Research Article

On the Graphs of Minimum Degree At Least 3 Having Minimum Sum-Connectivity Index

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For a graph G , its sum-connectivity index is denoted by $\chi(G)$ and is defined as the sum of the numbers $(d(u) + d(v))^{-1/2}$ over all edges uv of G , where $d(w)$ denotes the degree of a vertex $w \in V(G)$. In this study, we find a sharp lower bound on the sum-connectivity index of graphs having minimum degree of at least 3 under certain constraints and characterize the corresponding extremal graphs.

1. Introduction

In chemical graph theory, graph invariants are usually known as topological indices. Such indices play an important role in the study of quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR). Information about several good studied topological indices can be found in [1–7] and related references listed therein.

The well-known Randić-connectivity index (also known as the product-connectivity index), proposed by Randić [8] in 1975, is the most used topological index in QSPR and QSAR, which is defined as [8]

$$R(G) = \sum_{uv \in E(G)} (d(u)d(v))^{-1/2}. \quad (1)$$

The sum-connectivity index, a variant of the product-connectivity index, of a graph G is defined as [9]

$$\chi(G) = \sum_{uv \in E(G)} (d(u) + d(v))^{-1/2}. \quad (2)$$

In the previous studies [10–12], the sum-connectivity index of several graphs of molecules was calculated. In [13], relation between the sum-connectivity index and average distance was established. Extremal results concerning minimum sum-connectivity index and matching number were obtained in [14] for trees and connected unicyclic graphs, in [15] for connected bicyclic graphs, and in [16] for cacti. Results on trees with given matching number and maximum sum-connectivity index were found in [17]. Some mathematical aspects of χ index were studied in [9], particularly for the family of trees, which contains molecular

trees representing acyclic hydrocarbons. More detail about the χ index can be found in the recent survey [18] and recent articles [19, 20]. In the present study, we obtained the minimum general sum-connectivity index from the family of all n -vertex (where $n \geq 23$) graphs having minimum degree of at least 3 under certain constraints.

2. Main Results

Theorem 1. For $n \geq 23$, let G be an n -vertex graph with minimum degree of at least 3, such that G satisfies the following two conditions:

- (i) If G has a pair of adjacent vertices of degree 3, then this pair of adjacent vertices has at most one common neighbor
- (ii) If G contains no pair of adjacent vertices of degree 3, then G has a vertex of degree 3 whose all neighbors form a triangle

Then, it holds that

$$\chi(G) \geq \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}, \quad (3)$$

with the equality sign if and only if $G \cong K_3 + \overline{K}_{n-3}$.

3. Some Preliminaries Lemmas

In this section, some lemmas are given, which play a vital role in proving the main result.

Lemma 1 (See [21]). If $e \in E(G)$ is of maximal weight, then $\chi(G) > \chi(G - e)$.

Lemma 2. If $x, y, z, w \geq 3$, then the function f defined by

$$f(x, y, z, w) = \frac{1}{\sqrt{x+3}} + \frac{1}{\sqrt{y+3}} + \frac{1}{\sqrt{z+3}} + \frac{1}{\sqrt{w+3}} - \frac{1}{\sqrt{x+4}} - \frac{1}{\sqrt{y+4}} - \frac{1}{\sqrt{z+4}} - \frac{1}{\sqrt{w+4}}, \quad (4)$$

is strictly decreasing in all x, y, z , and w on the interval $[3, \infty)$.

Proof.

$$\frac{\partial f}{\partial x} = \frac{-1}{2\sqrt{(x+3)^3}} + \frac{1}{2\sqrt{(x+4)^3}} < 0, \quad (5)$$

shows that the function f is strictly decreasing in x . Because of the symmetry, we also conclude that f is strictly decreasing in y, z , and w . $\square$

Lemma 3. If n is an integer greater than 16, then the function f defined by

$$f(n) = \frac{3n-8}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} + \frac{1}{\sqrt{6}}, \quad (6)$$

is positive valued.

Proof. Clearly,

$$\frac{3n-5}{\sqrt{n+1}} - \frac{3n-5}{\sqrt{n+2}} > 0. \quad (7)$$

Also,

$$\frac{3}{\sqrt{2n-4}} - \frac{3}{\sqrt{2n-2}} > 0. \quad (8)$$

So,

$$\frac{-3}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} > 0, \quad (9)$$

for all $n > 53$.

From the above inequalities, we get that $f(n) > 0$ for all $n > 53$.

If $17 \leq n \leq 53$, then $f(n) > 0$ by using Mathematica. Hence, $f(n) > 0$ for all $n \geq 17$.

This completes the proof of the lemma. $\square$

Lemma 4. If n is an integer greater than 22, then the function f defined by

$$f(x) = \frac{1}{\sqrt{x+3}} - \frac{2}{\sqrt{x+4}} + \frac{x-1}{\sqrt{x+n-4}} - \frac{x-1}{\sqrt{x+n-3}}, \quad \text{where } x \geq 3, \quad (10)$$

is strictly increasing on the interval $[3, \infty)$.

Proof.

$$\frac{df}{dx} = \frac{-1}{2\sqrt{(x+3)^3}} + \frac{1}{\sqrt{(x+4)^3}} + \frac{x+2n-7}{2\sqrt{(x+n-4)^3}} - \frac{x+2n-5}{2\sqrt{(x+n-3)^3}}. \quad (11)$$

For $x \geq 3$, it holds that

$$\left(\frac{7}{6}\right)^{3/2} < (\sqrt{2})^2, \quad (12)$$

which together with inequality $1 + (1/x + 3) \leq 7/6$ imply that

$$\left(1 + \frac{1}{x+3}\right)^{3/2} \leq \left(\frac{7}{6}\right)^{3/2} < 2, \quad (13)$$

which further implies that

$$\frac{2}{(x+4)^{3/2}} - \frac{1}{(x+3)^{3/2}} > 0$$

$$\text{or } \frac{1}{(x+4)^{3/2}} - \frac{1}{2(x+3)^{3/2}} > 0. \quad (14)$$

Let

$$g(n) = \frac{x+2n-7}{2\sqrt{(x+n-4)^3}}, \quad (15)$$

then,

$$\frac{dg}{dn} = \frac{-2n+x+5}{4\sqrt{(x+n-4)^5}} < 0, \quad (16)$$

this shows that the function g is decreasing in n .

So,

$$\frac{x+2n-7}{2\sqrt{(x+n-4)^3}} - \frac{x+2n-5}{2\sqrt{(x+n-3)^3}} > 0. \quad (17)$$

Using these inequalities in the above equation, we get

$$\frac{df}{dx} > 0. \quad (18)$$

Hence, the function f is strictly increasing in x . $\square$

Lemma 5. Let

$$f(n) = \frac{3(n-4)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} + \frac{2}{\sqrt{n-1}}$$

$$- \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}}. \quad (19)$$

If n is an integer greater than 13, then $f(n) > 0$.

Proof. Given,

$$f(n) = \frac{3(n-4)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} + \frac{2}{\sqrt{n-1}}$$

$$- \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}}$$

$$> \frac{3(n-4)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} + \frac{2}{\sqrt{n-1}}$$

$$+ \frac{2}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{6}}. \quad (20)$$

As

$$\frac{3(n-3)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} > 0, \quad (21)$$

$$\frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} > 0,$$

also

$$\frac{2}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} > 0, \quad (22)$$

the expression is

$$-\frac{3}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} > 0, \quad (23)$$

for all $n > 53$.

From the above inequalities, we get that $f(n) > 0$ for all $n > 53$.

If $14 \leq n \leq 53$, then $f(n) > 0$ by using Mathematica. Hence, $f(n) > 0$ for all $n \geq 14$. $\square$

Lemma 6. If n is an integer greater than 22, then the function f defined by

$$f(x) = \frac{1}{\sqrt{x+3}} - \frac{2}{\sqrt{x+4}} + \frac{x-1}{\sqrt{x+n-3}}$$

$$- \frac{x-1}{\sqrt{x+n-2}}, \text{ where } x \geq 3, \quad (24)$$

is strictly increasing on the interval $[3, \infty)$.

Proof.

$$\frac{df}{dx} = \frac{-1}{2\sqrt{(x+3)^3}} + \frac{1}{\sqrt{(x+4)^3}} + \frac{x+2n-5}{2\sqrt{(x+n-3)^3}}$$

$$- \frac{x+2n-3}{2\sqrt{(x+n-2)^3}}. \quad (25)$$

For $x \geq 3$, it holds that

$$\left(\frac{7}{6}\right)^{3/2} < (\sqrt{2})^2, \quad (26)$$

which together with inequality $1 + (1/x + 3) \leq 7/6$ imply that

$$\left(1 + \frac{1}{x+3}\right)^{3/2} \leq \left(\frac{7}{6}\right)^{3/2} < 2, \quad (27)$$

which further implies that

$$\frac{2}{(x+4)^{3/2}} - \frac{1}{(x+3)^{3/2}} > 0$$

$$\text{or } \frac{1}{(x+4)^{3/2}} - \frac{1}{2(x+3)^{3/2}} > 0. \quad (28)$$

Let

$$g(n) = \frac{x+2n-5}{2\sqrt{(x+n-3)^3}}. \quad (29)$$

Then,

$$\frac{dg}{dn} = \frac{x-2n+3}{4\sqrt{(x+n-3)^5}} < 0. \quad (30)$$

So,

$$\frac{x+2n-5}{2\sqrt{(x+n-3)^3}} - \frac{x+2n-3}{2\sqrt{(x+n-2)^3}} > 0. \quad (31)$$

Using these inequalities in the above equation, we get

$$\frac{df}{dx} > 0. \quad (32)$$

Hence, the function f is strictly increasing in x . $\square$

Lemma 7. The function f defined is by

$$f(n) = \frac{3n-14}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} \\ + \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}}. \quad (33)$$

If n is an integer greater than 13, then $f(n)$ is positive valued.

Proof. Given,

$$f(n) = \frac{3n-14}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} \\ + \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}} \\ > \frac{3n-14}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} \\ + \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{6}}. \quad (34)$$

As

$$\frac{3(n-3)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} > 0, \\ \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} > 0, \quad (35)$$

also

$$\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} > 0, \quad (36)$$

so,

$$\frac{-3}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} > 0, \quad (37)$$

for all $n > 53$.

From the above inequalities, we get that $f(n) > 0$ for all $n > 53$.

If $14 \leq n \leq 53$, then $f(n) > 0$ by using Mathematica. Hence, $f(n) > 0$ for all $n \geq 14$. $\square$

Lemma 8. If $x \geq 3, y \geq 4$, then the function f defined by

$$f(x, y) = \frac{1}{\sqrt{x+y}} - \frac{1}{\sqrt{x+y-1}} + \frac{y-1}{\sqrt{y+3}} - \frac{y-2}{\sqrt{y+2}}, \quad (38)$$

is strictly increasing in x and strictly decreasing in y .

Proof.

$$\frac{\partial f}{\partial x} = \frac{-1}{2\sqrt{(x+y)^3}} + \frac{1}{2\sqrt{(x+y-1)^3}} > 0. \quad (39)$$

Now,

$$\frac{\partial f}{\partial y} = \frac{y+7}{2\sqrt{(y+3)^3}} - \frac{y+6}{2\sqrt{(y+2)^3}} + \frac{1}{\sqrt{(x+y-1)^3}} \\ - \frac{1}{2\sqrt{(x+y)^3}}, \quad (40)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-3}{4\sqrt{(x+y-1)^5}} + \frac{3}{4\sqrt{(x+y)^5}} < 0.$$

So, we have

$$\frac{\partial f}{\partial y} \leq \frac{\partial f(3, y)}{\partial y} = \frac{y+6}{2\sqrt{(y+3)^3}} - \frac{y+5}{2\sqrt{(y+2)^3}}. \quad (41)$$

Let

$$g(y) = \frac{y+6}{2\sqrt{(y+3)^3}}. \quad (42)$$

Then,

$$\frac{dg}{dy} = \frac{-y-12}{4\sqrt{(y+3)^5}} < 0, \quad (43)$$

which shows that g is strictly decreasing in y .

So,

$$\frac{y+6}{2\sqrt{(y+3)^3}} - \frac{y+5}{2\sqrt{(y+2)^3}} < 0. \quad (44)$$

By using this inequality in the above expression, we get

$$\frac{\partial f}{\partial y} < 0. \quad (45)$$

Hence, the function f is strictly decreasing in y . $\square$

Lemma 9. Let

$$f(n) = \frac{2(n-5)}{\sqrt{n+1}} - \frac{2(n-4)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} + \frac{1}{\sqrt{6}} \quad (46)$$

If n is an integer greater than 9, then $f(n) > 0$.

Proof. As

$$\begin{aligned} \frac{2(n-4)}{\sqrt{n+1}} - \frac{2(n-4)}{\sqrt{n+2}} &> 0, \\ \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} &> 0, \end{aligned} \quad (47)$$

so,

$$\frac{-2}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} > 0, \quad (48)$$

for all $n > 23$.

From the above inequalities, we get that $f(n) > 0$ for all $n > 23$.

If $10 \leq n \leq 23$, then $f(n) > 0$ by using Mathematica.

Hence, $f(n) > 0$ for all $n \geq 10$. $\square$

Lemma 10. If $x \geq 4$, then the function f defined by

$$f(x) = \frac{x}{\sqrt{x+3}} - \frac{x-1}{\sqrt{x+2}}, \quad (49)$$

is strictly decreasing on the interval $[4, \infty)$.

Proof.

$$\frac{df}{dx} = \frac{x+6}{2\sqrt{(x+3)^3}} - \frac{x+5}{2\sqrt{(x+2)^3}}. \quad (50)$$

Let

$$g(x) = \frac{x+6}{2\sqrt{(x+3)^3}}. \quad (51)$$

Then,

$$g'(x) = \frac{-x-12}{4\sqrt{(x+3)^5}} < 0. \quad (52)$$

So,

$$\frac{x+6}{2\sqrt{(x+3)^3}} - \frac{x+5}{2\sqrt{(x+2)^3}} < 0. \quad (53)$$

Using this in the above equation, we get

$$\frac{df}{dx} < 0. \quad (54)$$

Hence, the function f is strictly decreasing in x . $\square$

Lemma 11. If n is an integer greater than 9, then the function f defined by

$$\begin{aligned} f(n) = \frac{4n-14}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} \\ - \frac{n-3}{\sqrt{n}} + \frac{1}{\sqrt{6}} \end{aligned} \quad (55)$$

is positive valued.

Proof. As

$$\begin{aligned} \frac{3(n-3)}{\sqrt{n+1}} - \frac{3(n-3)}{\sqrt{n+2}} &> 0, \\ \frac{3}{\sqrt{2(n-2)}} - \frac{3}{\sqrt{2(n-1)}} &> 0, \end{aligned} \quad (56)$$

the expression is

$$\begin{aligned} \frac{n-2}{\sqrt{n+1}} - \frac{n-3}{\sqrt{n}} \\ = \sqrt{n+1} - \sqrt{n} + \frac{3}{\sqrt{n}} - \frac{3}{\sqrt{n+1}} > 0. \end{aligned} \quad (57)$$

So,

$$\frac{-3}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} > 0, \quad (58)$$

for all $n > 53$.

From the above inequalities, we get that $f(n) > 0$ for all $n > 53$.

If $10 \leq n \leq 53$, then $f(n) > 0$ by using Mathematica.

Hence, $f(n) > 0$ for all $n \geq 10$. $\square$

Lemma 12. If $x, y, z \geq 4$, then the function f defined by

$$\begin{aligned} f(x, y, z) = \frac{x-2}{\sqrt{x+3}} - \frac{x-3}{\sqrt{x+2}} + \frac{1}{\sqrt{x+y}} - \frac{1}{\sqrt{x+y-2}} \\ + \frac{y-2}{\sqrt{y+3}} - \frac{y-3}{\sqrt{y+2}} + \frac{1}{\sqrt{y+z}} - \frac{1}{\sqrt{y+z-2}} \\ + \frac{z-2}{\sqrt{z+3}} - \frac{z-3}{\sqrt{z+2}} + \frac{1}{\sqrt{x+z}} - \frac{1}{\sqrt{x+z-2}}, \end{aligned} \quad (59)$$

is decreasing in all x, y and z .

Proof.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{x+8}{2\sqrt{(x+3)^3}} - \frac{x+7}{2\sqrt{(x+2)^3}} - \frac{1}{2\sqrt{(x+y)^3}} - \frac{1}{2\sqrt{(x+z)^3}} \\ &\quad + \frac{1}{2\sqrt{(x+y-2)^3}} + \frac{1}{2\sqrt{(x+z-2)^3}}, \end{aligned} \quad (60)$$

which implies

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{3}{4\sqrt{(x+y)^5}} - \frac{3}{4\sqrt{(x+y-2)^5}} < 0, \\ \frac{\partial^2 f}{\partial x \partial z} &= \frac{3}{4\sqrt{(x+z)^5}} - \frac{3}{4\sqrt{(x+z-2)^5}} < 0. \end{aligned} \quad (61)$$

Since $\partial^2 f / \partial x \partial y < 0$ and $\partial^2 f / \partial x \partial z < 0$, it holds that

$$\begin{aligned} \frac{\partial f}{\partial x} &\leq \frac{\partial f(x, 4, 4)}{\partial x} \\ &= \frac{x+8}{2\sqrt{(x+3)^5}} - \frac{x+5}{2\sqrt{(x+2)^5}} - \frac{1}{\sqrt{(x+4)^3}}. \end{aligned} \quad (62)$$

Let

$$F(x) = \frac{x+8}{2\sqrt{(x+3)^3}} - \frac{x+5}{2\sqrt{(x+2)^3}} - \frac{1}{\sqrt{(x+4)^3}}, \quad (63)$$

where $x \geq 4$. We note that

$$F(x) = h(x+1) - h(x) + g(x) - g(x+1), \quad (64)$$

where

$$\begin{aligned} h(x) &= \frac{x+5}{2\sqrt{(x+2)^3}}, \\ g(x) &= \frac{1}{\sqrt{(x+3)^3}}. \end{aligned} \quad (65)$$

The functions h and g are continuous as well as differentiable on the closed interval $[x_o, x_o + 1]$ for every fixed number $x_o \geq 4$.

By virtue of Cauchy's mean value theorem, we observe that for every x_o , there exists a number $c_{x_o} \in (x_o, x_o + 1)$, such that

$$\begin{aligned} \frac{h(x_o + 1) - h(x_o)}{g(x_o + 1) - g(x_o)} &= \frac{h'(c_{x_o})}{g'(c_{x_o})} \\ &= \left(\frac{c_{x_o} + 3}{c_{x_o} + 2} \right)^{5/2} \left(\frac{c_{x_o} + 11}{6} \right), \end{aligned} \quad (66)$$

which is greater than one, and hence,

$$F(x_o) = h(x_o + 1) - h(x_o) + g(x_o) - g(x_o + 1) < 0, \quad (67)$$

because the function g is decreasing.

Therefore, for all $x \geq 4$, the function $F(x)$ is negative valued. Using $F(x) < 0$ in the above equation, we get

$$\frac{\partial f}{\partial x} < 0. \quad (68)$$

This shows that the function f is decreasing in x .

By symmetry, it follows that

$$\frac{\partial f}{\partial y} < 0, \quad (69)$$

$$\frac{\partial f}{\partial z} < 0,$$

which means that the function f is decreasing in y and z . Hence, the function f is decreasing in x, y , and z . $\square$

4. Proof of Theorem 1

Proof. We will prove the result by induction on n . For $n = 23$, by using the AutoGraphiX system [22], we find that $K_3 + \overline{K}_{n-3}$ has the maximum χ value among all n -vertex graphs having a minimum degree of at least 3, which implies that the result is true for $n = 23$. Now, we suppose that the theorem holds for all those k -vertex graphs which satisfy all the constraints of the theorem, where $23 \leq k \leq n - 1$.

If minimum degree of G is at least 4, then we take $v_1 v_2 \in E(G)$, such that $d(v_1) + d(v_2) \leq d(u) + d(v)$ for all $uv \in E(G)$. Clearly, the graph $G - v_1 v_2$ has minimum degree of at least 3, and from Lemma 1, it follows that $\chi(G) > \chi(G - v_1 v_2)$. Hence, it is sufficient to assume that the minimum degree of G is 3. Next, we consider all possible cases.

Case 1:

G contains at least one pair of adjacent vertices having degree 3 without common neighbors

Let $u_1, u_2 \in V(G)$ be two adjacent vertices having degree 3 without any common neighbor. Let v_1 and v_2 be the neighbors of u_1 different from u_2 , and w_1, w_2 be the

neighbors of u_2 different from u_1 . In this case, the vertex degrees $d(v_1), d(v_2), d(w_1)$, and $d(w_2)$ satisfy the inequality $3 \leq d(v_1), d(v_2), d(w_1), d(w_2) \leq n-2$. By

setting $G_1 \cong G - \{u_1\} + \{u_2v_1, u_2v_2\}$ and then by using Lemma 2, the inductive hypothesis, and Lemma 3, we have

$$\begin{aligned}
 \chi(G) &= \chi(G_1) + \frac{1}{\sqrt{d(u_1) + d(u_2)}} + \frac{1}{\sqrt{d(u_1) + d(v_1)}} + \frac{1}{\sqrt{d(u_1) + d(v_2)}} \\
 &\quad + \frac{1}{\sqrt{d(u_2) + d(w_1)}} - \frac{1}{\sqrt{d(u_2) + 1 + d(w_1)}} + \frac{1}{\sqrt{d(u_2) + d(w_2)}} \\
 &\quad - \frac{1}{\sqrt{d(u_2) + 1 + d(w_2)}} - \frac{1}{\sqrt{d(u_2) + 1 + d(v_1)}} - \frac{1}{\sqrt{d(u_2) + 1 + d(v_2)}} \\
 &= \chi(G_1) + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3 + d(v_1)}} + \frac{1}{\sqrt{3 + d(v_2)}} \\
 &\quad + \frac{1}{\sqrt{3 + d(w_1)}} - \frac{1}{\sqrt{4 + d(w_1)}} + \frac{1}{\sqrt{3 + d(w_2)}} \\
 &\quad - \frac{1}{\sqrt{4 + d(w_2)}} - \frac{1}{\sqrt{4 + d(v_1)}} - \frac{1}{\sqrt{4 + d(v_2)}} \\
 &\geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{4}{\sqrt{n+1}} - \frac{4}{\sqrt{n+2}} + \frac{1}{\sqrt{6}} \\
 &> \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
 \end{aligned} \tag{70}$$

Case 2:

G contains at least one pair of adjacent vertices of degree 3 with a common neighbor

Let $u_1, u_2 \in V(G)$ be two adjacent vertices of degree 3 having a common neighbor v . Let v_1 be the neighbor of u_1 different from u_2 , and v and w_1 be the neighbors of u_2 different from u_1, v .

Subcase i:

Let $d(v) = 3$, and there is an edge between v_1 and v . Clearly, it holds that $3 \leq d(v_1) \leq n-2$ and $3 \leq d(w_1) \leq n-3$. If we take $G_2 \cong G - \{u_1\} + \{u_2v_1, w_1v\}$, then by using Lemma 4, the inductive hypothesis, and Lemma 5, we get

$$\begin{aligned}
 \chi(G) &= \chi(G_2) + \frac{1}{\sqrt{d(u_1) + d(u_2)}} + \frac{1}{\sqrt{d(u_1) + d(v)}} + \frac{1}{\sqrt{d(u_1) + d(v_1)}} \\
 &\quad - \frac{1}{\sqrt{d(u_2) + d(v_1)}} - \frac{1}{\sqrt{d(v) + d(w_1) + 1}} \\
 &\quad + \frac{1}{\sqrt{d(u_2) + d(w_1)}} - \frac{1}{\sqrt{d(u_2) + d(w_1) + 1}}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{z \in N(w_1) \setminus \{u_2\}} \left(\frac{1}{\sqrt{d(z) + d(w_1)}} - \frac{1}{\sqrt{d(z) + d(w_1) + 1}} \right) \\
& \geq \chi(G_2) + \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{3 + d(w_1) + 1}} + \frac{1}{\sqrt{3 + d(w_1)}} \\
& \quad - \frac{1}{\sqrt{3 + d(w_1) + 1}} + \frac{d(w_1) - 1}{\sqrt{n - 4 + d(w_1)}} - \frac{d(w_1) - 1}{\sqrt{n - 4 + d(w_1) + 1}} \\
& \geq \chi(G_2) + \frac{2}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}} \\
& \geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{2}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}} \\
& > \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
\end{aligned} \tag{71}$$

Subcase ii:

Let $d(v) = 3$, and there is no edge between v_1 and v . Clearly, it holds that $3 \leq d(v_1) \leq n-3$. If $G_3 \cong G-$

$\{u_1\} + \{u_2v_1, v_1v\}$, then by using Lemma 6, the induction hypothesis, and Lemma 7, we obtained

$$\begin{aligned}
\chi(G) &= \chi(G_3) + \frac{1}{\sqrt{d(u_1) + d(u_2)}} + \frac{1}{\sqrt{d(u_1) + d(v)}} + \frac{1}{\sqrt{d(u_1) + d(v_1)}} \\
& \quad - \frac{1}{\sqrt{d(u_2) + d(v_1) + 1}} - \frac{1}{\sqrt{d(v) + d(v_1) + 1}} \\
& \quad + \sum_{z \in N(v_1) \setminus \{u_1\}} \left(\frac{1}{\sqrt{d(z) + d(v_1)}} - \frac{1}{\sqrt{d(z) + d(v_1) + 1}} \right) \\
& \geq \chi(G_3) + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{3 + d(v_1)}} - \frac{1}{\sqrt{3 + d(v_1) + 1}} \\
& \quad - \frac{1}{\sqrt{3 + d(v_1) + 1}} + \frac{d(v_1) - 1}{\sqrt{n - 3 + d(v_1)}} - \frac{d(v_1) - 1}{\sqrt{n - 3 + d(v_1) + 1}} \\
& \geq \chi(G_3) + \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}} \\
& \geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} + \frac{3}{\sqrt{6}} - \frac{2}{\sqrt{7}} \\
& > \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
\end{aligned} \tag{72}$$

Subcase iii:

Let $d(v) \geq 4$, and there is an edge between v_1 and v . Clearly, it holds that $3 \leq d(v_1) \leq n-2$. If we take

$G_4 \cong G - u_1 + u_2 v_1$, then by using Lemma 8, the inductive hypothesis, and Lemma 9, we have

$$\begin{aligned}
 \chi(G) &= \chi(G_4) + \frac{1}{\sqrt{d(u_1) + d(u_2)}} + \frac{1}{\sqrt{d(u_1) + d(v)}} + \frac{1}{\sqrt{d(u_1) + d(v_1)}} - \frac{1}{\sqrt{d(u_2) + d(v_1)}} \\
 &\quad + \frac{1}{\sqrt{d(u_2) + d(v)}} - \frac{1}{\sqrt{d(u_2) + d(v) - 1}} + \frac{1}{\sqrt{d(v_1) + d(v)}} - \frac{1}{\sqrt{d(v_1) + d(v) - 1}} \\
 &\quad + \sum_{z \in N(v) \setminus \{u_1, v_1, v_2\}} \left(\frac{1}{\sqrt{d(v) + d(z)}} - \frac{1}{\sqrt{d(v) - 1 + d(z)}} \right) \\
 &\geq \chi(G_4) + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{3 + d(v)}} - \frac{1}{\sqrt{3 + d(v) - 1}} + \frac{1}{\sqrt{d(v_1) + d(v)}} - \frac{1}{\sqrt{d(v_1) + d(v) - 1}} \\
 &\quad + \frac{d(v) - 3}{\sqrt{d(v) + d(z)}} + \frac{d(v) - 3}{\sqrt{d(v) - 1 + d(z)}} \\
 &\geq \chi(G_4) + \frac{n-1}{\sqrt{n+2}} - \frac{n-2}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} \\
 &\geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{n-1}{\sqrt{n+2}} - \frac{n-2}{\sqrt{n+1}} + \frac{1}{\sqrt{6}} \\
 &> \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
 \end{aligned} \tag{73}$$

Subcase iv:

Let $d(v) \geq 4$, and there is no edge between v_1 and v . Clearly, it holds that $3 \leq d(v_1) \leq n-3$. If we take

$G_5 \cong G - u_1 + u_2 v_1$, then by using Lemma 10, the inductive hypothesis, and Lemma 11, we get

$$\begin{aligned}
 \chi(G) &= \chi(G_5) + \frac{1}{\sqrt{d(u_1) + d(u_2)}} + \frac{1}{\sqrt{d(u_1) + d(v_1)}} + \frac{1}{\sqrt{d(u_1) + d(v)}} - \frac{1}{\sqrt{d(u_2) + d(v_1)}} + \frac{1}{\sqrt{d(u_2) + d(v)}} - \frac{1}{\sqrt{d(u_2) + d(v) - 1}} \\
 &\quad + \sum_{z \in N(v) \setminus \{u_1, u_2\}} \left(\frac{1}{\sqrt{d(v) + d(z)}} - \frac{1}{\sqrt{d(v) - 1 + d(z)}} \right) \\
 &\geq \chi(G_5) + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{3 + d(v)}} - \frac{1}{\sqrt{3 + d(v) - 1}} + \frac{d(v) - 2}{\sqrt{d(v) + d(z)}} - \frac{d(v) - 2}{\sqrt{d(v) - 1 + d(z)}} \\
 &\geq \chi(G_5) + \frac{n-2}{\sqrt{n+1}} - \frac{n-3}{\sqrt{n}} + \frac{1}{\sqrt{6}} \geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{n-2}{\sqrt{n+1}} - \frac{n-3}{\sqrt{n}} + \frac{1}{\sqrt{6}} \\
 &> \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
 \end{aligned} \tag{74}$$

G does not contain any pair of adjacent vertices of degree 3

Let $u \in V(G)$ be a vertex of degree 3 having neighbors u_1, u_2 , and u_3 , such that $u_1u_2, u_1u_3, u_2u_3 \in E(G)$; then clearly, $d(u_1), d(u_2), d(u_3) \geq 4$. If we take $G_6 \cong G - \{u\}$,

then by induction hypothesis and by using Lemma 12, we have

$$\begin{aligned}
\chi(G) &= \chi(G_6) + \frac{1}{\sqrt{d(u) + d(u_1)}} + \frac{1}{\sqrt{d(u) + d(u_2)}} + \frac{1}{\sqrt{d(u) + d(u_3)}} \\
&\quad + \frac{1}{\sqrt{d(u_1) + d(u_2)}} - \frac{1}{\sqrt{d(u_1) + d(u_2) - 2}} + \frac{1}{\sqrt{d(u_1) + d(u_3)}} \\
&\quad - \frac{1}{\sqrt{d(u_1) + d(u_3) - 2}} + \frac{1}{\sqrt{d(u_2) + d(u_3)}} - \frac{1}{\sqrt{d(u_2) + d(u_3) - 2}} \\
&\quad + \sum_{z \in N(u_1) \setminus \{u, u_2, u_3\}} \left(\frac{1}{\sqrt{d(u_1) + d(z)}} - \frac{1}{\sqrt{d(u_1) - 1 + d(z)}} \right) \\
&\quad + \sum_{z \in N(u_2) \setminus \{u, u_1, u_3\}} \left(\frac{1}{\sqrt{d(u_2) + d(z)}} - \frac{1}{\sqrt{d(u_2) - 1 + d(z)}} \right) \\
&\quad + \sum_{z \in N(u_3) \setminus \{u, u_1, u_2\}} \left(\frac{1}{\sqrt{d(u_3) + d(z)}} - \frac{1}{\sqrt{d(u_3) - 1 + d(z)}} \right) \\
&\geq \chi(G_6) + \frac{1}{\sqrt{3 + d(u_1)}} + \frac{1}{\sqrt{3 + d(u_2)}} + \frac{1}{\sqrt{3 + d(u_3)}} \\
&\quad + \frac{1}{\sqrt{d(u_1) + d(u_2)}} - \frac{1}{\sqrt{d(u_1) + d(u_2) - 2}} + \frac{1}{\sqrt{d(u_1) + d(u_3)}} \\
&\quad - \frac{1}{\sqrt{d(u_1) + d(u_3) - 2}} + \frac{1}{\sqrt{d(u_2) + d(u_3)}} - \frac{1}{\sqrt{d(u_2) + d(u_3) - 2}} \\
&\quad + \frac{d(u_1) - 3}{\sqrt{d(u_1) + 3}} - \frac{d(u_1) - 3}{\sqrt{d(u_1) + 3 - 1}} + \frac{d(u_2) - 3}{\sqrt{d(u_2) + 3}} \\
&\quad - \frac{d(u_2) - 3}{\sqrt{d(u_2) + 3 - 1}} + \frac{d(u_3) - 3}{\sqrt{d(u_3) + 3}} - \frac{d(u_3) - 3}{\sqrt{d(u_3) + 3 - 1}} \\
&\geq \chi(G_6) + \frac{3(n-3)}{\sqrt{n+2}} - \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-1)}} - \frac{3}{\sqrt{2(n-2)}} \\
&\geq \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-2)}} + \frac{3(n-3)}{\sqrt{n+2}} - \frac{3(n-4)}{\sqrt{n+1}} + \frac{3}{\sqrt{2(n-1)}} - \frac{3}{\sqrt{2(n-2)}} \\
&= \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}.
\end{aligned} \tag{75}$$

We observe that the equation

$$\chi(G) = \frac{3(n-3)}{\sqrt{n+2}} + \frac{3}{\sqrt{2(n-1)}}, \tag{76}$$

holds if and only if

$$d(u_1) = d(u_2) = d(u_3) = n - 1. \quad (77)$$

This completes the proof of Theorem 1. $\square$

5. Concluding Remarks

We have characterized the minimum sum-connectivity index from the family of all n -vertex graphs having minimum degree at least 3, under certain constraints. More precisely, we have been able to show that $K_3 + \overline{K}_{n-3}$ is the only graph having minimal sum-connectivity index in the family of all those n -vertex (where $n \geq 23$) graphs which have a minimum degree of at least 3 and satisfy the following two conditions:

- (i) If G has a pair of adjacent vertices of degree 3, then this pair of adjacent vertices has at most one common neighbor
- (ii) If G contains no pair of adjacent vertices of degree 3, then G has a vertex of degree 3 whose all neighbors form a triangle.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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