy sets, where the ambiguous information is presented in the form of interval. Later, Atanassov and Gargov [4] proposed the concept of interval-valued intuitionistic fuzzy sets, in which the degree of belongingness and nonbelongingness is expressed in the form of intervals.

Preceding, the idea of IFS, Yager [5] introduced the concept Pythagorean fuzzy set (PFS) with more flexible constraint  $0 \leq \lambda^2 + \mu^2 \leq 1$ . PFSs have more capacity than IFS and IVFS to model the ambiguity and to handle complex coarsens. The applications of IFS and PFS have gained a lot of attention in areas like multicriteria decision-making (MCDM) and image processing. Furthermore, Yager [6] developed the Pythagorean membership grades in decision-making.

In many real-life situations, one has to mention the degree of belongingness both in interval value and simple fuzzy value. Such types of situations cannot be handled by FS, IFS, PFS etc. Thus, to adequately treat such types of situations, Jun et al. [7] proposed the idea of cubic sets (CSs), which is the mixture of both interval-valued fuzzy sets and fuzzy sets. Furthermore, they investigated some properties and operations on CSs. Since, CSs are centered only on degree of belongingness, it may face trouble when degree of nonbelongingness comes into account. Thus, to investigate such type of obscurity, Kaur and Garg [8, 9] proposed the concept of cubic intuitionistic fuzzy set (CuIFS) which includes two parts, one representing the degree of belongingness and nonbelongingness in IVFS and other representing the degree of belongingness and nonbelongingness in simple FS. CuIFS is a more generalized form of IFS and IVIFS because it provides a more flexible environment to describe those situations where degree of belongingness or nonbelongingness fluctuates during the procedure of decision-making and it also enlarges the level of precision.

Since CuIFS fails to deal with information where sum of squares of upper interval-values of degree of belongingness and nonbelongingness and sum of squares of simple degree of belongingness and nonbelongingness are less than 1, to

deal with such problem, Abbas et al. [10] introduced the concept of CuPFSs and their application to MCDM with unknown weight information. Recently, Khan et al. [11] studied Pythagorean cubic fuzzy aggregation operators to aggregate cubic information and studied their application in MCDM. Ashraf et al. [12] presented the concept of cubic picture fuzzy sets. Furthermore, Mehmood et al. [13] proposed cubic hesitant fuzzy sets and its applications to MCDM. Muhiuddin and Al-roqi [14] introduced cubic soft sets with applications in BCK/BCI-algebras.

Graphs communicate information visually, but if the information is ambiguous and uncertain, then it can be recognized as fuzzy graphs (FGs). FGs have been shown to be a powerful tool for modeling complex problems such as communications, social networking, data hypothesis, man-made reasoning, system analysis, operations research, and economics. Kaufmann [15] initiated the idea of FG theory. Rosenfeld [16] gave the theoretical concept of FG by introducing relations between fuzzy sets and establishing the structure of FGs. Mordeson and Chang-Shyh [17] illustrated certain operations on FGs. Bhattacharya [18] added some remarks on FGs. Yeh and Bang [19] defined fuzzy relations, fuzzy graphs, and their applications to clustering analysis, fuzzy sets, and their applications in cognitive and decision process. Dogra [20] presented different types of products of FGs, and Al-Hawary [21] introduced complete FGs.

Afterward, Parvathi and Karunambigai [22] initiated the concept of intuitionistic fuzzy graphs (IFGs). Gani and Begum [23] illustrated degree, order, and size in IFGs. Akram et al. [24–26] introduced the idea of PFGs and presented specific types of PFGs, their direct sum, and their application in decision-making. Garg [23] presented Pythagorean fuzzy geometric aggregation operators for multiple attribute group decision analysis. Mandal and Ranadive [27] gave the idea of Pythagorean fuzzy preference relations and their applications in group decision-making systems. Wang and Garg [28] presented the algorithm for MADM with interactive Archimedean norm operations under Pythagorean fuzzy uncertainty. Furthermore, Akram and Naz [29] studied energy of Pythagorean fuzzy graphs (PFGs) with applications.

Moreover, Akram et al. [24] extended FGs to interval-valued fuzzy graphs (IVFGs), in which the ambiguous information is expressed in the form of interval. Naz et al. [30]

gave the concept of simplified IVFGs with application. Akram et al. [31] presented certain types of interval-valued fuzzy graphs. Mohamed and Ali [32–34] developed some products on interval-valued Pythagorean fuzzy graphs and defined strong interval-valued Pythagorean fuzzy graphs. Rashid et al. [35] put forward the idea of graphical structure of cubic sets and discuss some operations on them. Moreover, Muhiuddin et al. [36,37] extended the concept cubic graphs and presented their application.

Recently, Khan et al. [38] introduced graphical structures of cubic intuitionistic fuzzy information and its application in multiattribute decision-making (MADM). For other concepts, the readers are suggested to [39–42].

The objective of our proposed work is to apply graph terminology on cubic Pythagorean fuzzy sets. The presented work is devoted to elaborate the degree, order, and size of CuPFGs. Furthermore, cubic Pythagorean fuzzy preference relation (CuPFPR) is defined and to examine the rationality of the purposed work an application is presented. Aggregation operators are commonly used to compose all the inputted individual information into a single value. So, we use cubic Pythagorean fuzzy weighted averaging (CuPFWA) operator and cubic Pythagorean fuzzy weighted geometric (CuPFWG) operator to aggregate all cubic Pythagorean fuzzy preference relation matrices. Finally, we develop a CuPFG based MADM approach to handle situations in which the attributes' graphic structure is uncertain.

The proposed work is organized as follows: Section 2 presents the basic notions and definitions of CuPFGs, degree, total degree, order, size, and complete CuPFGs. In addition, certain operations including semistrong product, lexicographical product, and symmetric difference of two cubic Pythagorean fuzzy graphs are elaborated. Section 3 defines CuPFPR to compare the preference of the experts. Furthermore, the information of CuPFGs in MCDM is applied, and the proposed method is presented in a frame diagram. Section 4 concludes the entire work with certain remarks and further directions for future work.

## 2. Cubic Pythagorean Fuzzy Graphs

*Definition 1.* (see [10]). Let  $R$  be the universe of discourse. A cubic Pythagorean fuzzy set (CuPFS)  $A$ , defined on  $R$  is given as follows:

$$A = \{r, [\mathfrak{R}_{AL}(r), \mathfrak{R}_{AU}(r)], [\aleph_{AL}(r), \aleph_{AU}(r)], (\mu(r), \nu(r)) \mid r \in R\}, \quad (1)$$

where the functions  $\mathfrak{R}_{AL}: R \rightarrow D[0, 1]$ ,  $\mathfrak{R}_{AU}: R \rightarrow D[0, 1]$  and  $\aleph_{AL}: R \rightarrow D[0, 1]$ ,  $\aleph_{AU}: R \rightarrow D[0, 1]$  and  $\mu_A: R \rightarrow [0, 1]$ , and  $\nu_A: R \rightarrow [0, 1]$  denote the degree of membership and nonmembership of the element  $r \in R$ ,

respectively, such that  $0 \leq (\mathfrak{R}_{AU}(r))^2 + (\aleph_{AU}(r))^2 \leq 1$  and  $0 \leq (\mu(r))^2 + (\nu(r))^2 \leq 1$ .

*Definition 2.* A CuPFS  $B$  on  $R \times R$  is regarded as cubic Pythagorean fuzzy relation (CuPFR) in  $R$  indicated as

$$B = \{(r, s), [\mathfrak{R}_{AL}(r, s), \mathfrak{R}_{AU}(r, s)], [\aleph_{AL}(r, s), \aleph_{AU}(r, s)], (\mu(r, s), \nu(r, s)) \mid (r, s) \in R \times R\}, \quad (2)$$

where  $\mathfrak{R}_{BL}: R \rightarrow D[0, 1]$ ,  $\mathfrak{R}_{BU}: R \rightarrow D[0, 1]$  and  $\aleph_{BL}: R \rightarrow D[0, 1]$ ,  $\aleph_{BU}: R \rightarrow D[0, 1]$  and  $\mu_B: R \rightarrow [0, 1]$ , and  $\nu_B: R \rightarrow [0, 1]$  are such that  $0 \leq (\mathfrak{R}_{BU}(r, s))^2 + (\aleph_{BU}(r, s))^2 \leq 1$  and  $0 \leq (\mu(r, s))^2 + (\nu(r, s))^2 \leq 1$ , for all  $(r, s) \in E$ .

**Definition 3.** A cubic Pythagorean fuzzy graph (CuPFG)  $\tilde{G}$  on a nonempty set  $R$  is a pair  $\tilde{G} = (A, B)$ , where  $A$  is cubic Pythagorean fuzzy set and  $B$  is cubic Pythagorean fuzzy relation on  $R$  such that

$$\mathfrak{R}_{BL}(r, s) \leq \mathfrak{R}_{AL}(r) \wedge \mathfrak{R}_{AL}(s), \quad \mathfrak{R}_{BU}(r, s) \leq \mathfrak{R}_{AU}(r) \wedge \mathfrak{R}_{AU}(s), \text{ and } \aleph_{BL}(r, s) \leq \aleph_{AL}(r) \vee \aleph_{AL}(s)$$

$$\aleph_{BU}(r, s) \leq \aleph_{AU}(r) \vee \aleph_{AU}(s), \quad \mu_B(r, s) \leq \mu_A(r) \wedge \mu_A(s), \text{ and } \nu_B(r, s) \leq \nu_A(r) \vee \nu_A(s)$$

where  $0 \leq (\mathfrak{R}_{BU}(r, s))^2 + (\aleph_{BU}(r, s))^2 \leq 1$  and  $0 \leq (\mu(r, s))^2 + (\nu(r, s))^2 \leq 1$ , for all  $(r, s) \in E$ .

**Example 1.** Consider a graph  $G = (R, E)$ , where  $R = \{r_1, r_2, r_3, r_4\}$  is the set of vertices and  $E = \{r_1r_2, r_2r_3, r_3r_4, r_1r_4\}$  is the set of edges. The membership and nonmembership degrees of the vertices and edges are given in Figure 1.

**Definition 4.** The degree and total degree of a vertex  $r \in R$  in a CuPFG  $\tilde{G} = (A, B)$  is described as

$$d_G^-(r) = \langle (d\mathfrak{R}_{AL}(r), d\mathfrak{R}_{AU}(r), d\aleph_{AL}(r), d\aleph_{AU}(r)), (d\mu_A(r), d\nu_A(r)) \rangle, \tag{3}$$

$$td_G^-(r) = \langle (t d\mathfrak{R}_{AL}(r), t d\mathfrak{R}_{AU}(r), t d\aleph_{AL}(r), t d\aleph_{AU}(r)), t(d\mu_A(r), t d\nu_A(r)) \rangle, \tag{4}$$

where  $d\mathfrak{R}_{AL}(r) = \sum_{(r \neq s, s) \in E} \mathfrak{R}_{BL}(rs)$ ,  $d\mathfrak{R}_{AU}(r) = \sum_{(r \neq s, s) \in E} \mathfrak{R}_{BU}(rs)$ ,  $d\aleph_{AL}(r) = \sum_{(r \neq s, s) \in E} \aleph_{BL}(rs)$ ,  $d\aleph_{AU}(r) = \sum_{(r \neq s, s) \in E} \aleph_{BU}(rs)$ ,  $d\mu_A(r) = \sum_{(r \neq s, s) \in E} \mu_B(rs)$ ,  $d\nu_A(r) = \sum_{(r \neq s, s) \in E} \nu_B(rs)$ , and  $t d\mathfrak{R}_{AL}(r) = \sum_{(r \neq s, s) \in E} \mathfrak{R}_{BL}(rs) + \mathfrak{R}_{AL}(r)$ ,  $t d\mathfrak{R}_{AU}(r) = \sum_{(r \neq s, s) \in E} \mathfrak{R}_{BU}(rs) + \mathfrak{R}_{AU}(r)$ ,  $t d\aleph_{AL}(r) = \sum_{(r \neq s, s) \in E} \aleph_{BL}(rs) + \aleph_{AL}(r)$ ,  $t d\aleph_{AU}(r) = \sum_{(r \neq s, s) \in E} \aleph_{BU}(rs) + \aleph_{AU}(r)$ , and  $t d\mu_A(r) = \sum_{(r \neq s, s) \in E} \mu_B(rs) + \mu_A(r)$ ,  $t d\nu_A(r) = \sum_{(r \neq s, s) \in E} \nu_B(rs) + \nu_A(r)$ .

**Definition 5.** The order of CuPFG  $\tilde{G} = (A, B)$  is denoted as  $O(\tilde{G})$  and defined as

$$O(\tilde{G}) = \left( \left( \sum_{r \in R} \mathfrak{R}_{AL}(r), \sum_{r \in R} \mathfrak{R}_{AU}(r), \sum_{r \in R} \aleph_{AL}(r), \sum_{r \in R} \aleph_{AU}(r) \right), \left( \sum_{r \in R} \mu_A(r), \sum_{r \in R} \nu_A(r) \right) \right), \tag{5}$$

and the size of CuPFG is denoted as  $S(\tilde{G})$  and defined as

$$S(\tilde{G}) = \left( \left( \sum_{(r \neq s, s) \in E} \mathfrak{R}_{AL}(rs), \sum_{(r \neq s, s) \in E} \mathfrak{R}_{AU}(rs), \sum_{(r \neq s, s) \in E} \aleph_{AL}(rs), \sum_{(r \neq s, s) \in E} \aleph_{AU}(rs) \right), \left( \sum_{(r \neq s, s) \in E} \mu_A(rs), \sum_{(r \neq s, s) \in E} \nu_A(rs) \right) \right). \tag{6}$$

**Definition 6.** A CuPFG  $\tilde{G} = (A, B)$  of a graph  $G = (R, E)$  is said to be complete if it satisfies the following conditions:

$$\mathfrak{R}_{BL}(r, s) = \mathfrak{R}_{AL}(r) \wedge \mathfrak{R}_{AL}(s), \quad \mathfrak{R}_{BU}(r, s) = \mathfrak{R}_{AU}(r) \wedge \mathfrak{R}_{AU}(s), \text{ and } \aleph_{BL}(r, s) = \aleph_{AL}(r) \vee \aleph_{AL}(s),$$

$$\aleph_{BU}(r, s) = \aleph_{AU}(r) \vee \aleph_{AU}(s), \quad \mu_B(r, s) = \mu_A(r) \wedge \mu_A(s), \text{ and } \nu_B(r, s) = \nu_A(r) \vee \nu_A(s).$$

Now, we present certain operations on CuPFGs along with examples and some valuable results.

**Definition 7.** Let  $\tilde{G}_1 = (\aleph_1, \mathfrak{R}_1)$  and  $\tilde{G}_2 = (\aleph_2, \mathfrak{R}_2)$  be two CuPFGs of graphs  $G_1 = (R_1, E_1)$  and  $G_2 = (R_2, E_2)$ , respectively. The *semistrong product* of  $\tilde{G}_1$  and  $\tilde{G}_2$  is denoted as  $\tilde{G}_1 \bullet \tilde{G}_2 = (\aleph_1 \bullet \aleph_2, \mathfrak{R}_1 \bullet \mathfrak{R}_2)$  and is defined as follows:

$$(i) (\mathfrak{R}_{A_1L} \bullet \mathfrak{R}_{A_2L})(r_1, r_2) = \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_2L}(r_2)$$

$$(\mathfrak{R}_{A_1U} \bullet \mathfrak{R}_{A_2U})(r_1, r_2) = \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2)$$

$$(\aleph_{A_1L} \bullet \aleph_{A_2L})(r_1, r_2) = \aleph_{A_1L}(r_1) \vee \aleph_{A_2L}(r_2)$$

$$(\aleph_{A_1U} \bullet \aleph_{A_2U})(r_1, r_2) = \aleph_{A_1U}(r_1) \vee \aleph_{A_2U}(r_2)$$

$$(\mu_{A_1} \bullet \mu_{A_2})(r_1, r_2) = \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2)$$

$$(\nu_{A_1} \bullet \nu_{A_2})(r_1, r_2) = \nu_{A_1}(r_1) \vee \nu_{A_2}(r_2), \text{ for all } (r_1, r_2) \in R_1 \times R_2$$

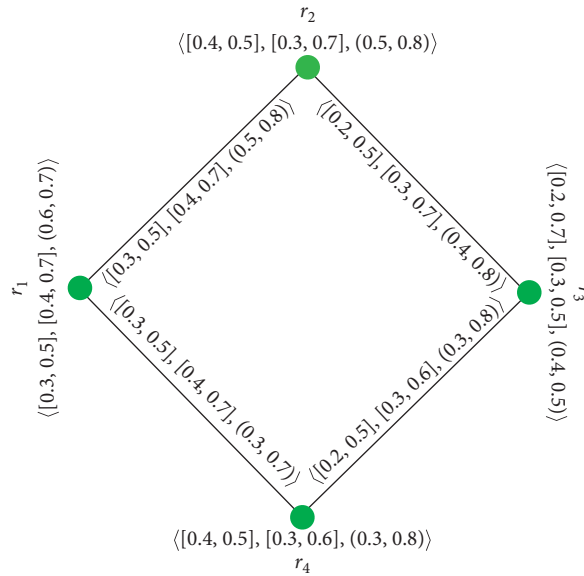
$$(ii) (\mathfrak{R}_{B_1L} \bullet \mathfrak{R}_{B_2L})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1L}(r) \wedge \mathfrak{R}_{B_2L}(r_2, s_2)$$

$$(\mathfrak{R}_{B_1U} \bullet \mathfrak{R}_{B_2U})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1U}(r) \wedge \mathfrak{R}_{B_2U}(r_2, s_2)$$

$$(\aleph_{B_1L} \bullet \aleph_{B_2L})((r, r_2), (r, s_2)) = \aleph_{A_1L}(r_1) \wedge \aleph_{B_2L}(r_2, s_2)$$

$$(\aleph_{B_1U} \bullet \aleph_{B_2U})((r, r_2), (r, s_2)) = \aleph_{A_1U}(r_1) \wedge \aleph_{B_2U}(r_2, s_2)$$

$$(\mu_{B_1} \bullet \mu_{B_2})((r, r_2), (r, s_2)) = \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2, s_2)$$

FIGURE 1: A cubic Pythagorean fuzzy graph  $\tilde{G}$ .

$$(\nu_{B_1} \bullet \nu_{B_2})((r, r_2), (r, s_2)) = \nu_{A_1} \quad (r_1) \wedge \nu_{B_2}(r_2, s_2)$$

for all  $r \in R_1$ , for all  $r_2 s_2 \in E_2$

$$(\nu_{B_1} \bullet \nu_{B_2})((r_1, r_2), (s_1, s_2)) = \nu_{B_1}(r_1, s_1) \wedge \nu_{B_2}(r_2, s_2)$$

for all  $r_1 s_1 \in E_1$ , for all  $r_2 s_2 \in E_2$

$$(iii) (\mathfrak{R}_{B_1 L} \bullet \mathfrak{R}_{B_2 L})((r_1, r_2), (s_1, s_2)) = \mathfrak{R}_{B_1 L}(r_1, s_1) \wedge \mathfrak{R}_{B_2 L}(r_2, s_2)$$

$$(\mathfrak{R}_{B_1 U} \bullet \mathfrak{R}_{B_2 U})((r_1, r_2), (s_1, s_2)) = \mathfrak{R}_{B_1 U}(r_1, s_1) \wedge \mathfrak{R}_{B_2 U}(r_2, s_2)$$

$$(\mathfrak{N}_{B_1 L} \bullet \mathfrak{N}_{B_2 L})((r_1, r_2), (s_1, s_2)) = \mathfrak{N}_{B_1 L}(r_1, s_1) \wedge \mathfrak{N}_{B_2 L}(r_2, s_2)$$

$$(\mathfrak{N}_{B_1 U} \bullet \mathfrak{N}_{B_2 U})((r_1, r_2), (s_1, s_2)) = \mathfrak{N}_{B_1 U}(r_1, s_1) \wedge \mathfrak{N}_{B_2 U}(r_2, s_2)$$

$$(\mu_{B_1} \bullet \mu_{B_2})((r_1, r_2), (s_1, s_2)) = \mu_{B_1}(r_1, s_1) \wedge \mu_{B_2}(r_2, s_2)$$

*Example 2.* Consider two CuPFGs  $\tilde{G}_1$  and  $\tilde{G}_2$  on  $R_1 = \{r_1, s_1\}$  and  $R_2 = \{r_2, s_2\}$ , respectively, as shown in Figure 2. Then, their semistrong product  $\tilde{G}_1 \bullet \tilde{G}_2$  is shown in Figure 3.

**Proposition 1.** The semistrong product  $\tilde{G}_1 \bullet \tilde{G}_2$  of two CuPFGs  $\tilde{G}_1$  and  $\tilde{G}_2$  is also a CuPFG.

*Definition 8.* Let  $\tilde{G}_1$  and  $\tilde{G}_2$  be two CuPFGs. Then, for any vertex  $(r_1, r_2) \in R_1 \times R_2$ ,

$$\begin{aligned} (d\mathfrak{R}_L)_{\tilde{G}_1 \bullet \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1 L} \bullet \mathfrak{R}_{B_2 L})((r_1, r_2), (s_1, s_2)) \\ &= \sum_{r_1=s_1, r_2 s_2 \in E_2} \mathfrak{R}_{A_1 L}(r_1) \wedge \mathfrak{R}_{B_2 L}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{R}_{B_1 L}(r_1 s_1) \wedge \mathfrak{R}_{B_2 L}(r_2 s_2), \end{aligned} \quad (7)$$

$$\begin{aligned} (d\mathfrak{R}_U)_{\tilde{G}_1 \bullet \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1 U} \bullet \mathfrak{R}_{B_2 U})((r_1, r_2), (s_1, s_2)) \\ &= \sum_{r_1=s_1, r_2 s_2 \in E_2} \mathfrak{R}_{A_1 U}(r_1) \wedge \mathfrak{R}_{B_2 U}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{R}_{B_1 U}(r_1 s_1) \wedge \mathfrak{R}_{B_2 U}(r_2 s_2), \end{aligned}$$

$$\begin{aligned} (d\mathfrak{N}_L)_{\tilde{G}_1 \bullet \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1 L} \bullet \mathfrak{N}_{B_2 L})((r_1, r_2), (s_1, s_2)) \\ &= \sum_{r_1=s_1, r_2 s_2 \in E_2} \mathfrak{N}_{A_1 L}(r_1) \wedge \mathfrak{N}_{B_2 L}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{N}_{B_1 L}(r_1 s_1) \wedge \mathfrak{N}_{B_2 L}(r_2 s_2), \end{aligned} \quad (8)$$

$$\begin{aligned} (d\mathfrak{N}_U)_{\tilde{G}_1 \bullet \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1 U} \bullet \mathfrak{N}_{B_2 U})((r_1, r_2), (s_1, s_2)) \\ &= \sum_{r_1=s_1, r_2 s_2 \in E_2} \mathfrak{N}_{A_1 U}(r_1) \wedge \mathfrak{N}_{B_2 U}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{N}_{B_1 U}(r_1 s_1) \wedge \mathfrak{N}_{B_2 U}(r_2 s_2), \end{aligned} \quad (9)$$

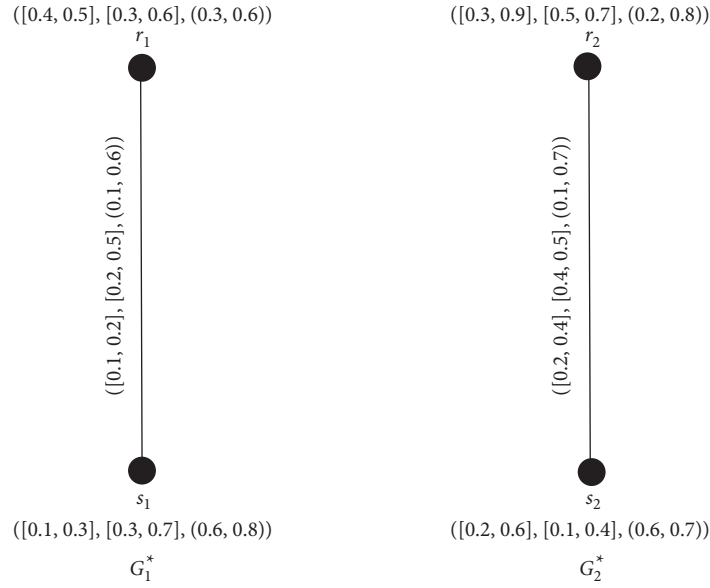


FIGURE 2: CuPFGs  $\check{G}_1$  and  $\check{G}_2$ .

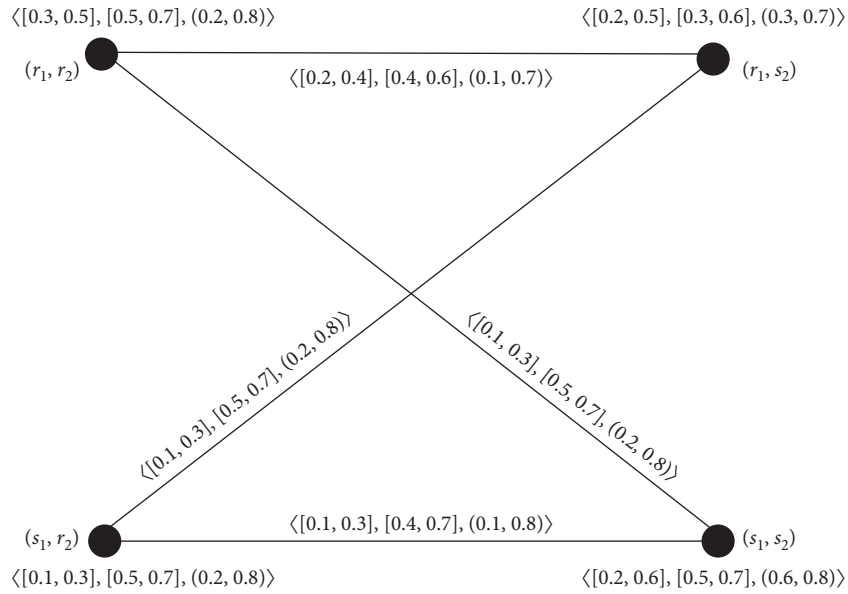


FIGURE 3: Semistrong product of two CuPFGs  $\check{G}_1$  and  $\check{G}_2$ .

$$\begin{aligned}
 (d\mu)_{\check{G}_1 \bullet \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\mu_{B_1 L} \bullet \mu_{B_2 L})(r_1, r_2, (s_1, s_2)) \\
 &= \sum_{r_1 = s_1, r_2, s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2, s_2) + \sum_{r_1, s_1 \in E_1, r_2, s_2 \in E_2} \mu_{B_1}(r_1, s_1) \wedge \mu_{B_2}(r_2, s_2),
 \end{aligned}
 \tag{10}$$

$$\begin{aligned}
 (d\nu)_{\check{G}_1 \bullet \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2)(s_1, s_2) \in R_1 \times R_2} (\nu_{B_1} \bullet \nu_{B_2})(r_1, r_2, (s_1, s_2)) \\
 &= \sum_{r_1 = s_1, r_2, s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2, s_2) + \sum_{r_1, s_1 \in E_1, r_2, s_2 \in E_2} \nu_{B_1}(r_1, s_1) \wedge \nu_{B_2}(r_2, s_2).
 \end{aligned}
 \tag{11}$$

**Theorem 1.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. If  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L}$ ,  $\mathfrak{N}_{A_1L} \leq \mathfrak{N}_{B_2L}$ ,  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2U}$ ,  $\mathfrak{N}_{A_1U} \leq \mathfrak{N}_{B_2U}$ ,  $\mu_{A_1} \geq \mu_{B_2}$ ,  $\nu_{A_1} \leq \nu_{B_2}$  and  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}$ ,  $\mathfrak{N}_{A_2L} \leq \mathfrak{N}_{B_1L}$ ,  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1U}$ ,  $\mathfrak{N}_{A_2U} \leq \mathfrak{N}_{B_1U}$ ,  $\mu_{A_2} \geq \mu_{B_1}$ ,  $\nu_{A_2} \leq \nu_{B_1}$ , then  $d_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) = d_{\check{G}_1}^{\check{G}_1}(r_1) + d_{\check{G}_2}^{\check{G}_2}(r_2)$ , for all  $(r_1, r_2) \in R_1 \times R_2$ .

**Definition 9.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. Then, for any vertex  $(r_1, r_2) \in R_1 \times R_2$ ,

$$\begin{aligned} (t d\mathfrak{R}_L)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \bullet \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1L} \bullet \mathfrak{R}_{A_2L})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{R}_{B_1L}(r_1 s_1) \wedge \mathfrak{R}_{B_2L}(r_2 s_2) + \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_2L}(r_2), \end{aligned} \quad (12)$$

$$\begin{aligned} (t d\mathfrak{R}_U)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1U} \bullet \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1U} \bullet \mathfrak{R}_{A_2U})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{B_2U}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{R}_{B_1U}(r_1 s_1) \wedge \mathfrak{R}_{B_2U}(r_2 s_2) + \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2), \end{aligned} \quad (13)$$

$$\begin{aligned} (t d\mathfrak{N}_L)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1L} \bullet \mathfrak{N}_{B_2L})((r_1, r_2), (s_1, s_2)) + (\mathfrak{N}_{A_1L} \bullet \mathfrak{N}_{A_2L})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{B_2L}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{N}_{B_1L}(r_1 s_1) \wedge \mathfrak{N}_{B_2L}(r_2 s_2) + \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{A_2L}(r_2), \end{aligned} \quad (14)$$

$$\begin{aligned} (t d\mathfrak{N}_U)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1U} \bullet \mathfrak{N}_{B_2U})((r_1, r_2), (s_1, s_2)) + (\mathfrak{N}_{A_1U} \bullet \mathfrak{N}_{A_2U})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mathfrak{N}_{A_1U}(r_1) \wedge \mathfrak{N}_{B_2U}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mathfrak{N}_{B_1U}(r_1 s_1) \wedge \mathfrak{N}_{B_2U}(r_2 s_2) + \mathfrak{N}_{A_1U}(r_1) \wedge \mathfrak{N}_{A_2U}(r_2), \end{aligned} \quad (15)$$

$$\begin{aligned} (t d\mu)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mu_{B_1} \bullet \mu_{B_2})((r_1, r_2), (s_1, s_2)) + (\mu_{A_1} \bullet \mu_{A_2})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \mu_{B_1}(r_1 s_1) \wedge \mu_{B_2}(r_2 s_2) + \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2), \end{aligned} \quad (16)$$

$$\begin{aligned} (t d\nu)_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\nu_{B_1} \bullet \nu_{B_2})((r_1, r_2), (s_1, s_2)) + (\nu_{A_1} \bullet \nu_{A_2})(r_1, r_2) \\ &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2 s_2) + \sum_{r_1 s_1 \in E_1, r_2 s_2 \in E_2} \nu_{B_1}(r_1 s_1) \wedge \nu_{B_2}(r_2 s_2) + \nu_{A_1}(r_1) \wedge \nu_{A_2}(r_2). \end{aligned} \quad (17)$$

**Theorem 2.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. If

(i)  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L}$  and  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}$ , then

$$(t d\mathfrak{R}_{AL})_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) = (t d\mathfrak{R}_{AL})_{\check{G}_1}^{\check{G}_1}(r_1) + (t d\mathfrak{R}_{AL})_{\check{G}_2}^{\check{G}_2}(r_2) - (\mathfrak{R}_{A_1L}(r_1) \vee \mathfrak{R}_{A_2L}(r_2)). \quad (18)$$

(ii)  $\mathfrak{R}_{A_1U} \geq \mathfrak{R}_{B_2U}$  and  $\mathfrak{R}_{A_2U} \geq \mathfrak{R}_{B_1U}$ , then

$$(t d\mathfrak{R}_{AU})_{\check{G}_1 \bullet \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) = (t d\mathfrak{R}_{AU})_{\check{G}_1}^{\check{G}_1}(r_1) + (t d\mathfrak{R}_{AU})_{\check{G}_2}^{\check{G}_2}(r_2) - (\mathfrak{R}_{A_1U}(r_1) \vee \mathfrak{R}_{A_2U}(r_2)). \quad (19)$$

(iii)  $\aleph_{A_1L} \geq \aleph_{B_2L}$  and  $\aleph_{A_2L} \geq \aleph_{B_1L}$ , then

$$(t d\aleph_{AL})_{\check{G}_1 \bullet \check{G}_2}^{\check{}}(r_1, r_2) = (t d\aleph_{AL})_{\check{G}_1}^{\check{}}(r_1) + (t d\aleph_{AL})_{\check{G}_2}^{\check{}}(r_2) - (\aleph_{A_1L}(r_1) \vee \aleph_{A_2L}(r_2)). \quad (20)$$

(iv)  $\aleph_{A_1U} \geq \aleph_{B_2U}$  and  $\aleph_{A_2U} \geq \aleph_{B_1U}$ , then

$$(t d\aleph_{AU})_{\check{G}_1 \bullet \check{G}_2}^{\check{}}(r_1, r_2) = (t d\aleph_{AU})_{\check{G}_1}^{\check{}}(r_1) + (t d\aleph_{AU})_{\check{G}_2}^{\check{}}(r_2) - (\aleph_{A_1U}(r_1) \vee \aleph_{A_2U}(r_2)). \quad (21)$$

(v)  $\mu_{A_1} \geq \mu_{B_2}$  and  $\mu_{A_2} \geq \mu_{B_1}$ , then

$$(t d\mu_A)_{\check{G}_1 \bullet \check{G}_2}^{\check{}}(r_1, r_2) = (t d\mu_A)_{\check{G}_1}^{\check{}}(r_1) + (t d\mu_A)_{\check{G}_2}^{\check{}}(r_2) - (\mu_{A_1}(r_1) \vee \mu_{A_2}(r_2)). \quad (22)$$

(vi)  $\nu_{A_1} \geq \nu_{B_2}$  and  $\nu_{A_2} \geq \nu_{B_1}$ , then

$$(t d\nu_A)_{\check{G}_1 \bullet \check{G}_2}^{\check{}}(r_1, r_2) = (t d\nu_A)_{\check{G}_1}^{\check{}}(r_1) + (t d\nu_A)_{\check{G}_2}^{\check{}}(r_2) - (\nu_{A_1}(r_1) \vee \nu_{A_2}(r_2)). \quad (23)$$

*proof.* It is easy to prove by using Definition 9 and Theorem 2.  $\square$

*Definition 10.* Let  $\check{G}_1 = (\aleph_1, \mathfrak{R}_1)$  and  $\check{G}_2 = (\aleph_2, \mathfrak{R}_2)$  be two CuPFGs of graphs  $G_1 = (R_1, E_1)$  and  $G_2 = (R_2, E_2)$  respectively. The lexicographical product of  $G_1$  and  $G_2$  is denoted as  $G_1 \circ G_2 = (\aleph_1 \circ \aleph_2, \mathfrak{R}_1 \circ \mathfrak{R}_2)$  and is defined as follows:

- (i)  $(\aleph_{A_1L} \circ \aleph_{A_2L})(r_1, r_2) = \aleph_{A_1L}(r_1) \wedge \aleph_{A_2L}(r_2)$   
 $(\mathfrak{R}_{A_1U} \circ \mathfrak{R}_{A_2U})(r_1, r_2) = \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2)$   
 $(\aleph_{A_1L} \circ \aleph_{A_2L})(r_1, r_2) = \aleph_{A_1L}(r_1) \wedge \aleph_{A_2L}(r_2)$   
 $(\aleph_{A_1U} \circ \aleph_{A_2U})(r_1, r_2) = \aleph_{A_1U}(r_1) \wedge \aleph_{A_2U}(r_2)$   
 $(\mu_{A_1} \circ \mu_{A_2})(r_1, r_2) = \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2)$   
 $(\nu_{A_1} \circ \nu_{A_2})(r_1, r_2) = \nu_{A_1}(r_1) \wedge \nu_{A_2}(r_2)$   
 for all  $(r_1, r_2) \in R_1 \times R_2$
- (ii)  $(\mathfrak{R}_{B_1L} \circ \mathfrak{R}_{B_2L})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1L}(r) \wedge \mathfrak{R}_{B_2L}(r_2, s_2)$   
 $(\mathfrak{R}_{B_1U} \circ \mathfrak{R}_{B_2U})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1U}(r) \wedge \mathfrak{R}_{B_2U}(r_2, s_2)$   
 $(\aleph_{B_1L} \circ \aleph_{B_2L})((r, r_2), (r, s_2)) = \aleph_{A_1L}(r) \wedge \aleph_{B_2L}(r_2, s_2)$   
 $(\aleph_{B_1U} \circ \aleph_{B_2U})((r, r_2), (r, s_2)) = \aleph_{A_1U}(r) \wedge \aleph_{B_2U}(r_2, s_2)$   
 $(\mu_{B_1} \circ \mu_{B_2})((r, r_2), (r, s_2)) = \mu_{A_1}(r) \wedge \mu_{B_2}(r_2, s_2)$   
 $(\nu_{B_1} \circ \nu_{B_2})((r, r_2), (r, s_2)) = \nu_{A_1}(r) \wedge \nu_{B_2}(r_2, s_2)$   
 for all  $r \in R_1$ , for all  $r_2, s_2 \in E_2$
- (iii)  $(\mathfrak{R}_{A_1L} \circ \mathfrak{R}_{B_2L})((r_1, z), (s_1, z)) = \mathfrak{R}_{B_1L}(r_1, s_1) \wedge \mathfrak{R}_{A_2L}(z)$

$$(\mathfrak{R}_{A_1U} \circ \mathfrak{R}_{B_2U})((r_1, z), (s_1, z)) = \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{A_2U}(z)$$

$$(\aleph_{A_1L} \circ \aleph_{B_2L})((r_1, z), (s_1, z)) = \aleph_{B_1L}(r_1, s_1) \wedge \aleph_{A_2L}(z)$$

$$(\aleph_{A_1U} \circ \aleph_{B_2U})((r_1, z), (s_1, z)) = \aleph_{B_1U}(r_1, s_1) \wedge \aleph_{A_2U}(z)$$

$$(\mu_{B_1} \circ \mu_{B_2})((r_1, z), (s_1, z)) = \mu_{B_1}(r_1, s_1) \wedge \mu_{A_2}(z)$$

$$(\nu_{B_1} \circ \nu_{B_2})((r_1, z), (s_1, z)) = \nu_{B_1}(r_1, s_1) \wedge \nu_{A_2}(z)$$

for all  $z \in R_2$ , for all  $r_1, s_1 \in E_1$

$$(iv) (\mathfrak{R}_{B_1L} \circ \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)) = \mathfrak{R}_{B_1L}(r_1, s_1) \wedge \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{A_2L}(s_2)$$

$$(\mathfrak{R}_{B_1U} \circ \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)) = \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{A_2U}(s_2)$$

$$(\aleph_{B_1L} \circ \aleph_{B_2L})((r_1, r_2), (s_1, s_2)) = \aleph_{B_1L}(r_1, s_1) \wedge \aleph_{A_2L}(r_2) \wedge \aleph_{A_2L}(s_2)$$

$$(\aleph_{B_1U} \circ \aleph_{B_2U})((r_1, r_2), (s_1, s_2)) = \aleph_{B_1U}(r_1, s_1) \wedge \aleph_{A_2U}(r_2) \wedge \aleph_{A_2U}(s_2)$$

$$(\mu_{B_1} \circ \mu_{B_2})((r_1, r_2), (s_1, s_2)) = \mu_{B_1}(r_1, s_1) \wedge \mu_{B_2}(r_2) \wedge \mu_{B_2}(s_2)$$

$$(\nu_{B_1} \circ \nu_{B_2})((r_1, r_2), (s_1, s_2)) = \nu_{B_1}(r_1, s_1) \wedge \nu_{B_2}(r_2) \wedge \nu_{B_2}(s_2) \text{ for all } r_1, s_1 \in E_1, r_2 \neq s_2$$

*Example 3.* Consider two CuPFGs  $\check{G}_1$  and  $\check{G}_2$  on  $R_1 = \{r, s\}$  and  $R_2 = \{t, u, v\}$ , respectively, as shown in Figure 4. Their lexicographical product  $G_1 \circ G_2$  is shown in Figure 5.

**Proposition 2.** The lexicographical product  $\check{G}_1 \circ \check{G}_2$  of two CuPFGs  $G_1$  and  $G_2$  is also a CuPFG.



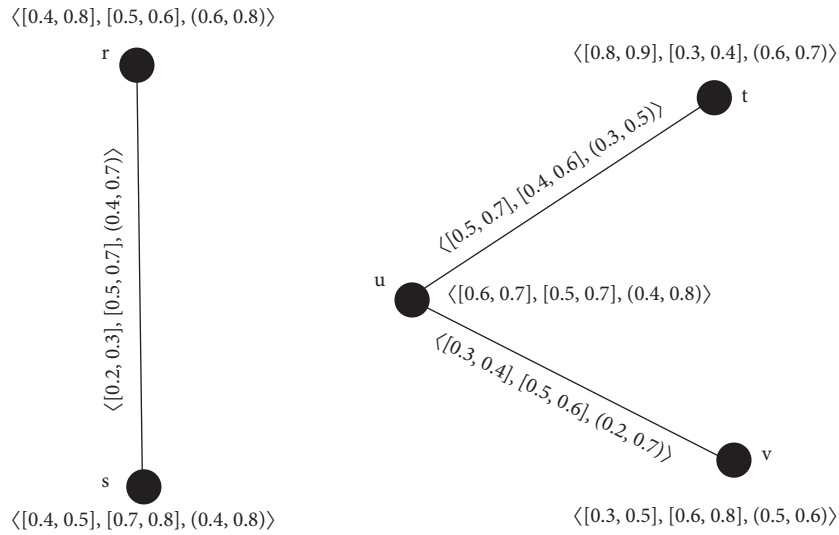


FIGURE 4: CuPFGs  $\check{G}_1$  and  $\check{G}_2$ .

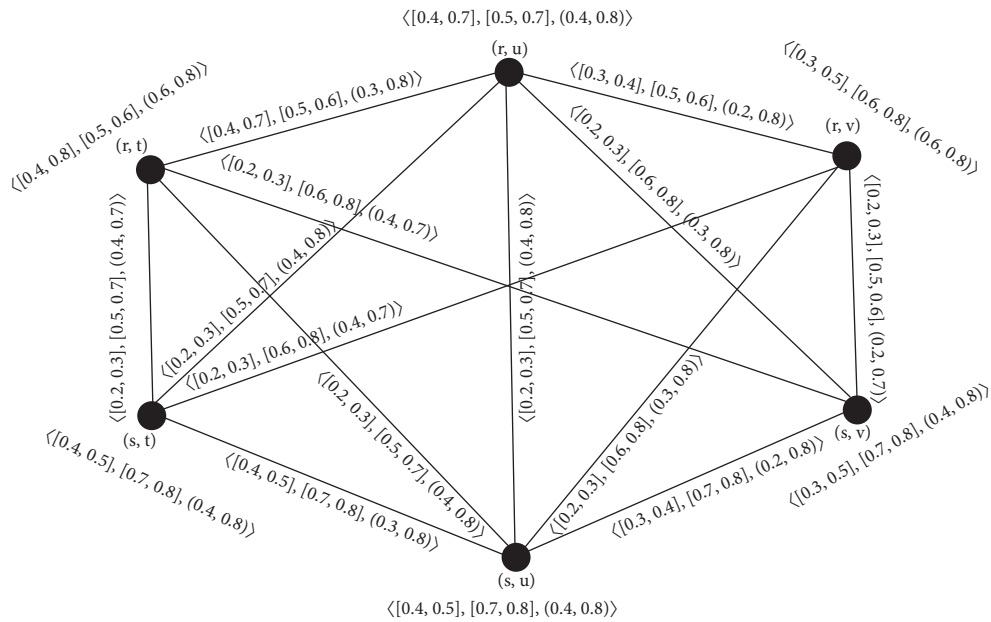


FIGURE 5: Lexicographical product of two CuPFGs  $\check{G}_1$  and  $\check{G}_2$ .

*Definition 11.* Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. Then, for any vertex  $(r_1, r_2) \in R_1 \times R_2$ ,

$$\begin{aligned}
 (d\mathfrak{R}_L)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \circ \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2, s_2) + \sum_{r_2=s_2, r_1s_1 \in E_1} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{B_1L}(r_1s_1) \\
 &\quad + \sum_{r_1s_1 \in E_1, r_2 \neq s_2} \mathfrak{R}_{B_1L}(r_1s_1) \wedge \mathfrak{R}_{B_2L}(r_2) \wedge \mathfrak{R}_{B_2L}(s_2),
 \end{aligned}$$

(24)



$$\begin{aligned}
 (d\mathfrak{R}_U)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1U} \circ \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{B_2U}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{B_1U}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{B_2U}(r_2) \wedge \mathfrak{R}_{B_2U}(s_2),
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 (d\aleph)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\aleph_{B_1} \circ \aleph_{B_2})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \aleph_{A_1L}(r_1) \wedge \aleph_{B_2L}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \aleph_{A_2L}(r_2) \wedge \aleph_{B_1L}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \aleph_{B_1L}(r_1, s_1) \wedge \aleph_{B_2L}(r_2) \wedge \aleph_{B_2L}(s_2),
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 (d\aleph_U)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\aleph_{B_1U} \circ \aleph_{B_2U})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \aleph_{A_1U}(r_1) \wedge \aleph_{B_2U}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \aleph_{A_2U}(r_2) \wedge \aleph_{B_1U}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \aleph_{B_1U}(r_1, s_1) \wedge \aleph_{B_2U}(r_2) \wedge \aleph_{B_2U}(s_2),
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 (d\mu_L)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mu_{B_1L} \circ \mu_{B_2L})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mu_{A_2}(r_2) \wedge \mu_{B_1}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mu_{B_1}(r_1, s_1) \wedge \mu_{B_2}(r_2) \wedge \mu_{B_2}(s_2),
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 (d\nu)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\nu_{B_1} \circ \nu_{B_2})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \nu_{A_2}(r_2) \wedge \nu_{B_1}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \nu_{B_1}(r_1, s_1) \wedge \nu_{B_2}(r_2) \wedge \nu_{B_2}(s_2).
 \end{aligned} \tag{29}$$

**Theorem 3.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. If  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L}$ ,  $\aleph_{A_1L} \leq \aleph_{B_2L}$ ,  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2U}$ ,  $\aleph_{A_1U} \leq \aleph_{B_2U}$ ,  $\mu_{A_1} \geq \mu_{B_2}$ ,  $\nu_{A_1} \leq \nu_{B_2}$  and  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}$ ,  $\aleph_{A_2L} \leq \aleph_{B_1L}$ ,  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1U}$ ,  $\aleph_{A_2U} \leq \aleph_{B_1U}$ ,  $\mu_{A_2} \geq \mu_{B_1}$ ,  $\nu_{A_2} \leq \nu_{B_1}$ , then  $d_{\check{G}_1 \circ \check{G}_2}^-(r_1, r_2) =$

$a_2(d_{\check{G}_1}^-(r_1) + d_{\check{G}_2}^-(r_2))$ , where,  $a_2 = |R_2|$  for all  $(r_1, r_2) \in R_1 \times R_2$ .

*proof.* From the Definition 11 of vertex degree of  $\check{G}_1 \circ \check{G}_2$ ,

$$\begin{aligned}
 (d\mathfrak{R}_L)_{\check{G}_1 \circ \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \circ \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)) \\
 &= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{B_1L}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{R}_{B_1L}(r_1) \wedge \mathfrak{R}_{B_1L, s_1} \wedge \mathfrak{R}_{B_2L}(r_2, s_2) \\
 &= \sum_{r_2, s_2 \in E_2} \mathfrak{R}_{A_2L}(r_2, s_2) + \sum_{r_1, s_1 \in E_1} \mathfrak{R}_{A_1L}(r_1, s_1) \\
 &\quad + \sum_{r_1, s_1 \in E_1} \mathfrak{R}_{A_1L}(r_1, s_1) \text{ (by using } \mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L} \text{ and } \mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}) \\
 &= a_2(d\mathfrak{R}_L)_{\check{G}_1}^-(r_1) + (d\mathfrak{R}_L)_{\check{G}_2}^-(r_2),
 \end{aligned} \tag{30}$$

where  $a_2 = |R_2|$ . By following the same steps, we have

$$\begin{aligned}
(d\mathfrak{R}_U)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= a_2(d\mathfrak{R}_U)_{\check{G}_1}^{\check{G}_1}(r_1) + (d\mathfrak{R}_U)_{\check{G}_2}^{\check{G}_2}(r_2), \\
(d\mathfrak{N}_L)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= a_2(d\mathfrak{N}_L)_{\check{G}_1}^{\check{G}_1}(r_1) + (d\mathfrak{N}_L)_{\check{G}_2}^{\check{G}_2}(r_2), \\
(d\mathfrak{N}_U)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= a_2(d\mathfrak{N}_U)_{\check{G}_1}^{\check{G}_1}(r_1) + (d\mathfrak{N}_U)_{\check{G}_2}^{\check{G}_2}(r_2), \\
(d\mu)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= a_2(d\mu)_{\check{G}_1}^{\check{G}_1}(r_1) + (d\mu)_{\check{G}_2}^{\check{G}_2}(r_2), \\
(d\nu)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= a_2(d\nu)_{\check{G}_1}^{\check{G}_1}(r_1) + (d\nu)_{\check{G}_2}^{\check{G}_2}(r_2).
\end{aligned} \tag{31}$$

□

*Definition 12.* Let us take two CuPFGs  $\check{G}_1$  and  $\check{G}_2$ , then the total degree for any vertex  $(r_1, r_2) \in R_1 \times R_2$  is defined as

$$\begin{aligned}
(t d\mathfrak{R}_L)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \circ \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1L} \circ \mathfrak{R}_{A_2L})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{B_1L}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{R}_{B_1L}(r_1) \wedge \mathfrak{R}_{B_1L}(s_1) \wedge \mathfrak{R}_{B_2L}(r_2, s_2) + \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_2L}(r_2),
\end{aligned} \tag{32}$$

$$\begin{aligned}
(t d\mathfrak{R}_U)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1U} \circ \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1U} \circ \mathfrak{R}_{A_2U})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{B_2U}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{B_1U}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{R}_{B_1U}(r_1) \wedge \mathfrak{R}_{B_1U}(s_1) \wedge \mathfrak{R}_{B_2U}(r_2, s_2) + \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2),
\end{aligned} \tag{33}$$

$$\begin{aligned}
(t d\mathfrak{N}_L)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1L} \circ \mathfrak{N}_{B_2L})((r_1, r_2), (s_1, s_2)) + (\mathfrak{N}_{A_1L} \circ \mathfrak{N}_{A_2L})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{B_2L}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{N}_{A_2L}(r_2) \wedge \mathfrak{N}_{B_1L}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{N}_{B_1L}(r_1) \wedge \mathfrak{N}_{B_1L}(s_1) \wedge \mathfrak{N}_{B_2L}(r_2, s_2) + \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{A_2L}(r_2),
\end{aligned} \tag{34}$$

$$\begin{aligned}
(t d\mathfrak{N}_U)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1U} \circ \mathfrak{N}_{B_2U})((r_1, r_2), (s_1, s_2)) + (\mathfrak{N}_{A_1U} \circ \mathfrak{N}_{A_2U})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{N}_{A_1U}(r_1) \wedge \mathfrak{N}_{B_2U}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{N}_{A_2U}(r_2) \wedge \mathfrak{N}_{B_1U}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mathfrak{N}_{B_1U}(r_1) \wedge \mathfrak{N}_{B_1U}(s_1) \wedge \mathfrak{N}_{B_2U}(r_2, s_2) + \mathfrak{N}_{A_1U}(r_1) \wedge \mathfrak{N}_{A_2U}(r_2),
\end{aligned} \tag{35}$$

$$\begin{aligned}
(t d\mu)_{\check{G}_1 \circ \check{G}_2}^{\check{G}_1, \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mu_{B_1} \circ \mu_{B_2})((r_1, r_2), (s_1, s_2)) + (\mu_{A_1} \circ \mu_{A_2})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mu_{A_2}(r_2) \wedge \mu_{B_1}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \mu_{B_1}(r_1) \wedge \mu_{B_1}(s_1) \wedge \mu_{B_2}(r_2, s_2) + \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2),
\end{aligned} \tag{36}$$

$$\begin{aligned}
(t \, dv)_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) &= \sum_{(r_1, r_2) \in R_1 \times R_2} (\nu_{B_1} \circ \nu_{B_2})((r_1, r_2), (s_1, s_2)) + (\nu_{A_1} \circ \nu_{A_2})(r_1, r_2) \\
&\cdot \sum_{r_1=s_1, r_2, s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2, s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \nu_{A_2}(r_2) \wedge \nu_{B_1}(r_1, s_1) \\
&+ \sum_{r_1, s_1 \in E_1, r_2 \neq s_2} \nu_{B_1}(r_1) \wedge \nu_{B_1}(s_1) \wedge \nu_{B_2}(r_2, s_2) + \nu_{A_1}(r_1) \wedge \nu_{A_2}(r_2).
\end{aligned} \tag{37}$$

**Theorem 4.** Let us take two CuPFGs  $\check{G}_1$  and  $\check{G}_2$ . If

(i)  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L}$  and  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}$ , then

$$(t \, d\mathfrak{R}_{AL})_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\mathfrak{R}_{AL})_{\check{G}_1}^{\check{}}(r_1) + (t \, d\mathfrak{R}_{AL})_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\mathfrak{R}_{A_1L}(r_1) - (\mathfrak{R}_{A_1L}(r_1) \vee \mathfrak{R}_{A_2L}(r_2)). \tag{38}$$

(ii)  $\mathfrak{R}_{A_1U} \geq \mathfrak{R}_{B_2U}$  and  $\mathfrak{R}_{A_2U} \geq \mathfrak{R}_{B_1U}$ , then

$$(t \, d\mathfrak{R}_{AU})_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\mathfrak{R}_{AU})_{\check{G}_1}^{\check{}}(r_1) + (t \, d\mathfrak{R}_{AU})_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\mathfrak{R}_{A_1U}(r_1) - (\mathfrak{R}_{A_1U}(r_1) \vee \mathfrak{R}_{A_2U}(r_2)). \tag{39}$$

(iii)  $\aleph_{A_1L} \geq \aleph_{B_2L}$  and  $\aleph_{A_2L} \geq \aleph_{B_1L}$ , then

$$(t \, d\aleph_{AL})_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\aleph_{AL})_{\check{G}_1}^{\check{}}(r_1) + (t \, d\aleph_{AL})_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\aleph_{A_1L}(r_1) - (\aleph_{A_1L}(r_1) \vee \aleph_{A_2L}(r_2)). \tag{40}$$

(iv)  $\aleph_{A_1U} \geq \aleph_{B_2U}$  and  $\aleph_{A_2U} \geq \aleph_{B_1U}$ , then

$$(t \, d\aleph_{AU})_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\aleph_{AU})_{\check{G}_1}^{\check{}}(r_1) + (t \, d\aleph_{AU})_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\aleph_{A_1U}(r_1) - (\aleph_{A_1U}(r_1) \vee \aleph_{A_2U}(r_2)). \tag{41}$$

(v)  $\mu_{A_1} \geq \mu_{B_2}$  and  $\mu_{A_2} \geq \mu_{B_1}$ , then

$$(t \, d\mu_A)_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\mu_A)_{\check{G}_1}^{\check{}}(r_1) + (t \, d\mu_A)_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\mu_{A_1}(r_1) - (\mu_{A_1}(r_1) \vee \mu_{A_2}(r_2)). \tag{42}$$

(vi)  $\nu_{A_1} \geq \nu_{B_2}$  and  $\nu_{A_2} \geq \nu_{B_1}$ , then

$$(t \, d\nu_A)_{\check{G}_1 \circ \check{G}_2}^{\check{}}(r_1, r_2) = a_2 (t \, d\nu_A)_{\check{G}_1}^{\check{}}(r_1) + (t \, d\nu_A)_{\check{G}_2}^{\check{}}(r_2) - (a_2 - 1)\nu_{A_1}(r_1) - (\nu_{A_1}(r_1) \vee \nu_{A_2}(r_2)). \tag{43}$$

where  $a_2 = |R_2|$ .

*proof.* It is easy to prove by using Definition 12 and Theorem 4.  $\square$

**Definition 13.** Let  $\check{G}_1 = (\aleph_1, \mathfrak{R}_1)$  and  $\check{G}_2 = (\aleph_2, \mathfrak{R}_2)$  be two CuPFGs of graphs  $G_1 = (R_1, E_1)$  and  $G_2 = (R_2, E_2)$ , respectively. The symmetric difference of  $\check{G}_1$  and  $\check{G}_2$  is denoted by  $G_1 \oplus G_2 = (\aleph_1 \oplus \aleph_2, \mathfrak{R}_1 \oplus \mathfrak{R}_2)$  and is defined as follows:

$$\begin{aligned}
(i) \quad &(\mathfrak{R}_{A_1L} \oplus \mathfrak{R}_{A_2L})(r_1, r_2) = \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_2L}(r_2) \\
&(\mathfrak{R}_{A_1U} \oplus \mathfrak{R}_{A_2U})(r_1, r_2) = \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2)
\end{aligned}$$

$$(\aleph_{A_1L} \oplus \aleph_{A_2L})(r_1, r_2) = \aleph_{A_1L}(r_1) \wedge \aleph_{A_2L}(r_2)$$

$$(\aleph_{A_1U} \oplus \aleph_{A_2U})(r_1, r_2) = \aleph_{A_1U}(r_1) \wedge \aleph_{A_2U}(r_2)$$

$$(\mu_{A_1} \oplus \mu_{A_2})(r_1, r_2) = \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2)$$

$$(\nu_{A_1} \oplus \nu_{A_2})(r_1, r_2) = \nu_{A_1}(r_1) \wedge \nu_{A_2}(r_2) \quad \text{for all } (r_1, r_2) \in \check{R}$$

$$(ii) \quad (\mathfrak{R}_{B_1L} \oplus \mathfrak{R}_{B_2L})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1L}(r) \wedge \mathfrak{R}_{B_2L}(r_2, s_2)$$

$$(\mathfrak{R}_{B_1U} \oplus \mathfrak{R}_{B_2U})((r, r_2), (r, s_2)) = \mathfrak{R}_{A_1U}(r) \wedge \mathfrak{R}_{B_2U}(r_2, s_2)$$

$$(\aleph_{B_1L} \oplus \aleph_{B_2L})((r, r_2), (r, s_2)) = \aleph_{A_1L}(r) \wedge \aleph_{B_2L}(r_2, s_2)$$

$$(\mathfrak{N}_{B_1U} \oplus \mathfrak{N}_{B_2U})((r, r_2), (r, s_2)) = \mathfrak{N}_{A_1U}(r) \wedge \mathfrak{N}_{B_2U}(r_2, s_2)$$

$$(\mu_{B_1U} \oplus \mu_{B_2U})((r, r_2), (r, s_2)) = \mathfrak{N}_{A_1U}(r) \wedge \mathfrak{N}_{B_2U}(r_2, s_2)$$

$$(\nu_{B_1} \oplus \nu_{B_2})((r, r_2), (r, s_2)) = \nu_{A_1}(r_1) \vee \nu_{B_2}(r_2, s_2),$$

for all  $r \in R_1$ , for all  $r_2 s_2 \in E_2$

$$(iii) (\mathfrak{R}_{A_1L} \oplus \mathfrak{R}_{B_2L})((r_1, z), (s_1, z)) = \mathfrak{R}_{B_1L}(r_1, s_1) \wedge \mathfrak{R}_{A_2L}(z)$$

$$(\mathfrak{R}_{A_1U} \oplus \mathfrak{R}_{B_2U})((r_1, z), (s_1, z)) = \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{A_2U}(z)$$

$$(\mathfrak{N}_{A_1L} \oplus \mathfrak{N}_{B_2L})((r_1, z), (s_1, z)) = \mathfrak{N}_{B_1L}(r_1, s_1) \wedge \mathfrak{N}_{A_2L}(z)$$

$$(\mathfrak{N}_{A_1U} \oplus \mathfrak{N}_{B_2U})((r_1, z), (s_1, z)) = \mathfrak{N}_{B_1U}(r_1, s_1) \wedge \mathfrak{N}_{A_2U}(z)$$

$$(\mu_{B_1} \oplus \mu_{B_2})((r_1, z), (s_1, z)) = \mu_{B_1}(r_1, s_1) \wedge \mu_{A_2}(z)$$

$$(\nu_{B_1} \oplus \nu_{B_2})((r_1, z), (s_1, z)) = \nu_{B_1}(r_1, s_1) \wedge \nu_{A_2}(z) \text{ for all } z \in R_2, \text{ for all } r_1 s_1 \in E_1$$

$$(iv) (\mathfrak{R}_{B_1L} \oplus \mathfrak{R}_{B_2L})(r_1, r_2), ((s_1, s_2)) = \mathfrak{R}_{B_1L}(r_1, s_1) \wedge \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{A_2L}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \mathfrak{R}_{B_2L}(r_2, s_2) \wedge \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_1L}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

$$(\mathfrak{R}_{B_1U} \oplus \mathfrak{R}_{B_2U})(r_1, r_2), ((s_1, s_2)) = \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{A_2U}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \mathfrak{R}_{B_2U}(r_2, s_2) \wedge \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_1U}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

$$(\mathfrak{N}_{B_1L} \oplus \mathfrak{N}_{B_2L})(r_1, r_2), ((s_1, s_2)) = \mathfrak{N}_{B_1L}(r_1, s_1) \wedge \mathfrak{N}_{A_2L}(r_2) \wedge \mathfrak{N}_{A_2L}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \mathfrak{N}_{B_2L}(r_2, s_2) \wedge \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{A_1L}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

$$(\mathfrak{N}_{B_1U} \oplus \mathfrak{N}_{B_2U})(r_1, r_2), ((s_1, s_2)) = \mathfrak{N}_{B_1U}(r_1, s_1) \wedge \mathfrak{N}_{A_2U}(r_2) \wedge \mathfrak{N}_{A_2U}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \mathfrak{N}_{B_2U}(r_2, s_2) \wedge \mathfrak{N}_{A_1U}(r_1) \wedge \mathfrak{N}_{A_1U}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

$$(\mu_{B_1} \oplus \mu_{B_2})(r_1, r_2), ((s_1, s_2)) = \mu_{B_1}(r_1, s_1) \wedge \mu_{A_2}(r_2) \wedge \mu_{A_2}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \mu_{B_2}(r_2, s_2) \wedge \mu_{A_1}(r_1) \wedge \mu_{A_1}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

$$(\nu_{B_1} \oplus \nu_{B_2})(r_1, r_2), ((s_1, s_2)) = \nu_{B_1}(r_1, s_1) \wedge \nu_{A_2}(r_2) \wedge \nu_{A_2}(s_2) \quad \forall r_1 s_1 \notin E_1, r_2 s_2 \in E_2 \text{ or } = \nu_{B_2}(r_2, s_2) \wedge \nu_{A_1}(r_1) \wedge \nu_{A_1}(s_1) \quad \forall r_2 s_2 \notin E_2, r_1 s_1 \in E_1$$

*Example 4.* Consider two CuPFGs  $\tilde{G}_1$  and  $\tilde{G}_2$  on  $R_1 = \{r, s\}$  and  $R_2 = \{t, u, v\}$ , respectively, as shown in Figure 4. Their symmetric difference  $\tilde{G}_1 \oplus \tilde{G}_2$  is shown in Figure 6.

**Proposition 3.** *The symmetric difference  $\tilde{G}_1 \oplus \tilde{G}_2$  of two CuPFGs  $G_1$  and  $G_2$  of  $G_1$  and  $G_2$  is also a CuPFG.*

*Definition 14.* Let  $\tilde{G}_1$  and  $\tilde{G}_2$  be two CuPFGs. Then, for any vertex  $(r_1, r_2) \in V_1 \times V_2$ ,

$$\begin{aligned} (d\mathfrak{R}_L)_{\tilde{G}_1 \oplus \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \oplus \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)), \\ &= \sum_{r_1 = s_1, r_2 s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2, s_2) + \sum_{r_2 = s_2, r_1 s_1 \in E_1} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{B_1L}(r_1, s_1) \\ &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \mathfrak{R}_{B_1L}(r_1, s_1) \wedge \mathfrak{R}_{B_2L}(r_2) \wedge \mathfrak{R}_{B_2L}(s_2) \\ &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \mathfrak{R}_{B_2L}(r_2, s_2) \wedge \mathfrak{R}_{B_1L}(r_1) \wedge \mathfrak{R}_{B_1L}(s_1), \end{aligned} \quad (44)$$

$$\begin{aligned} (d\mathfrak{R}_U)_{\tilde{G}_1 \oplus \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1U} \oplus \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)), \\ &= \sum_{r_1 = s_1, r_2 s_2 \in E_2} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{B_2U}(r_2, s_2) + \sum_{r_2 = s_2, r_1 s_1 \in E_1} \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{B_1U}(r_1, s_1) \\ &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \mathfrak{R}_{B_1U}(r_1, s_1) \wedge \mathfrak{R}_{B_2U}(r_2) \wedge \mathfrak{R}_{B_2U}(s_2) \\ &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \mathfrak{R}_{B_2U}(r_2, s_2) \wedge \mathfrak{R}_{B_1U}(r_1) \wedge \mathfrak{R}_{B_1U}(s_1), \end{aligned} \quad (45)$$

$$\begin{aligned} (d\mathfrak{N}_L)_{\tilde{G}_1 \oplus \tilde{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2), (s_1, s_2) \in R_1 \times R_2} (\mathfrak{N}_{B_1L} \oplus \mathfrak{N}_{B_2L})((r_1, r_2), (s_1, s_2)), \\ &= \sum_{r_1 = s_1, r_2 s_2 \in E_2} \mathfrak{N}_{A_1L}(r_1) \wedge \mathfrak{N}_{B_2L}(r_2, s_2) + \sum_{r_2 = s_2, r_1 s_1 \in E_1} \mathfrak{N}_{A_2L}(r_2) \wedge \mathfrak{N}_{B_1L}(r_1, s_1) \\ &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \mathfrak{N}_{B_1L}(r_1, s_1) \wedge \mathfrak{N}_{B_2L}(r_2) \wedge \mathfrak{N}_{B_2L}(s_2) \\ &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \mathfrak{N}_{B_2L}(r_2, s_2) \wedge \mathfrak{N}_{B_1L}(r_1) \wedge \mathfrak{N}_{B_1L}(s_1), \end{aligned} \quad (46)$$

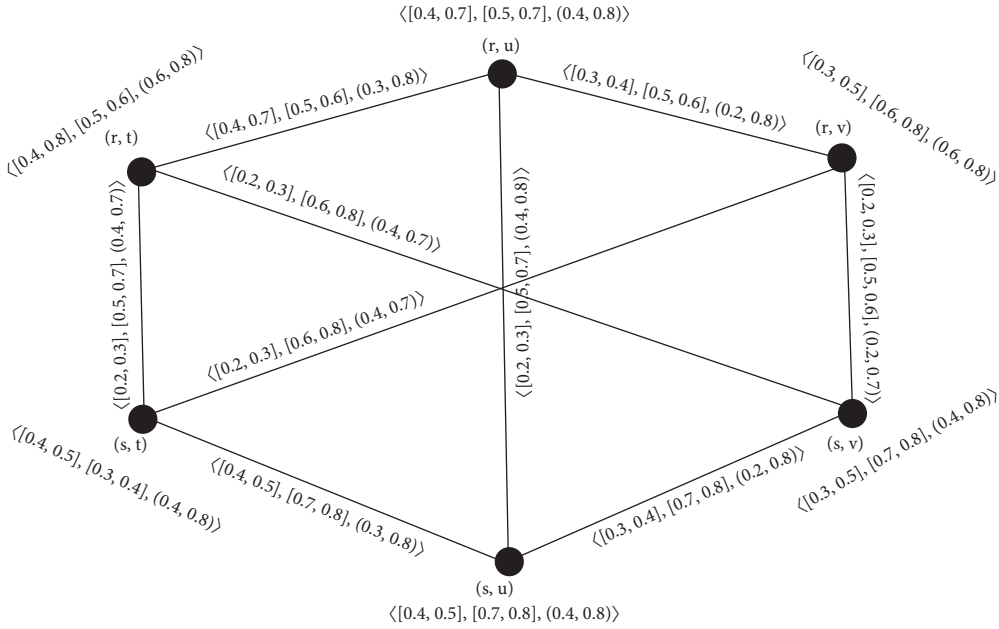


FIGURE 6: Symmetric difference of two CuPFGs  $\check{G}_1$  and  $\check{G}_2$ .

$$\begin{aligned}
 (d\aleph_U)_{\check{G}_1 \oplus \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\aleph_{B_1 U} \oplus \aleph_{B_2 U})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \aleph_{A_1 U}(r_1) \wedge \aleph_{B_2 U}(r_2 s_2) + \sum_{r_2=s_2, r_1 s_1 \in E_1} \aleph_{A_2 U}(r_2) \wedge \aleph_{B_1 U}(r_1 s_1) \\
 &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \aleph_{B_1 U}(r_1 s_1) \wedge \aleph_{B_2 U}(r_2) \wedge \aleph_{B_2 U}(s_2) \\
 &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \aleph_{B_2 U}(r_2 s_2) \wedge \aleph_{B_1 U}(r_1) \wedge \aleph_{B_1 U}(s_1),
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 (d\mu_L)_{\check{G}_1 \oplus \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mu_{B_1 L} \oplus \mu_{B_2 L})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2 s_2) + \sum_{r_2=s_2, r_1 s_1 \in E_1} \mu_{A_2}(r_2) \wedge \mu_{B_1}(r_1 s_1) \\
 &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \mu_{B_1}(r_1 s_1) \wedge \mu_{B_2}(r_2) \wedge \mu_{B_2}(s_2) \\
 &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \mu_{B_2}(r_2 s_2) \wedge \mu_{B_1}(r_1) \wedge \mu_{B_1}(s_1),
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 (d\nu)_{\check{G}_1 \oplus \check{G}_2}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\nu_{B_1} \oplus \nu_{B_2})((r_1, r_2), (s_1, s_2)), \\
 &= \sum_{r_1=s_1, r_2=s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2 s_2) + \sum_{r_2=s_2, r_1 s_1 \in E_1} \nu_{A_2}(r_2) \wedge \nu_{B_1}(r_1 s_1) \\
 &+ \sum_{r_1 s_1 \in E_1, r_2 s_2 \notin E_2} \nu_{B_1}(r_1 s_1) \wedge \nu_{B_2}(r_2) \wedge \nu_{B_2}(s_2) \\
 &+ \sum_{r_2 s_2 \in E_2, r_1 s_1 \notin E_1} \nu_{B_2}(r_2 s_2) \wedge \nu_{B_1}(r_1) \wedge \nu_{B_1}(s_1).
 \end{aligned} \tag{49}$$

**Theorem 5.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. If  $\aleph_{A_1 L} \geq \aleph_{B_2 L}$ ,  $\aleph_{A_1 L} \leq \aleph_{B_2 L}$ ,  $\aleph_{A_1 L} \geq \aleph_{B_2 U}$ ,  $\aleph_{A_1 U} \leq \aleph_{B_2 U}$ ,  $\mu_{A_1} \geq \mu_{B_2}$ ,  $\nu_{A_1} \leq \nu_{B_2}$  and  $\aleph_{A_2 L} \geq \aleph_{B_1 L}$ ,  $\aleph_{A_2 L} \leq \aleph_{B_1 L}$ ,  $\aleph_{A_2 L} \geq \aleph_{B_1 U}$ ,  $\aleph_{A_2 U} \leq$

$\aleph_{B_1 U}$ ,  $\mu_{A_2} \geq \mu_{B_1}$ ,  $\nu_{A_2} \leq \nu_{B_1}$ , then  $d_{\check{G}_1 \oplus \check{G}_2}(r_1, r_2) = b_2 d_{\check{G}_2}(r_1) + b_1 d_{\check{G}_1}(r_2)$ , where  $b_2 = |R_2| - d_{\check{G}_2}(r_2)$  and  $b_1 = |R_1| - d_{\check{G}_1}(r_1)$ , for all  $(r_1, r_2) \in R_1 \times R_2$ .

*Definition 15.* Let  $\tilde{G}_1$  and  $\tilde{G}_2$  be two CuPFGs. Then, for any vertex  $(r_1, r_2) \in R_1 \times R_2$ ,

$$\begin{aligned}
(t d\mathfrak{R}_L)_{\tilde{G}_1 \oplus \tilde{G}_2}^{\sim}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1L} \oplus \mathfrak{R}_{B_2L})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1L} \oplus \mathfrak{R}_{A_2L})(r_1, r_2) \\
&= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{B_2L}(r_2 s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{B_1L}(r_1 s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \mathfrak{R}_{A_2L}(r_2) \wedge \mathfrak{R}_{A_2L}(s_2) \wedge \mathfrak{R}_{B_1L}(r_1 s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_1L}(s_1) \wedge \mathfrak{R}_{B_2L}(r_2 s_2) + \mathfrak{R}_{A_1L}(r_1) \wedge \mathfrak{R}_{A_2L}(r_2),
\end{aligned} \tag{50}$$

$$\begin{aligned}
(t d\mathfrak{R}_U)_{\tilde{G}_1 \oplus \tilde{G}_2}^{\sim}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mathfrak{R}_{B_1U} \oplus \mathfrak{R}_{B_2U})((r_1, r_2), (s_1, s_2)) + (\mathfrak{R}_{A_1U} \oplus \mathfrak{R}_{A_2U})(r_1, r_2) \\
&= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{B_2U}(r_2 s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{B_1U}(r_1 s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \mathfrak{R}_{A_2U}(r_2) \wedge \mathfrak{R}_{A_2U}(s_2) \wedge \mathfrak{R}_{B_1U}(r_1 s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_1U}(s_1) \wedge \mathfrak{R}_{B_2U}(r_2 s_2) + \mathfrak{R}_{A_1U}(r_1) \wedge \mathfrak{R}_{A_2U}(r_2),
\end{aligned} \tag{51}$$

$$\begin{aligned}
(t d\aleph_L)_{\tilde{G}_1 \oplus \tilde{G}_2}^{\sim}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\aleph_{B_1L} \oplus \aleph_{B_2L})((r_1, r_2), (s_1, s_2)) + (\aleph_{A_1L} \oplus \aleph_{A_2L})(r_1, r_2) \\
&= \sum_{r_1=s_1, r_2, s_2 \in E_2} \aleph_{A_1L}(r_1) \wedge \aleph_{B_2L}(r_2 s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \aleph_{A_2L}(r_2) \wedge \aleph_{B_1L}(r_1 s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \aleph_{A_2L}(r_2) \wedge \aleph_{A_2L}(s_2) \wedge \aleph_{B_1L}(r_1 s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \aleph_{A_1L}(r_1) \wedge \aleph_{A_1L}(s_1) \wedge \aleph_{B_2L}(r_2 s_2) + \aleph_{A_1L}(r_1) \wedge \aleph_{A_2L}(r_2),
\end{aligned} \tag{52}$$

$$\begin{aligned}
(t d\aleph_U)_{\tilde{G}_1 \oplus \tilde{G}_2}^{\sim}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\aleph_{B_1U} \oplus \aleph_{B_2U})((r_1, r_2), (s_1, s_2)) + (\aleph_{A_1U} \oplus \aleph_{A_2U})(r_1, r_2) \\
&= \sum_{r_1=s_1, r_2, s_2 \in E_2} \aleph_{A_1U}(r_1) \wedge \aleph_{B_2U}(r_2 s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \aleph_{A_2U}(r_2) \wedge \aleph_{B_1U}(r_1 s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \aleph_{A_2U}(r_2) \wedge \aleph_{A_2U}(s_2) \wedge \aleph_{B_1U}(r_1 s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \aleph_{A_1U}(r_1) \wedge \aleph_{A_1U}(s_1) \wedge \aleph_{B_2U}(r_2 s_2) + \aleph_{A_1U}(r_1) \wedge \aleph_{A_2U}(r_2),
\end{aligned} \tag{53}$$

$$\begin{aligned}
(t d\mu)_{\tilde{G}_1 \oplus \tilde{G}_2}^{\sim}(r_1, r_2) &= \sum_{(r_1, r_2) (s_1, s_2) \in R_1 \times R_2} (\mu_{B_1} \oplus \mu_{B_2})((r_1, r_2), (s_1, s_2)) + (\mu_{A_1} \oplus \mu_{A_2})(r_1, r_2) \\
&= \sum_{r_1=s_1, r_2, s_2 \in E_2} \mu_{A_1}(r_1) \wedge \mu_{B_2}(r_2 s_2) + \sum_{r_2=s_2, r_1, s_1 \in E_1} \mu_{A_2}(r_2) \wedge \mu_{B_1}(r_1 s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \mu_{A_2}(r_2) \wedge \mu_{A_2}(s_2) \wedge \mu_{B_1}(r_1 s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \mu_{A_1}(r_1) \wedge \mu_{A_1}(s_1) \wedge \mu_{B_2}(r_2 s_2) + \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2),
\end{aligned} \tag{54}$$

$$\begin{aligned}
(t \, dv)_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= \sum_{(r_1, r_2) \left( \begin{smallmatrix} (s_1, s_2) \in R_1 \times R_2 \end{smallmatrix} \right)} (\mu_{B_1} \oplus \mu_{B_2})((r_1, r_2), (s_1, s_2)) + (\nu_{A_1} \oplus \nu_{A_2})(r_1, r_2) \\
&= \sum_{r_1 = s_1, r_2, s_2 \in E_2} \nu_{A_1}(r_1) \wedge \nu_{B_2}(r_2, s_2) + \sum_{r_2 = s_2, r_1, s_1 \in E_1} \mu_{A_2}(r_2) \wedge \mu_{B_1}(r_1, s_1) \\
&\quad + \sum_{r_1, s_1 \in E_1, r_2, s_2 \notin E_2} \nu_{A_2}(r_2) \wedge \nu_{A_2}(s_2) \wedge \mu_{B_1}(r_1, s_1) \\
&\quad + \sum_{r_2, s_2 \in E_2, r_1, s_1 \notin E_1} \nu_{A_1}(r_1) \wedge \nu_{A_1}(s_1) \wedge \mu_{B_2}(r_2, s_2) + \mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2).
\end{aligned} \tag{55}$$

**Theorem 6.** Let  $\check{G}_1$  and  $\check{G}_2$  be two CuPFGs. If

(i)  $\mathfrak{R}_{A_1L} \geq \mathfrak{R}_{B_2L}$  and  $\mathfrak{R}_{A_2L} \geq \mathfrak{R}_{B_1L}$ , then

$$\begin{aligned}
(t \, d\mathfrak{R}_{AL})_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= b_2 (t \, d\mathfrak{R}_{AL})_{\check{G}_1}^{\check{}}(r_1) + b_1 (t \, d\mathfrak{R}_{AL})_{\check{G}_2}^{\check{}}(r_2) \\
&\quad - (b_2 - 1)\mathfrak{R}_{A_1L}(r_1) - (b_1 - 1)\mathfrak{R}_{A_2L}(r_2) - (\mathfrak{R}_{A_1L}(r_1) \vee \mathfrak{R}_{A_2L}(r_2)).
\end{aligned} \tag{56}$$

(ii)  $\mathfrak{R}_{A_1U} \geq \mathfrak{R}_{B_2U}$  and  $\mathfrak{R}_{A_2U} \geq \mathfrak{R}_{B_1U}$ , then

$$\begin{aligned}
(t \, d\mathfrak{R}_{AU})_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= b_2 (t \, d\mathfrak{R}_{AU})_{\check{G}_1}^{\check{}}(r_1) + b_1 (t \, d\mathfrak{R}_{AU})_{\check{G}_2}^{\check{}}(r_2) \\
&\quad - (b_2 - 1)\mathfrak{R}_{A_1U}(r_1) - (b_1 - 1)\mathfrak{R}_{A_2U}(r_2) - (\mathfrak{R}_{A_1U}(r_1) \vee \mathfrak{R}_{A_2U}(r_2)).
\end{aligned} \tag{57}$$

(iii)  $\aleph_{A_1L} \geq \aleph_{B_2L}$  and  $\aleph_{A_2L} \geq \aleph_{B_1L}$ , then

$$\begin{aligned}
(t \, d\aleph_{AL})_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= b_2 (t \, d\aleph_{AL})_{\check{G}_1}^{\check{}}(r_1) + b_1 (t \, d\aleph_{AL})_{\check{G}_2}^{\check{}}(r_2) - (b_2 - 1)\aleph_{A_1L}(r_1) \\
&\quad - (b_1 - 1)\aleph_{A_2L}(r_2) - (\aleph_{A_1L}(r_1) \vee \aleph_{A_2L}(r_2)).
\end{aligned} \tag{58}$$

(iv)  $\aleph_{A_1U} \geq \aleph_{B_2U}$  and  $\aleph_{A_2U} \geq \aleph_{B_1U}$ , then

$$\begin{aligned}
(t \, d\aleph_{AU})_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= b_2 (t \, d\aleph_{AU})_{\check{G}_1}^{\check{}}(r_1) + b_1 (t \, d\aleph_{AU})_{\check{G}_2}^{\check{}}(r_2) - (b_2 - 1)\aleph_{A_1U}(r_1) \\
&\quad - (b_1 - 1)\aleph_{A_2U}(r_2) - (\aleph_{A_1U}(r_1) \vee \aleph_{A_2U}(r_2)).
\end{aligned} \tag{59}$$

(v)  $\mu_{A_1} \geq \mu_{B_2}$  and  $\mu_{A_2} \geq \mu_{B_1}$ , then

$$\begin{aligned}
(t \, d\mu_A)_{\check{G}_1 \oplus \check{G}_2}^{\check{}}(r_1, r_2) &= b_2 (t \, d\mu_A)_{\check{G}_1}^{\check{}}(r_1) + b_1 (t \, d\mu_A)_{\check{G}_2}^{\check{}}(r_2) - (b_2 - 1)\mu_{A_1}(r_1) - (b_1 - 1)\mu_{A_2}(r_2) \\
&\quad - (\mu_{A_1}(r_1) \vee \mu_{A_2}(r_2)).
\end{aligned} \tag{60}$$

(vi)  $\nu_{A_1} \geq \nu_{B_2}$  and  $\nu_{A_2} \geq \nu_{B_1}$ , then



$$(t \, dv_A)_{\tilde{G}_1 \oplus \tilde{G}_2}(r_1, r_2) = b_2(t \, dv_A)_{\tilde{G}_1}(r_1) + b_1(t \, dv_A)_{\tilde{G}_2}(r_2) - (b_2 - 1)v_{A_1}(r_1) - (b_1 - 1)v_{A_2}(r_2) - (v_{A_1}(r_1) \vee v_{A_2}(r_2)), \quad (61)$$

where  $b_2 = |R_2| - d_{\tilde{G}_2}(r_2)$  and  $b_1 = |R_1| - d_{\tilde{G}_1}(r_1)$ , for all  $(r_1, r_2) \in R_1 \times R_2$ .

*proof.* It is easy to prove by using Definition 15 and Theorem 5. □

### 3. Application

The purpose of decision-making activity is to solve an issue ended up with satisfactory results. In current section, we investigate MCDM problem in cubic PF environment, demonstrating the validity of our proposed work in realistic scenarios. To express the experts' preference among the given alternatives, preference relation is the most beneficial technique by which the ranking of alternatives can be obtained. For set of alternatives  $E = \{E_1, E_2, E_3, \dots, E_n\}$ , the decision makers compare each one alternative to other and obtained preference relations, respectively. If every element in the preference relations belongs to a CuPFs then the concept of the cubic Pythagorean fuzzy preference relation (CuPFPR) can be taken as follows.

*Definition 16.* A CuPFPR on the set of alternatives  $E = \{E_1, E_2, E_3, \dots, E_n\}$  is represented by a matrix  $Z = (e_{jk})_{n \times n}$ , for  $j \neq k$ , where

$$(e_{jk})_{n \times n} = \langle [\mathfrak{R}_{jk}^L, \mathfrak{R}_{jk}^U], [\aleph_{jk}^L, \aleph_{jk}^U], (\mu_{jk}, \nu_{jk}), \quad (62)$$

for all  $(j, k = 1, 2, 3, \dots, n)$ . Such that,  $\mathfrak{R}_{jk}^L$  and  $\mathfrak{R}_{jk}^U$  represent the lower and upper limits of interval-valued preference degree to which the object  $E_j$  is preferred to  $E_k$ ,  $\aleph_{jk}^L$  and  $\aleph_{jk}^U$  represent the lower and upper limits of interval-valued degree to which the object  $E_j$  is not preferred to  $E_k$ ,  $\mu_{jk}$  is the simple fuzzy valued degree of preference, and  $\nu_{jk}$  is the simple fuzzy valued degree of nonpreference of the alternative  $E_j$  over  $E_k$ , respectively, with the following conditions:

$$\begin{aligned} &\mathfrak{R}_{jk}^L, \mathfrak{R}_{jk}^U, \aleph_{jk}^L, \aleph_{jk}^U, \mu_{jk}, \nu_{jk} \in [0, 1], (\mathfrak{R}_{jk}^U)^2 + (\aleph_{jk}^U)^2 \leq 1 \\ &\text{and } (\mu_{jk})^2 + (\nu_{jk})^2 \leq 1 \\ &\mathfrak{R}_{jk}^L = \aleph_{kj}^L, \mathfrak{R}_{jk}^U = \aleph_{kj}^U, \mu_{jk} = \nu_{kj}. \end{aligned}$$

#### 3.1. Selection of the Best Captain for a Cricket Team.

Teams often consist of multitalented individuals who have the right skills and experience to deliver results. But a team leader (captain) is always needed to direct the energy of the members of team toward more effectiveness. The captain of a team provides a road map outlining the steps the team needs to achieve the preferred destination. He helps everybody to identify the roles that best fit skills and experiences. Successful captains are committed to the success of their team

and its players. Captain should not just be the best players of the team or the player that the coach likes the most, but it is about having certain qualities that make that player a good leader.

The selection of a captain is one of the most important and critical decisions that can be made as its influence is deep across all players of the team. A poor process and little thought invite risk and disaster, while a well-considered and inviting process can establish a foundation for progress and success. Keeping in view the role of captain, the experts aim is to choose the best among four candidates  $\{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4\}$  on the basis of certain criterion:

$C_1$ : personal traits: a courageous leader is someone who is not afraid to honestly and openly seek out feedback from those around them and always motivate his team. He is self-confident and a good instructor.

$C_2$ : previous performance and experience: the past performance of a player defines his expertise and experience helps him to guide others.

$C_3$ : professional skills: the best captain cares passionately about team's success and about their teammates with his leadership qualities. He is a good planner, courageous, and consistent.

The experts of the game assigned the weight vector,  $W = \{0.4, 0.3, 0.3\}^T$ , where  $\sum_{m=1}^3 w_m = 1$ , demonstrating the significance of that particular criterion  $C_l$  ( $l = 1, 2, 3$ ). The decision-making experts of the game pairwise compare the four players  $\mathfrak{S}_h$  ( $h = 1, 2, 3, 4$ ) under consideration with respect to the three criteria  $C_l$  ( $l = 1, 2, 3$ ) and provides information according to their preference in the form of cubic Pythagorean fuzzy preference relation (CuPFPR)  $Z^{(l)} = e_{jk4 \times 4}^{(l)}$ , where  $(e_{jk}^l)_{4 \times 4} = \langle [\mathfrak{R}_{jk}^L, \mathfrak{R}_{jk}^U], [\aleph_{jk}^L, \aleph_{jk}^U], (\mu_{jk}, \nu_{jk}) \rangle$  is cubic Pythagorean fuzzy element assigned by the experts, such that interval-valued terms present the preference and rejection grade of each attribute in past and simple fuzzy terms represents the preference and rejection grade of each attribute in present. The CuPFPRs,  $Z^{(l)} = e_{jk4 \times 4}^{(l)}$ , are shown in Tables 1–3 as follows.

The cubic Pythagorean fuzzy influenced networks corresponding to CuPFPRs  $Z^{(l)}$  ( $l = 1, 2, 3$ ) given in Tables 1–3, are shown in Figure 7.

Now, we find aggregated CuPFPR by applying CuPFWA operator.

The Pythagorean fuzzy influenced graph, corresponding to the aggregated CuPFPR given in Table 4, is displayed in Figure 8. Under the constraint on degree of truthness  $\eta_{jk} \geq 0.5$  ( $i, j = 1, 2, 3, 4$ ) on both interval value and simple value, a partial influenced graph is constructed and depicted in Figure 9. The out-degrees out-d  $(\mathfrak{S}_h)$  ( $h = 1, 2, 3, 4$ ) of all considered players in the partial graph are calculated as follows:

TABLE 1: CuPFPR with respect to the criteria “personal traits.”

$Z^1$	$\mathfrak{F}_1$	$\mathfrak{F}_2$	$\mathfrak{F}_3$	$\mathfrak{F}_4$
$\mathfrak{F}_1$	—	$\langle [0.6, 0.7], [0.5, 0.7], (0.6, 0.7) \rangle$	$\langle [0.8, 0.9], [0.3, 0.4], (0.7, 0.5) \rangle$	$\langle [0.4, 0.8], [0.5, 0.6], (0.9, 0.4) \rangle$
$\mathfrak{F}_2$	$\langle [0.8, 0.9], [0.3, 0.4], (0.8, 0.6) \rangle$	—	$\langle [0.6, 0.8], [0.4, 0.5], (0.9, 0.3) \rangle$	$\langle [0.7, 0.8], [0.5, 0.6], (0.7, 0.5) \rangle$
$\mathfrak{F}_3$	$\langle [0.3, 0.5], [0.7, 0.8], (0.3, 0.9) \rangle$	$\langle [0.5, 0.6], [0.7, 0.8], (0.8, 0.4) \rangle$	—	$\langle [0.6, 0.7], [0.5, 0.7], (0.8, 0.6) \rangle$
$\mathfrak{F}_4$	$\langle [0.7, 0.8], [0.5, 0.6], (0.7, 0.6) \rangle$	$\langle [0.8, 0.9], [0.3, 0.4], (0.5, 0.8) \rangle$	$\langle [0.3, 0.4], [0.8, 0.9], (0.8, 0.5) \rangle$	—

TABLE 2: CuPFPR with respect to the criteria “previous performance and experience.”

$Z^2$	$\mathfrak{F}_1$	$\mathfrak{F}_2$	$\mathfrak{F}_3$	$\mathfrak{F}_4$
$\mathfrak{F}_1$	—	$\langle [0.3, 0.4], [0.8, 0.9], (0.3, 0.8) \rangle$	$\langle [0.4, 0.5], [0.7, 0.8], (0.7, 0.6) \rangle$	$\langle [0.2, 0.3], [0.8, 0.9], (0.7, 0.5) \rangle$
$\mathfrak{F}_2$	$\langle [0.7, 0.9], [0.3, 0.4], (0.5, 0.8) \rangle$	—	$\langle [0.6, 0.7], [0.5, 0.6], (0.5, 0.7) \rangle$	$\langle [0.5, 0.7], [0.6, 0.7], (0.8, 0.6) \rangle$
$\mathfrak{F}_3$	$\langle [0.6, 0.7], [0.5, 0.6], (0.6, 0.7) \rangle$	$\langle [0.3, 0.4], [0.4, 0.5], (0.7, 0.6) \rangle$	—	$\langle [0.6, 0.8], [0.4, 0.6], (0.9, 0.4) \rangle$
$\mathfrak{F}_4$	$\langle [0.3, 0.4], [0.8, 0.9], (0.8, 0.5) \rangle$	$\langle [0.7, 0.8], [0.4, 0.5], (0.7, 0.4) \rangle$	$\langle [0.8, 0.9], [0.3, 0.4], (0.4, 0.9) \rangle$	—

TABLE 3: CuPFPR with respect to the criteria “professional skills.”

$Z^3$	$\mathfrak{F}_1$	$\mathfrak{F}_2$	$\mathfrak{F}_3$	$\mathfrak{F}_4$
$\mathfrak{F}_1$	—	$\langle [0.8, 0.9], [0.3, 0.4], (0.8, 0.3) \rangle$	$\langle [0.5, 0.8], [0.5, 0.6], (0.8, 0.5) \rangle$	$\langle [0.3, 0.4], [0.7, 0.8], (0.4, 0.9) \rangle$
$\mathfrak{F}_2$	$\langle [0.6, 0.7], [0.5, 0.7], (0.8, 0.5) \rangle$	—	$\langle [0.6, 0.7], [0.5, 0.6], (0.7, 0.6) \rangle$	$\langle [0.4, 0.6], [0.6, 0.7], (0.5, 0.8) \rangle$
$\mathfrak{F}_3$	$\langle [0.5, 0.6], [0.6, 0.7], (0.7, 0.6) \rangle$	$\langle [0.6, 0.7], [0.5, 0.6], (0.6, 0.4) \rangle$	—	$\langle [0.6, 0.7], [0.5, 0.6], (0.6, 0.5) \rangle$
$\mathfrak{F}_4$	$\langle [0.4, 0.5], [0.5, 0.7], (0.5, 0.8) \rangle$	$\langle [0.5, 0.8], [0.4, 0.6], (0.4, 0.8) \rangle$	$\langle [0.4, 0.5], [0.6, 0.7], (0.5, 0.4) \rangle$	—

$$\begin{aligned}
 \text{out} - d(\mathfrak{F}_1) &= \langle [1.0956, 1.3665], [1.1294, 1.3801], (1.2598, 1.2071) \rangle, \\
 \text{out} - d(\mathfrak{F}_2) &= \langle [1.8400, 2.2780], [1.4037, 1.7545], (2.0531, 1.8709) \rangle, \\
 \text{out} - d(\mathfrak{F}_3) &= \langle [0.6, 0.7286], [0.4733, 0.6446], (0.7602, 0.5213) \rangle, \\
 \text{out} - d(\mathfrak{F}_4) &= \langle [1.1264, 1.4029], [0.9953, 1.2663], (1.1759, 1.3917) \rangle.
 \end{aligned} \tag{63}$$

According to membership value of out-d ( $\mathfrak{F}_h$ ) ( $h = 1, 2, 3, 4$ ), we obtain the optimal ranking order of the four players  $\mathfrak{F}_h$  ( $h = 1, 2, 3, 4$ ) as follows:

$$\mathfrak{F}_2 > \mathfrak{F}_4 > \mathfrak{F}_1 > \mathfrak{F}_3. \tag{64}$$

Thus,  $\mathfrak{F}_4$  is decided as the best for captancy of the team.

Now, we find aggregated CuPFPR by applying CuPFWG operator.

$$\begin{aligned}
 \text{out} - d(\mathfrak{F}_1) &= \langle [1.2828, 1.5558], [0.9448, 1.1943], (1.3689, 1.0932) \rangle, \\
 \text{out} - d(\mathfrak{F}_2) &= \langle [1.9033, 2.3333], [1.3648, 1.6890], (2.2221, 1.7036) \rangle, \\
 \text{out} - d(\mathfrak{F}_3) &= \langle [0.6, 0.7352], [0.4676, 0.6382], (0.8042, 0.5030) \rangle, \\
 \text{out} - d(\mathfrak{F}_4) &= \langle [1.8239, 2.1913], [1.4790, 1.8470], (1.9008, 1.8269) \rangle.
 \end{aligned} \tag{65}$$

According to membership value of out-d ( $\mathfrak{F}_h$ ) ( $h = 1, 2, 3, 4$ ), we obtain ranking order of the four players  $\mathfrak{F}_h$  ( $h = 1, 2, 3, 4$ ) as

$$\mathfrak{F}_2 > \mathfrak{F}_4 > \mathfrak{F}_1 > \mathfrak{F}_3. \tag{66}$$

Hence, in both cases  $\mathfrak{F}_4$  is decided as the best for captancy of the team.

The Pythagorean fuzzy influenced graph, corresponding to the aggregated CuPFPR given in Table 5 is displayed in Figure 10. Within the constraint on degree of truthness  $\eta_{jk} \geq 0.5$  ( $j, k = 1, 2, 3, 4$ ) on both interval value and simple value, a partial influenced graph is constructed as depicted in Figure 11. The out-degrees out-d ( $\mathfrak{F}_h$ ) ( $h = 1, 2, 3, 4$ ) of all considered players in the partial graph are calculated as

Our proposed method in MCDM of CuPFGs is outlined in Algorithm 1.

Furthermore, for better understanding the organization of the proposed method is presented in Figure 12.

3.2. Comparative Analysis. In order to demonstrate the effectiveness and validity of the propose method, a comparative analysis with already existing techniques is carried

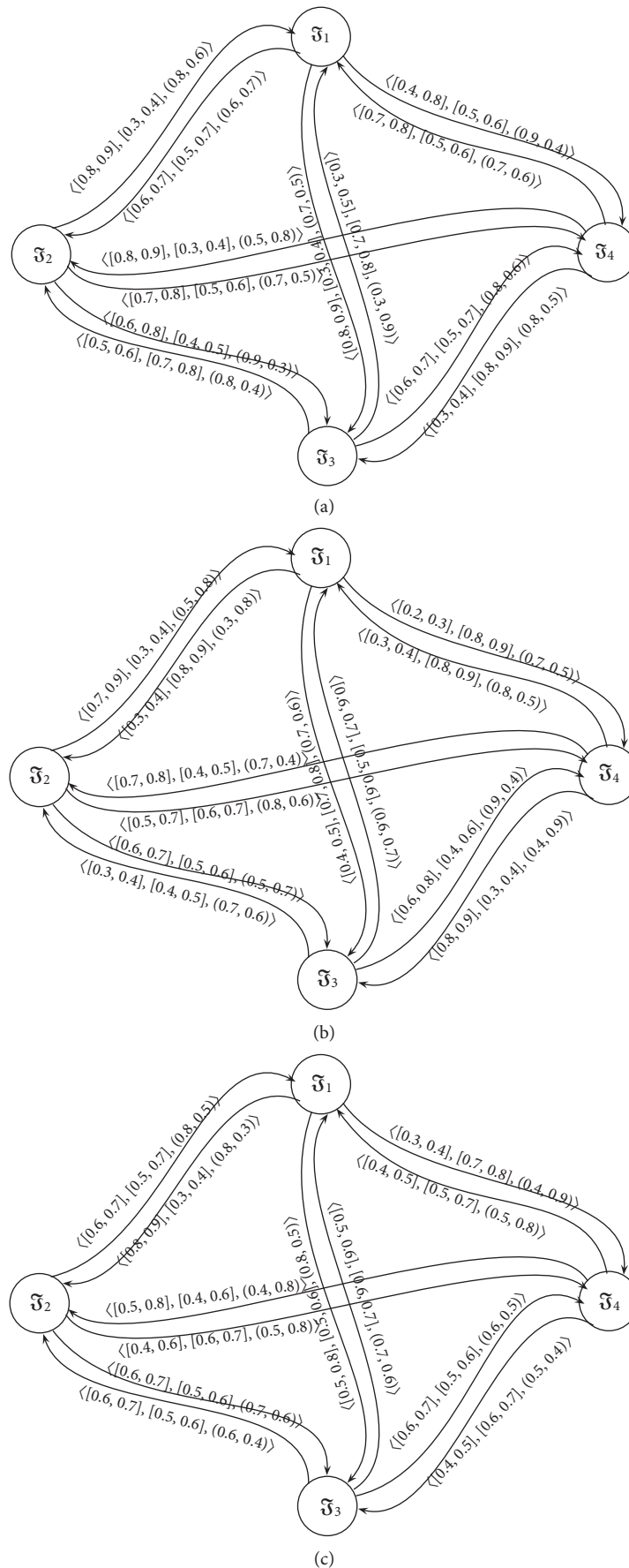


FIGURE 7: Cubic Pythagorean fuzzy influenced networks corresponding to CuPFPRs.

TABLE 4: Aggregated cubic Pythagorean fuzzy preference relation by CuPFWA operator.

$Z$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{S}_1$	—	$\langle [0.53124, 0.6382], [0.6019, 0.7480], (0.5312, 0.6734) \rangle$	$\langle [0.5643, 0.7283], [0.5275, 0.6321], (0.7286, 0.5337) \rangle$	$\langle [0.2980, 0.4842], [0.6812, 0.7911], (0.6544, 0.6929) \rangle$
$\mathfrak{S}_2$	$\langle [0.7050, 0.8346], [0.3759, 0.5261], (0.6947, 0.6597) \rangle$	—	$\langle [0.6, 0.7384], [0.4639, 0.5639], (0.6997, 0.5584) \rangle$	$\langle [0.5349, 0.7050], [0.5639, 0.6645], (0.6586, 0.6528) \rangle$
$\mathfrak{S}_3$	$\langle [0.4305, 0.5842], [0.6222, 0.7246], (0.4762, 0.7951) \rangle$	$\langle [0.4531, 0.5564], [0.6492, 0.7662], (0.7050, 0.4752) \rangle$	—	$\langle [0.6, 0.7286], [0.4734, 0.6645], (0.7602, 0.5213) \rangle$
$\mathfrak{S}_4$	$\langle [0.4590, 0.5643], [0.6311, 0.7646], (0.6586, 0.6597) \rangle$	$\langle [0.6675, 0.8386], [0.3642, 0.5016], (0.5172, 0.7319) \rangle$	$\langle [0.4389, 0.5454], [0.6595, 0.7752], (0.5643, 0.6972) \rangle$	—

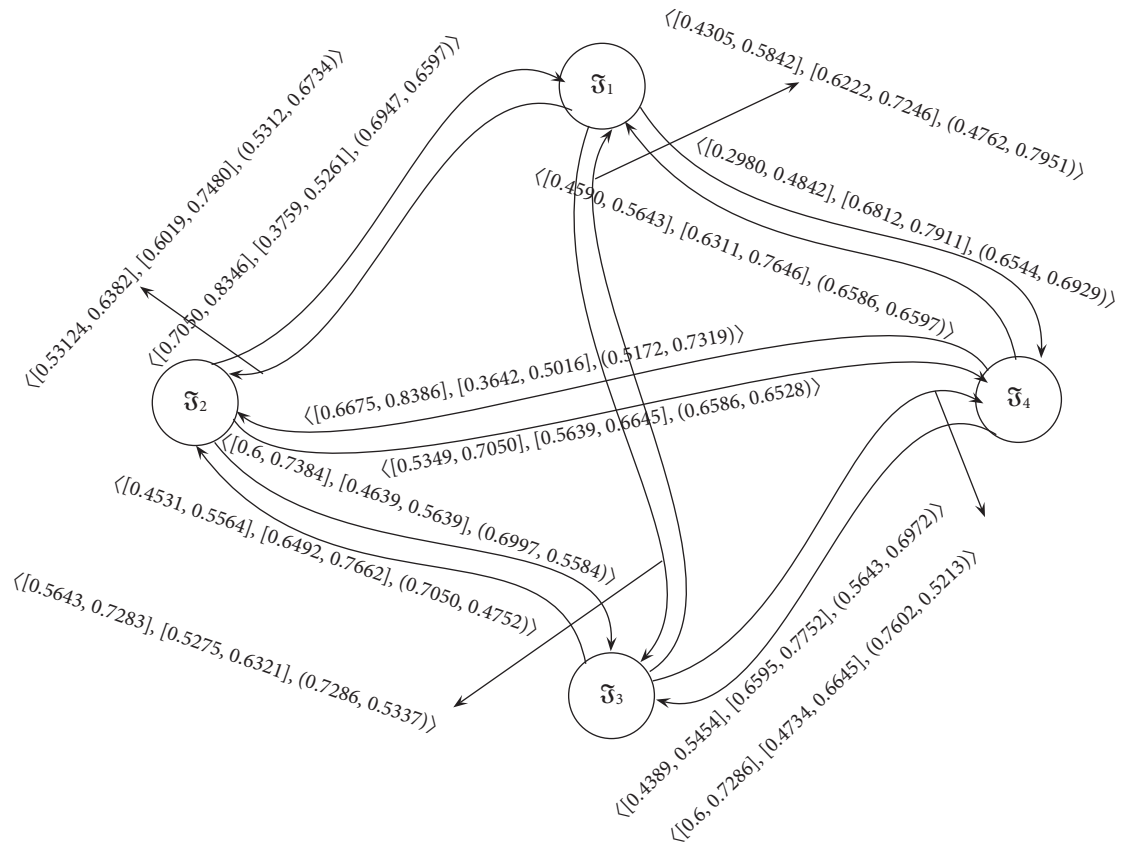


FIGURE 8: Influenced graph of aggregated CuPFPR.

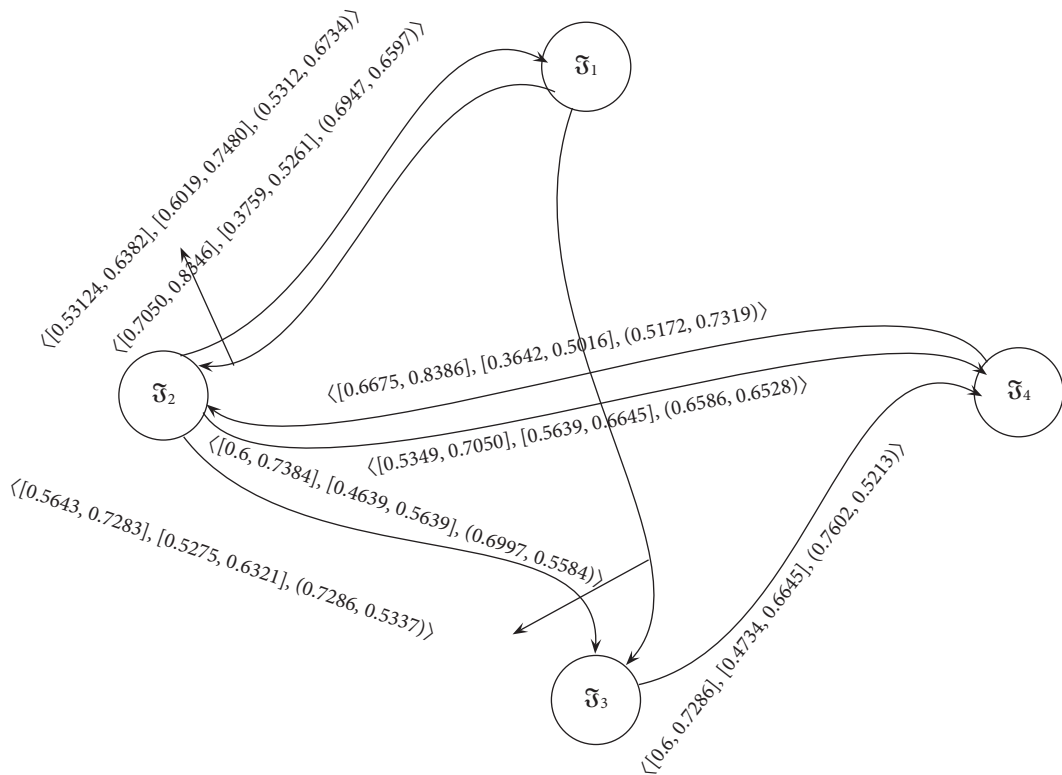


FIGURE 9: Partial influenced graph of aggregated CuPFPR.

TABLE 5: Aggregated cubic Pythagorean fuzzy preference relation by CuPFWG operator.

$Z$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{S}_1$	—	$\langle [0.6336, 0.7480], [0.4939, 0.6382], (0.6336, 0.5650) \rangle$	$\langle [0.6492, 0.8078], [0.4509, 0.5562], (0.7353, 0.5281) \rangle$	$\langle [0.3231, 0.6220], [0.6369, 0.7387], (0.7752, 0.5455) \rangle$
$\mathfrak{S}_2$	$\langle [0.7246, 0.8628], [0.3497, 0.4731], (0.7425, 0.6193) \rangle$	—	$\langle [0.6, 0.7459], [0.4573, 0.5578], (0.7838, 0.4762) \rangle$	$\langle [0.5788, 0.7246], [0.5578, 0.6581], (0.6959, 0.6081) \rangle$
$\mathfrak{S}_3$	$\langle [0.4768, 0.6025], [0.6042, 0.7050], (0.5584, 0.7390) \rangle$	$\langle [0.4920, 0.5928], [0.5643, 0.6681], (0.7246, 0.4517) \rangle$	—	$\langle [0.6, 0.7458], [0.4676, 0.6382], (0.8042, 0.5030) \rangle$
$\mathfrak{S}_4$	$\langle [0.5433, 0.6492], [0.5757, 0.7097], (0.6959, 0.5619) \rangle$	$\langle [0.7085, 0.8492], [0.3565, 0.4830], (0.5573, 0.6498) \rangle$	$\langle [0.5721, 0.6929], [0.5468, 0.6544], (0.6492, 0.5578) \rangle$	—

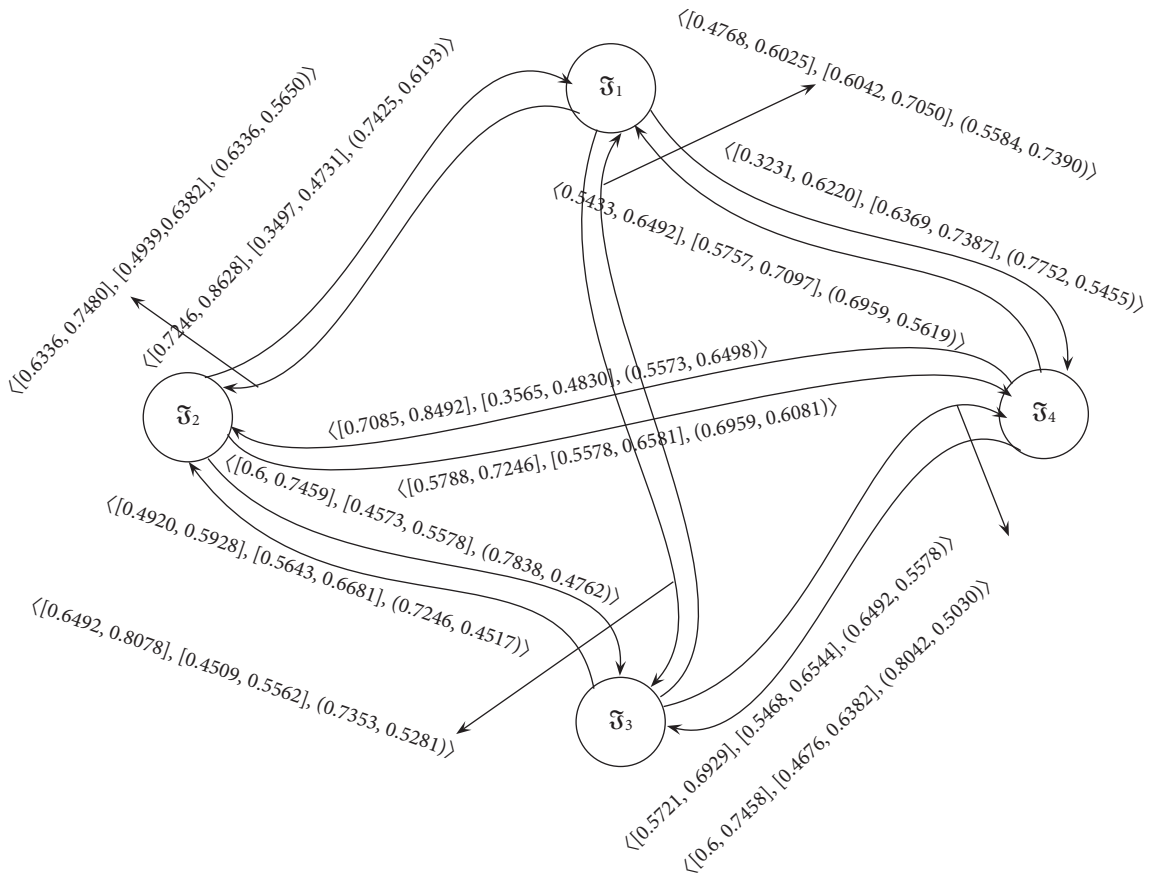


FIGURE 10: Influenced graph of aggregated CuPFPR.

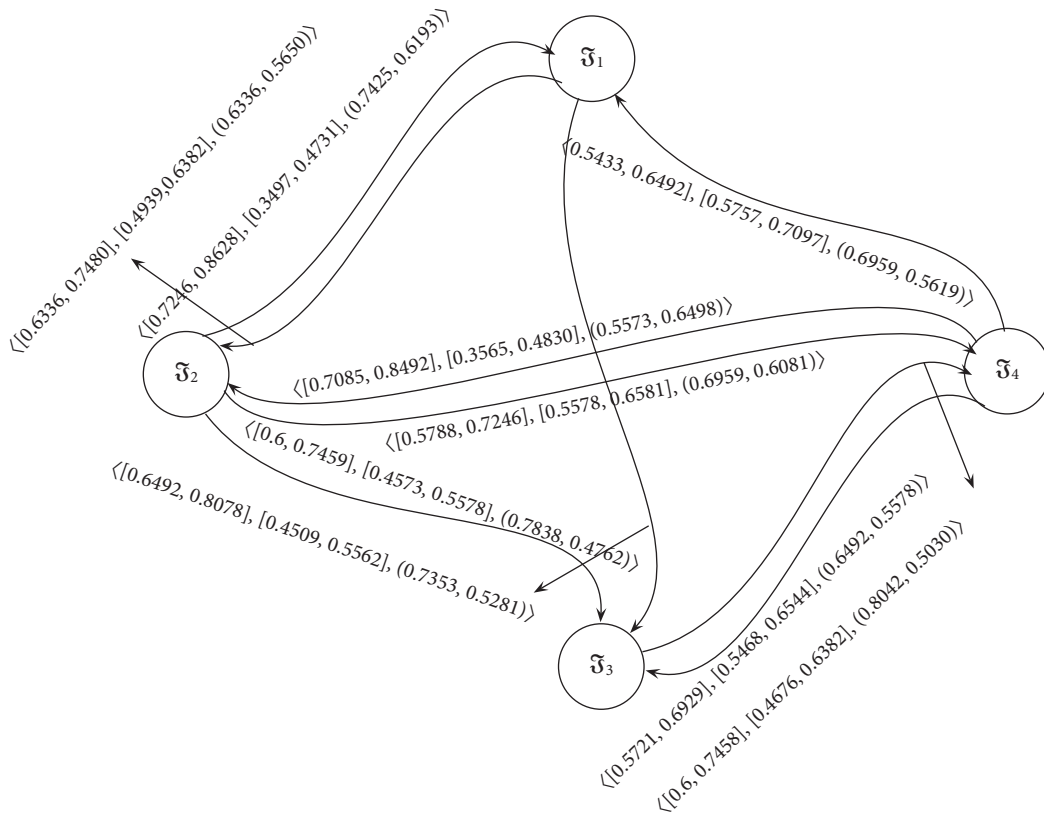


FIGURE 11: Partial influenced graph of aggregated CuPFPR.



- (1) Input: first of all, take discrete set of alternatives  $E = \{E_1, E_2, E_3, \dots, E_n\}$  and attributes  $C = \{C_1, C_2, C_3, \dots, C_n\}$  under some parameters in order to achieve the target
- (2) Let the cubic Pythagorean fuzzy preference relation(CuPFPR) is of the form:  
 $Z = (e_{jk})_{4 \times 4} = \langle [\mathfrak{R}_{jk}^L, \mathfrak{R}_{jk}^U], [\mathfrak{N}_{jk}^L, \mathfrak{N}_{jk}^U], (\mu_{jk}, \nu_{jk}) \rangle (j, k = 1, 2, 3, \dots, n)$  of each alternative  $E_j$  over the  $E_k$  with respect to the all considered criteria  $C_l (l = 1, 2, 3 \dots, n)$
- (3) Apply cubic Pythagorean fuzzy weighted averaging(CuPFWA) operator [10]  

$$(e_j) = CuPFWA(e_{j1}, e_{j2}, e_{j3}, \dots, e_{jn}) = \langle \left[ \prod_{j=1}^n (\mathfrak{R}_{e_j}^L)^{w_j}, \mathfrak{R}_{j=1}^n (\mu_{e_j}^U)^{w_j} \right], \left[ \sqrt{1 - \prod_{j=1}^n (1 - (\mathfrak{N}_{e_j}^L)^{2w_j}), \sqrt{1 - \prod_{j=1}^n (1 - (\mathfrak{N}_{e_j}^U)^{2w_j})} \right], \left( \prod_{j=1}^n (\mu_{e_j})^{w_j}, \sqrt{1 - \prod_{j=1}^n (1 - (\nu_{e_j})^2)^{w_j}} \right) \rangle$$

and cubic Pythagorean fuzzy weighted geometric(CuPFWG) operator [10]  

$$(e_j) = CuPFWG(e_{j1}, e_{j2}, e_{j3}, \dots, e_{jn}) \langle \left[ \sqrt{1 - \prod_{j=1}^n (1 - (\mathfrak{N}_{e_j}^L)^{2w_j}), \sqrt{1 - \prod_{j=1}^n (1 - (\mathfrak{N}_{e_j}^U)^{2w_j})} \right], \left[ \prod_{j=1}^n (\mathfrak{R}_{e_j}^L)^{w_j}, \mathfrak{R}_{j=1}^n (\mu_{e_j}^U)^{w_j} \right], \left( \sqrt{1 - \prod_{j=1}^n (1 - (\nu_{e_j})^2)^{w_j}}, \prod_{j=1}^n (\mu_{e_j})^{w_j} \right) \rangle$$

to aggregate the individual CuPFPR
- (4) Construct the collective averaged CuPFR  
 $(e_{jk})_{4 \times 4} = \langle [\mathfrak{R}_{jk}^L, \mathfrak{R}_{jk}^U], [\mathfrak{N}_{jk}^L, \mathfrak{N}_{jk}^U], (\mu_{jk}, \nu_{jk}) \rangle (j, k = 1, 2, 3, 4)$ ,  
 computed by CuPFWA and CuPFWG operator
- (5) Draw an influenced model relating the collective CuPFPR
- (6) Draw a partially influenced model relating the collective CuPFPR by using constraint on degree of truthness  $\eta_{jk} \geq 0.5$
- (7) Investigate the out-degrees of all alternatives  $E_j$  in partial influenced network
- (8) Rank each of the alternative on the basis of out-degrees  $\eta_j$   
 Output: the alternative having maximum out-degree is selected as the best one

ALGORITHM 1

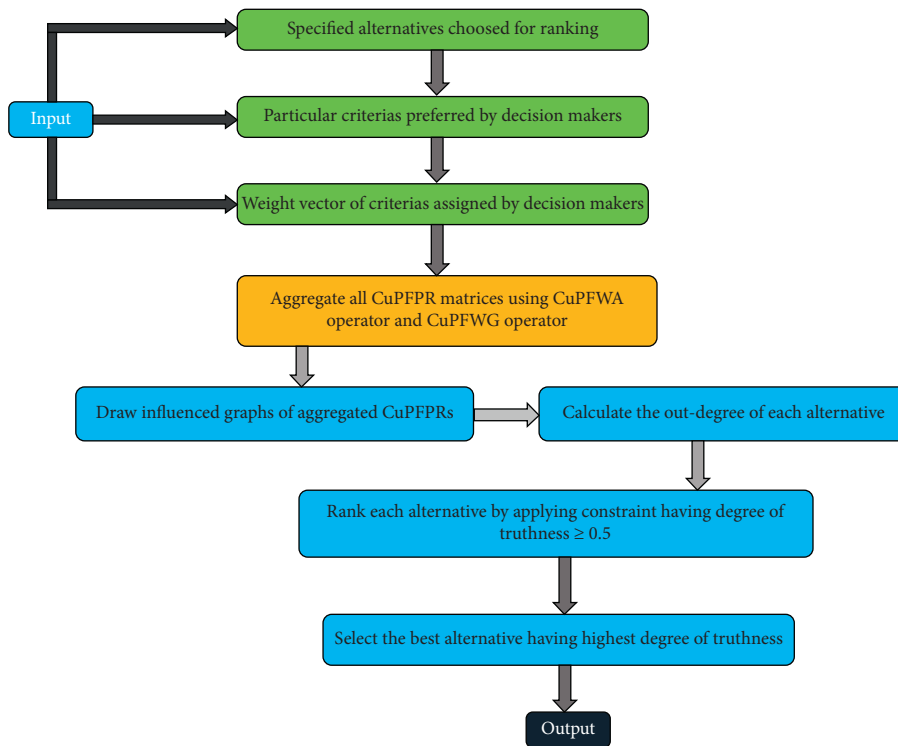


FIGURE 12: Flowchart of the purposed work.

out in this section. The Pythagorean fuzzy weighted average (PFWA) operator and the Pythagorean fuzzy weighted geometric (PFWG) operator introduced by Yager [6] are applied for solving the above decision-making problem. For comparison with previous techniques, we have to take information in Pythagorean fuzzy environment. The results

corresponding to these approaches are summarized in Table 6.

Hence, the final result for the ranking of the alternatives of the existing technique (PFWG operator and PFWA operator) is the same as the proposed method which indicates the rationality and effectiveness of the proposed technique.

TABLE 6: Comparative analysis with existing techniques.

Techniques	Out-degrees of the alternatives	Rankings of the alternatives
Existing PFWA [6] technique	(1.2598,1.2071) (2.0531,1.8709) (0.7602,0.5213) (1.1759,1.3917)	$\mathfrak{S}_2 > \mathfrak{S}_4 > \mathfrak{S}_1 > \mathfrak{S}_3$
Existing PFWG [6] technique	(1.3689,1.0932) (2.2221,1.7036),(0.8042,0.5030) (1.9008,1.8269)	$\mathfrak{S}_2 > \mathfrak{S}_4 > \mathfrak{S}_1 > \mathfrak{S}_3$
Our proposed technique using CuPFWA [10]	$\langle [1.0956, 1.3665], [1.1294, 1.3801], (1.2598, 1.2071) \rangle \langle [1.8400, 2.2780], [1.4037, 1.7545], (2.0531, 1.8709) \rangle$	
Our proposed technique using CuPFWA [10]	$\langle [0.6, 0.7286], [0.4733, 0.6446], (0.7602, 0.5213) \rangle \langle [1.1264, 1.4029], [0.9953, 1.2663], (1.1759, 1.3917) \rangle$	$\mathfrak{S}_2 > \mathfrak{S}_4 > \mathfrak{S}_1 > \mathfrak{S}_3$
Our proposed technique using CuPFWG [10]	$\langle [1.2828, 1.5558], [0.9448, 1.1943], (1.3689, 1.0932) \rangle \langle [1.9033, 2.3333], [1.3648, 1.6890], (2.2221, 1.7036) \rangle$	
Our proposed technique using CuPFWG [10]	$\langle [0.6, 0.7352], [0.4676, 0.6382], (0.8042, 0.5030) \rangle \langle [1.8239, 2.1913], [1.4790, 1.8470], (1.9008, 1.8269) \rangle$	$\mathfrak{S}_2 > \mathfrak{S}_4 > \mathfrak{S}_1 > \mathfrak{S}_3$

Obviously, the method presented in this paper is more accurate, flexible, and generalized. The existing method in Pythagorean fuzzy environment is a special case of the proposed method. The originality of this decision-making approach is that we have described cubic Pythagorean fuzzy preference relations among the alternatives and developed MADM model with the interrelated alternatives by operating the corresponding graphical structures with cubic Pythagorean fuzzy information.

**4. Conclusion**

Fuzzy graph theory has effectively operated to solve a variety of issues, where information is uncertain or vague, such as producing, social networking, man-made reasoning, image processing, and network security. Several types of graphs were introduced in the past few years to model the vagueness and uncertainties in information: cubic fuzzy graphs is one of them where the data are in the form of intervals as well as simple fuzzy numbers. This research paper proposes the novel concept of CuPFGs, where the flexibility of information is increased within the fuzzy environment. Furthermore, we present several terms related to CuPFGs and develop certain operations including semistrong product, lexicographical product, and symmetric difference of two CuPFGs. In addition, some essential and mandatory constraints are illustrated. Lastly, the term CuPFPR is defined, and to show the validity of our proposed work, an application in a cubic Pythagorean fuzzy environment is solved. The presented work is more suitable to handle different issues, but it has certain limitations. In coming years, one can further extend the concept of CuPFGs to cubic Fermatean fuzzy graphs.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare no conflicts of interest.

**References**

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
- [3] M. B. Gorzalczy, "A method of inference in approximate reasoning based on interval-valued fuzzy set," *Fuzzy Sets and Systems*, vol. 21, no. 1, pp. 1–17, 1987.
- [4] K. T. Atanassov and V Gargov, "Interval valued intuitionistic fuzzy sets," *Intuitionistic Fuzzy Sets*, vol. 35, pp. 139–177, 1999.
- [5] R. R. Yager, "Pythagorean fuzzy subsets," in *Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, pp. 57–61, Alberta, Canada, 2013.
- [6] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2013.
- [7] Y. B. Jun, C. S. Kim, and K. O. Yang, "Cubic sets," *Annals of fuzzy mathematics and informatics*, vol. 4, no. 1, pp. 83–98, 2011.
- [8] H. Garg and G. Kaur, "Cubic intuitionistic fuzzy sets and its fundamental properties," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 33, no. 6, 2019.
- [9] H. Garg and G. Kaur, "Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis," *Granular Computing*, vol. 5, no. 2, pp. 169–184, 2020.
- [10] S. Z. Abbas, M. S. Ali Khan, S. Abdullah, H. Sun, and F. Hussain, "Cubic Pythagorean fuzzy sets and their application to multi-attribute decision making with unknown weight information," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 1, pp. 1529–1544, 2019.
- [11] F. Khan, M. S. A. Khan, M. Shahzad, and S. Abdullah, "Pythagorean cubic fuzzy aggregation operators and their application to multi-criteria decision making problems," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 1, pp. 595–607, 2019.

- [12] S. Ashraf, S. Abdullah, and A. Qadir, "Novel concept of cubic picture fuzzy sets," *Journal of New Theory*, vol. 24, pp. 59–72, 2018.
- [13] F. Mehmood, T. Mahmood, and Q. Khan, "Cubic hesitant fuzzy sets and their applications to multi criteria decision making," *International Journal of Algebra and Statistics*, vol. 5, no. 1, pp. 19–51, 2016.
- [14] G. Muhiuddin and A. M. Al-roqi, "Cubic soft sets with applications in BCK/BCI-algebras," *Annals of Fuzzy Mathematics and Informatics*, vol. 8, no. 2, pp. 291–304, 2014.
- [15] A. Kauffman, "Introduction to la theorie des sous-ensembles flous, masson et cie," 1973.
- [16] A. Rosenfeld, "FUZZY graphs††the support of the office of computing activities, national science foundation, under grant GJ-32258X, is gratefully acknowledged, as is the help of Shelly Rowe in preparing this paper," in *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, pp. 77–95, Academic Press, Cambridge, MA, USA, 1975.
- [17] J. N. Mordeson and P. Chang-Shyh, *Operations on fuzzy graphs*, *Information sciences*, vol. 79, no. 3-4, pp. 159–170, 1994.
- [18] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987.
- [19] R. T. Yeh and S. Y. Bang, "Fuzzy relations, fuzzy graphs, and their applications to clustering analysis the research reported here is supported in part by the National Science Foundation under Grant No. GJ-31528," *Fuzzy sets and their applications to Cognitive and Decision Processes*, vol. 1, pp. 125–149, 1975.
- [20] S. Dogra, "Different types of product of fuzzy graphs," *Progress in Nonlinear Dynamics and Chaos*, vol. 3, no. 1, pp. 41–56, 2015.
- [21] T. Al-Hawary, "Complete fuzzy graphs," *International Journal of Mathematical Combinatorics*, vol. 4, p. 26, 2011.
- [22] R. Parvathi and M. G. Karunambigai, "Intuitionistic fuzzy graphs in computational intelligence, theory and applications," 2006.
- [23] H. Garg, "Novel neutrality operation-based Pythagorean fuzzy geometric aggregation operators for multiple attribute group decision analysis," *International Journal of Intelligent Systems*, vol. 34, no. 10, pp. 2459–2489, 2019.
- [24] M. Akram, F. Ilyas, and A. Borumand Saeid, "Certain notions of Pythagorean fuzzy graphs," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 6, pp. 5857–5874, 2019.
- [25] M. Akram, J. M. Dar, and S. Naz, "Certain graphs under Pythagorean fuzzy environment," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 127–144, 2019.
- [26] M. Akram, A. Habib, F. Ilyas, and J. Dar, "Specific types of Pythagorean fuzzy graphs and application to decision-making," *Mathematical and Computational Applications*, vol. 23, no. 3, p. 42, 2018.
- [27] P. Mandal and A. S. Ranadive, "Pythagorean fuzzy preference relations and their applications in group decision-making systems," *International Journal of Intelligent Systems*, vol. 34, no. 7, pp. 1700–1717, 2019.
- [28] L. Wang and H. Garg, "Algorithm for multiple attribute decision-making with interactive archimedean norm operations under pythagorean fuzzy uncertainty," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 503–527, 2021.
- [29] M. Akram and S. Naz, "Energy of Pythagorean fuzzy graphs with applications," *Mathematics*, vol. 6, no. 8, p. 136, 2018.
- [30] M. Akram, S. Naz, and B. Davvaz, "Simplified interval-valued Pythagorean fuzzy graphs with application," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 229–253, 2019.
- [31] M. Akram, N. O. Alshehri, and W. A. Dudek, "Certain types of interval-valued fuzzy graphs," *Journal of Applied Mathematics*, vol. 24, 2013.
- [32] S. Y. Mohamed and A. M. Ali, "Interval-valued Pythagorean fuzzy graph," *Journal of Computer and Mathematical Sciences*, vol. 9, no. 10, pp. 1497–1511, 2018.
- [33] S. Mohamed Ali and A. M. Ali, "Some products on interval-valued Pythagorean fuzzy graph," *Malaya Journal of Matematik*, vol. 7, no. 3, pp. 566–571, 2019.
- [34] S. Y. Mohamed and A. M. Ali, "Strong interval-valued pythagorean fuzzy graph," *Journal of Applied Science and Computations*, vol. 5, no. 10, pp. 669–713, 2018.
- [35] S. Rashid, N. Yaqoob, M. Akram, and M. Gulistan, "Cubic graphs with application," *International Journal of Analysis and Applications*, vol. 16, no. 5, pp. 733–750, 2018.
- [36] G. Muhiuddin, M. Mohseni Takallo, Y. B. Jun, and R. A. Borzooei, "Cubic graphs and their application to a traffic flow problem," *International Journal of Computational Intelligence Systems*, vol. 13, no. 1, pp. 1265–1280, 2020.
- [37] H. Rashmanlou, G. Muhiuddin, S. Amanathulla, F. Mofidnakhai, and M. Pal, "A study on cubic graphs with novel application," *Journal of Intelligent and Fuzzy Systems*, vol. 40, no. 1, pp. 89–101, 2021.
- [38] S. U. Khan, N. Jan, K. Ullah, and L. Abdullah, "Graphical structures of cubic intuitionistic fuzzy information," *Journal of Mathematics*, vol. 21, 2021.
- [39] D. Ajay and P. Chellamani, "Pythagorean neutrosophic fuzzy graphs," *International Journal of Neutrosophic Science*, vol. 11, no. 2, pp. 108–114, 2020.
- [40] M. Akram and W. A. Dudek, "Interval-valued fuzzy graphs," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 289–299, 2011.
- [41] A. N. Gani and S. S. Begum, "Degree, order and size in intuitionistic fuzzy graphs," *International Journal of Algorithms, Computing and Mathematics*, vol. 3, no. 3, pp. 11–16, 2010.
- [42] S. Naz, S. Ashraf, and M. Akram, "A novel approach to decision-making with Pythagorean fuzzy information," *Mathematics*, vol. 6, no. 6, p. 95, 2018.