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Research Article

A New Class of the Power Function Distribution: Theory and Inference with an Application to Engineering Data

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In this study, a new class that generates optimal univariate models called a new exponentiated-G class of distributions is developed. Numerous complementary statistical properties are derived and discussed in detail for the newly exponentiated power function (EPF) distribution. All possible shapes of the probability density and hazard rate functions are sketched for selected values of parameters. Six accredited estimation methods are discussed, and their performance is assessed and compared by a simulation study. The applicability of the new class is evaluated by analyzing the automotive engineering sector data.

1. Introduction

Modeling complicated problems is an enigma for applied researchers and practitioners. They seem to be worried about dealing with a variety of lifetime datasets that particularly follows physical and natural sciences. For this, they are searching for simple and efficient models. Consequently, a power function (PF) distribution is explored. It is a simple lifetime model as exponential and Pareto distributions. The PF distribution is a special case of the beta distribution. For more details about the classical work of the PF distribution, see Dallas [1] who developed a relationship between the Pareto and PF variables through an inverse transformation. Furthermore, for a deep understanding of the characterization of the PF distribution, see some credible work of Meniconi and Barry [2], Saran and Pandey [3], Chang [4], Tavangar [5], and Ahsanullah et al. [6].

In the most recent times, attention towards the generalization of probability distributions has grown phenomenally high. For more insight, see the trustworthy work of Cordeiro and Brito [7], Zaka and Akhter [8], Al Mutairi et al. [9], Tahir et al. [10], Shahzad et al. [11], Ahsan-ul-Haq et al. [12], Okorie et al. [13], Abdul-Moniem [14], Hassan et al. [15], Zaka et al. [16], Arshad et al. [17], Arshad et al. [18, 19], Al-Mutairi [20], Alzaatreh et al. [21], Gleaton and Lynch [22], Bourguignon et al. [23], Afify et al. [24], Tahir et al. [25], Aldahlan et al. [26], Aslam et al. [27], Balogun et al. [28], Afify et al. [29], Mansour et al. [30], Mahdavi and Kundu [31], Nassar et al. [32], Ijaz et al. [33], Klakattawi and Aljuhani [34], Afify et al. [35], Alsubie et al. [36], Ahmad et al. [37], and Nofal et al. [38].

To the best of our knowledge, the new class has not been discussed before, and it is the first time to explore the scenario particularly observed in the automobile sector. We develop a new class of distributions called the new exponentiated-G (NE-G) family and study one of its special submodels using the PF distributions as a baseline model. The studied model is called the exponentiated power function (EPF) distribution. The present study has some motivations as follows: (a) to develop new optimal models; (b) to advance the characteristics of the baseline models; (c) the density and hazard rate functions possess unimodal and bathtub-shaped curves, respectively; (d) to model the real-time scenario in the automobile sector.

This paper is outlined in the following sections. The development of the new class is presented in Section 2. A comprehensive discussion on mathematical and reliability measures is completed in Sections 3 and 4, respectively. Miscellaneous measures are discussed in Section 5. Six accredited estimation methods are discussed in Section 6. Simulation results are presented in Section 7. A lifetime application of the EPF distribution is discussed in Section 8, and finally, some conclusions are reported in Section 9.

2. The New Exponentiated-G Class

Tahir and Cordeiro [39] (Remarks 2 (ii)) developed the exponentiated generalized negative binomial G class which is defined by the CDF:

$$F(x;\beth)=\frac{1-\left(1+\eta\gamma G(x;\beth)^{\beta}\right)^{-(1/\alpha)}}{1-(1+\eta\gamma)^{-(1/\alpha)}}. \tag{1}$$

Chesneau et al. [40] reparameterized the parameters of (1) and provided the following CDF:

$$F(x;\beth)=\frac{(1+\eta G(x;\beth))^{1+(1/\eta)}-1}{(1+\eta)^{1+(1/\eta)}-1}. \tag{2}$$

In this section, we provide a new generator of distributions that is very simple and capable of generating optimal alternative models. The new generator is called the new exponentiated-G class, and it is specified by the CDF:

$$F(x;\beth)=\frac{(1+G(x;\beth))^{\alpha}-1}{2^{\alpha}-1},\quad x\in\mathfrak{R}, \tag{3}$$

where $G(x;\beth)\epsilon\,(0,1)$ is a CDF of any baseline (parent) distribution. It is based on a parametric vector $\beth>0$ depending on $(r\times 1)$, $\alpha>0$ is a shape parameter, and $(1/2^{\alpha}-1)$ is a normalizing constant. $F(0;\beth)=0$, and $F(\infty;\beth)=1$. It is noted that the new class reduces to the baseline model with $\alpha=1$.

The probability density function (PDF) ($f(x)$) and the hazard rate function (HRF) ($h(x)$) of the new class reduce to

$$f(x;\beth)=\frac{\alpha g(x;\beth)(1+G(x;\beth))^{\alpha-1}}{2^{\alpha}-1},\quad x\in\mathfrak{R}, \tag{4}$$

$$h(x;\beth)=\frac{\alpha g(x;\beth)(1+G(x;\beth))^{\alpha-1}}{2^{\alpha}-(1+G(x;\beth))^{\alpha-1}},\quad x\in\mathfrak{R}, \tag{5}$$

where $g(x;\beth)=dG(x;\beth)/dx$.

The quantile function (QF) ($Q(x)$) of the new class takes the form.

$$Q(x)=G^{-1}\left((1+(2^{\alpha}-1)p)^{\frac{1}{\alpha}}-1\right),\quad p\in(0,1). \tag{6}$$

2.1. EPF Distribution. In this section, we discuss some useful characteristics of the EPF distribution. The PF distribution is specified by the following CDF and PDF:

$$G(x)=\left(\frac{x}{g_0}\right)^{\beta},\quad \beta>0,0<x\leq g_0, \tag{7}$$

and

$$g(x)=\frac{\beta}{(g_0)^{\beta}}x^{\beta-1},\quad \beta>0,\ 0<x\leq g_0. \tag{8}$$

To this end, we define the analytical expressions of CDF and PDF of the new EPF distribution with two shape parameters α and β. The CDF of the EPF distribution has the form (for $0<x\leq g_0$)

$$F(x)=\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1},\quad \alpha,\beta>0. \tag{9}$$

Its PDF reduces to

$$f(x)=\frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)}\left(\frac{1}{x^{1-\beta}}\right)\left(1+\left(\frac{x}{g_0}\right)^{\beta}\right)^{-(1-\alpha)},\quad \alpha,\beta>0, \tag{10}$$

where $(1/2^{\alpha}-1)$ is the normalizing constant and g_0 is a possible maximum assured life of a component.

The EPF distribution brings an additional shape parameter ($\alpha>0$) that modulates the skewness and kurtosis tail weights of the baseline model. We note that a new parameter may offer a better fit to the unimodal, increasing, U-shaped, and bathtub-shaped failure rate data. The EPF distribution reduces to the baseline model (power function) for $\alpha=1$.

2.2. Asymptotic Properties of PDF and CDF. Asymptotes of the CDF and PDF at $x\longrightarrow 0$ are given by

$$F(x)|_{x\longrightarrow 0}\sim 0 \text{ and } f(x)|_{x\longrightarrow 0}\sim 0. \tag{11}$$

Asymptotes of the CDF and PDF at $x\longrightarrow g$ are given by

$$F(x)|_{x\longrightarrow g_0}\sim 1 \text{ and } f(x)|_{x\longrightarrow g_0}\sim\frac{\alpha\beta(2)^{\alpha-1}}{g_0(2^{\alpha}-1)}. \tag{12}$$

The derived expressions explore a dynamic effect of α on the asymptotes of $F(x)$ and $f(x)$.

2.3. Shapes of PDF. Here, we discuss different shapes of the PDF of the EPF distribution. Figure 1 presents some different curves of the PDF for various choices of EPF parameters. We note that these curves can be decreasing, decreasing-increasing, and can be upside-down bathtub for $g_0=2$.

3. Moments and Related Measures

Theorem 1. *Let X follow the EPF distribution with two shape parameters* $(\alpha,\beta>0)$*; then, the r-th ordinary moment* $(\mu_r^{'})$ *of X has the form*

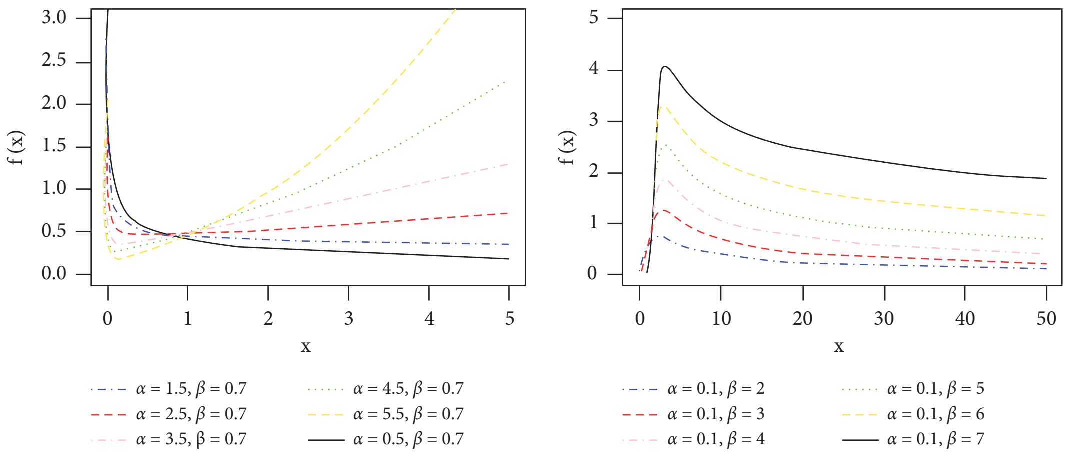

Figure 1: Some curves of PDF for EPF distribution.

$$\mu_r' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \tag{13}$$

Proof. The following expression follows using (6) as

$$\mu_r' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \int_0^{g_0} x^r x^{\beta-1} \left(1 + \left(\frac{x}{g_0}\right)^\beta\right)^{\alpha-1} dx\,. \tag{14}$$

The prior expression can be rewritten by simplifying the expression $(1 + (x/g_0)^\beta)^{\alpha-1}$ as follows:

$$\mu_r' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} \frac{1}{(g_0)^{\beta(j+1)}} \int_0^{g_0} x^{r+\beta(j+1)-1} dx\,. \tag{15}$$

After some algebra, the r-th ordinary moment of X reduces to

$$\mu_r' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}, \tag{16}$$

where $\psi_j = \binom{\alpha-1}{j}(1/(g_0)^{\beta(j+1)}), \Lambda_{r,j,\beta} = r + \beta(j+1)$.

Table 1 presents some numerical results for the first-four ordinary moments $(\mu_1', \mu_2', \mu_3', \mu_4')$, variance $= \sigma^2$, skewness $= \gamma_1$, and kurtosis $= \gamma_2$ for different values of the EPF parameters.

Table 1 shows flexible and versatile behavior for moments, variance and alongside the skewness and kurtosis. The results indicate that the EPF distribution can be discussed for leptokurtic and skewed datasets. □

Corollary 1. *The first and second ordinary moments and the inverse moment* (μ_{-w}') *can be obtained by substituting* $r = 1, 2,$ *and* $-w$, *in (7), respectively. The analytical expressions of mean, variance, and inverse moment are given, respectively, by*

$$E(X) = \mu_1' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{1,j,\beta}}}{\Lambda_{1,j,\beta}},$$

$$Var(X) = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \left(\sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{2,j,\beta}}}{\Lambda_{2,j,\beta}} - \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \left(\sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{1,j,\beta}}}{\Lambda_{1,j,\beta}}\right)\right), \tag{17}$$

and

$$\mu_{-w}' = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{-w,j,\beta}}}{\Lambda_{-w,j,\beta}}. \tag{18}$$

Corollary 2. *The factorial generating function is obtained directly followed by* $F_x(s) = E(1+s)^x = E(e^{x\ln(1+s)}) = \sum_{r=0}^{\infty}((\ln(1+s))^r/r!)\mu_r'$, *and it can be written for X as follows:*

$$F_x(s) = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{r=0}^{\infty} \frac{(\ln(1+s))^r}{r!} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \tag{19}$$

Theorem 2. *If* $X \sim EPF(\alpha, \beta)$, *then the moment generating function (MGF)* $(M_X(s))$ *of X is given by*

$$M_X(s) = \frac{\alpha\beta}{(g_0)^\beta (2^\alpha - 1)} \sum_{r=0}^{\infty} \frac{s^r}{r!} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}, \quad r = 1, 2, \ldots.. \tag{20}$$

Proof. The MGF $M_X(s)$ is defined as

$$M_X(s) = \int_0^{g_0} e^{sx} f(x) dx\,. \tag{21}$$

Table 1: Numerical analysis for moments, variance, skewness, and kurtosis.

Parameters			μ'_1	μ'_2	μ'_3	μ'_4	σ^2	γ_1	γ_2
		$\beta = 1.9$	1.3586	2.0621	3.3315	5.5933	0.3245	0.0011	0.0340
		$\beta = 0.9$	1.0044	1.3503	2.0370	3.2723	0.6662	0.0001	0.2197
$g_0 = 2$	$\alpha = 1.5$	$\beta = 0.1$	0.3096	0.4915	1.0025	2.2783	0.4591	6.0526	7.7215
		$\beta = 0.2$	0.5586	0.9290	1.9286	4.4235	0.8209	2.4066	3.4680
		$\beta = 0.3$	0.7638	1.3215	2.7872	6.4478	1.1123	1.2730	2.0790
	$\beta = 0.3$	$\alpha = 0.5$	0.6226	1.0334	2.1415	4.9061	0.8990	1.9872	2.9723
		$\alpha = 0.1$	0.5687	0.9272	1.9070	4.3511	0.8164	2.3676	3.4399
$g_0 = 3$	$\beta = 0.3$	$\alpha = 1.1$	0.7064	1.2029	2.5196	5.8065	1.0265	1.5233	2.3952
	$\beta = 0.09$	$\alpha = 1.1$	0.2282	0.3561	0.7221	1.6358	0.3387	9.0647	11.1686
	$\beta = 0.08$	$\alpha = 0.01$	0.1667	0.2553	0.5141	1.1608	0.2460	13.3676	16.0774

Also, $e^{sx} = \sum_{r=0}^{\infty}((sx)^r/r!)$.

Hence, the MGF of X is obtained as

$$M_X(s) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{r=0}^{\infty} \frac{s^r}{r!} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \quad (22)$$

□

Theorem 3. .

If $X \sim EPF(\alpha,\beta)$, then the characteristic function $(\varphi_X(s))$ of X is given by

$$\varphi_X(s) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}; \; r = 1, 2, \ldots an\, d\, i = \sqrt{-1}. \quad (23)$$

Proof. The characteristic function $\varphi_X(s)$ is defined as

$$\varphi_X(s) = \int_0^{g_0} e^{isx} f(x)dx\,. \quad (24)$$

Hence, the characteristic function of X is obtained as

$$\varphi_X(s) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \quad (25)$$

Vitality function is defined as

$$V(x) = \frac{1}{S(x)} \int_x^{g_0} x f(x)dx\,. \quad (26)$$

It is obtained for X as

$$V(x) = \frac{1}{1-F(x)} \left(\frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{1,j,\beta}} - (x)^{\Lambda_{1,j,\beta}}}{\Lambda_{1,j,\beta}} \right). \quad (27)$$

The conditional moments are defined as

$$E(x^r | X > t) = \frac{1}{\overline{F}(t)} \int_t^{g_0} x^r f(x)dx\,. \quad (28)$$

It is obtained for X as

$$E(x^r | X > t) = \frac{1}{1-F(t)} \left(\frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \psi_j \frac{(g_0)^{\Lambda_{r,j,\beta}} - t^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}} \right). \quad (29)$$

□

3.1. Incomplete Moments and Associated Measures

Theorem 4. *If $X \sim EPF(\alpha,\beta)$, then the r-th lower incomplete moments of X are*

$$\Phi_r(t) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \psi_j \frac{t^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \quad (30)$$

Proof. The r-th incomplete moments $\Phi_r(t)$ are defined as

$$\Phi_r(t) = \int_0^t x^r f(x)dx\,. \quad (31)$$

It is obtained for X as

$$\Phi_r(t) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} \frac{1}{(g_0)^{\beta(j+1)}} \int_0^t x^{r+\beta(j+1)-1} dx\,. \quad (32)$$

Hence, $\Phi_r(t)$ of X reduces to

$$\Phi_r(t) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \psi_j \frac{t^{\Lambda_{r,j,\beta}}}{\Lambda_{r,j,\beta}}. \quad (33)$$

□

Corollary 3. *The first incomplete moment is obtained by substituting r = 1 in equation (33) as*

$$\Phi_1(t) = \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)} \sum_{j=0}^{\infty} \psi_j \frac{t^{\Lambda_{1,j,\beta}}}{\Lambda_{1,j,\beta}}. \quad (34)$$

The residual life function is defined as

$$R_t(x) = \frac{S(x+t)}{S(t)}. \quad (35)$$

Hence, the residual life function (RLF) and its associated CDF of X are given by

$$R_{t(x)} = \frac{2^{\alpha} - \left(1 + (x + t/g)^{\beta}\right)^{\alpha}}{2^{\alpha} - \left(1 + (t/(g_0))^{\beta}\right)^{\alpha}}, \quad (36)$$

and

$$F_{R(t)} = \frac{\left(1 + (x + t/g)^{\beta}\right)^{\alpha} - \left(1 + (t/g)^{\beta}\right)^{\alpha}}{2^{\alpha} - \left(1 + (t/(g_0))^{\beta}\right)^{\alpha}}, \quad (37)$$

respectively.

Furthermore, the reversed RLF is defined as $\overline{R}_t(x) = (S(x-t)/S(t))$. *Hence, reversed RLF and its associated CDF of X take the forms:*

$$\begin{aligned}\overline{R}_t(x) &= \frac{2^{\alpha} - \left(1 + (x - t/(g_0))^{\beta}\right)^{\alpha}}{2^{\alpha} - \left(1 + (t/(g_0))^{\beta}\right)^{\alpha}},\\ F_{\overline{R}(t)} &= \frac{\left(1 + (x - t/(g_0))^{\beta}\right)^{\alpha} - \left(1 + (t/(g_0))^{\beta}\right)^{\alpha}}{2^{\alpha} - \left(1 + (t/(g_0))^{\beta}\right)^{\alpha}}.\end{aligned} \quad (38)$$

The mean RLF is defined as $MRL = (1 - \Phi_1(t)/S(t) - t)$. *It is obtained for X as*

$$MRL = \frac{1}{S(t) - t}\left(1 - \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha} - 1)} \sum_{j=0}^{\infty} \psi_j \frac{t^{\Lambda_{1,j,\beta}}}{\Lambda_{1,j,\beta}}\right). \quad (39)$$

The mean inactivity time (MIT) is defined as $MIT = t - (\Phi_1(t)/F(t))$. *It is obtained for X as*

$$\mathrm{MIT} = t - \frac{\alpha\beta \sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{(g_0)^{\beta}\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1}. \quad (40)$$

The strong mean inactivity time (SMIT) of a device is defined as $SMIT = t^2 - (\Phi_2(t)/F(t))$. *It is obtained for X as*

$$\mathrm{MIT} = t^2 - \frac{\alpha\beta \sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{(g_0)^{\beta}\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1}. \quad (41)$$

The mean past lifetime (MPL) of a device is defined as $MPL = x - (\int_0^x t f(t) dt/F(x))$. *It is obtained for X as*

$$\mathrm{MPL} = \frac{x\left(\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1\right) - (2^{\alpha} - 1)\left(\alpha\beta/(g_0)^{\beta}(2^{\alpha} - 1)\sum_{j=0}^{\infty} \psi_j \left(x^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)\right)}{\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1}. \quad (42)$$

Furthermore, the Lorenz (t), Bonferroni B(t), and Zenga Z(t) inequality curves have a significant role not only in the study of economics, the distribution of income, poverty, or wealth, but also they have a vital role in the fields of insurance, demography, medicine, and reliability engineering.

Theorem 5. *If* $X \sim EPF(\alpha, \beta)$, *then the Lorenz inequality curve of X is*

$$L(t) = \frac{\sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}. \quad (43)$$

Proof. Lorenz inequality curve is defined as

$$L(t) = \frac{\Phi_1(t)}{\mu_1'}. \quad (44)$$

It is obtained for X, using equations (17) and (34), as

$$L(t) = \frac{\sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}. \quad (45)$$

□

Theorem 6. *If* $X \sim EPF(\alpha, \beta)$, *then the Bonferroni inequality curve of X is as follows:*

$$B(t) = \frac{(2^{\alpha} - 1)\sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)\left(\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1\right)}. \quad (46)$$

Proof. The Bonferroni inequality curve is defined as

$$B(x) = \frac{L(t)}{F(x)}. \quad (47)$$

It is obtained for X, using $L(t)$ and the CDF of the EPF model, as

$$B(t) = \frac{(2^{\alpha} - 1)\sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)\left(\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1\right)}. \quad (48)$$

□

Theorem 7. *If* $X \sim EPF(\alpha, \beta)$, *then the Zenga inequality curve of X reduces to*

$$Z(t) = \frac{\sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right) - t\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{t\left(\sum_{j=0}^{\infty} \psi_j \left((g_0)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right) - \sum_{j=0}^{\infty} \psi_j \left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)\right)}. \quad (49)$$

Proof. Zenga inequality curve is defined as

$$Z(t) = \frac{L(t) - t}{t(1 - L(t))}. \quad (50)$$

It is obtained for X as

$$Z(t) = \frac{\sum_{j=0}^{\infty}\psi_j\left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right) - t\sum_{j=0}^{\infty}\psi_j\left(\left(g_0\right)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)}{t\left(\sum_{j=0}^{\infty}\psi_j\left(\left(g_0\right)^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right) - \sum_{j=0}^{\infty}\psi_j\left(t^{\Lambda_{1,j,\beta}}/\Lambda_{1,j,\beta}\right)\right)}. \quad (51)$$

□

4. Reliability Function and Associated Measures

Probability distributions consider as a backbone for reliability engineering to analyze and predict the lifetime of a component/device. In this section, numerous notable reliability measures are discussed.

4.1. Survival Function. The survival function of X takes the form:

$$S(x) = \frac{2^{\alpha} - \left(1 + \left((x/g_0)\right)^{\beta}\right)^{\alpha}}{2^{\alpha} - 1}. \quad (52)$$

4.2. Hazard Rate Function. The HRF (in demography), failure rate function (in engineering), and sometimes it is called the force of mortality (in economics). The HRF of X is

$$h(x) = \frac{\alpha\beta x^{\beta-1}\left(1 + (x/g_0)^{\beta}\right)^{\alpha-1}}{(g_0)^{\beta}\left(2^{\alpha} - \left(1 + (x/g_0)^{\beta}\right)^{\alpha}\right)}. \quad (53)$$

4.3. Mean Time between Failures. Mean time between failures (MTBF) is defined as $(-t/\log(S(x)))$ Hence, it is obtained for X as

$$MTBF = \frac{-t}{\log\left(\left(2^{\alpha} - 1\right)^{-1}\left(2^{\alpha} - \left(1 + (x/g_0)^{\beta}\right)^{\alpha}\right)\right)}. \quad (54)$$

4.4. Cumulative HRF. The cumulative HRF is defined as $h_c(x) = -\log(S(x))$. Hence, the cumulative HRF of X has the form:

$$h_c(x) = -\log\left(\frac{2^{\alpha} - \left(1 + (x/g_0)^{\beta}\right)^{\alpha}}{2^{\alpha} - 1}\right). \quad (55)$$

Figure 2 presents the different curves of the EPF of HRF for various choices of its parameters. We note that it possesses increasing U-shaped and bathtub shape curves for $g_0 = 3$.

4.5. Reverse HRF. The reverse HRF is defined as $h_r(x) = f(x)/S(x)$. It is obtained for X as

$$h_r(x) = \frac{\alpha\beta x^{\beta-1}\left(1 + (x/g_0)^{\beta}\right)^{\alpha-1}}{(g_0)^{\beta}\left(2^{\alpha} - \left(1 + (x/g_0)^{\beta}\right)^{\alpha}\right)}. \quad (56)$$

4.6. Odds Ratio. The odds ratio is defined as $O(x) = F(x)/f(x)$. Hence, the odds ratio of X is given by

$$O(x) = \frac{(g_0)^{\beta}\left(\left(1 + (x/g_0)^{\beta}\right)^{\alpha} - 1\right)}{\alpha\beta x^{\beta-1}\left(1 + (x/g_0)^{\beta}\right)^{\alpha-1}}. \quad (57)$$

4.7. Mills Ratio. The mill's ratio is defined as $M(x) = S(x)/f(x)$. Hence, the mill's ratio of X is

$$M(x) = \frac{(g_0)^{\beta}\left(2^{\alpha} - \left(1 + (x/g_0)^{\beta}\right)^{\alpha}\right)}{\alpha\beta x^{\beta-1}\left(1 + (x/g_0)^{\beta}\right)^{\alpha-1}}. \quad (58)$$

5. Miscellaneous Measures

5.1. Quantile Function

Theorem 8. *If $X \sim EPF(\alpha, \beta)$, then the QF of X is given by*

$$QF = g_0\left(\left(\left(2^{\alpha} - 1\right)p + 1\right)^{1/\alpha} - 1\right)^{1/\beta}. \quad (59)$$

Proof. The QF is defined by

$$QF = F^{-1}\left(x_p\right) = P\left(X \le x_p\right), \quad p \in (0, 1). \quad (60)$$

The p-th QF of X is obtained, by inverting the CDF (7), as

$$QF = g_0\left(\left(\left(2^{\alpha} - 1\right)p + 1\right)^{(1/\alpha)} - 1\right)^{1/\beta}. \quad (61)$$

□

Corollary 4. *The 1st quartile (Q_1), median (Q_2), and 3rd quartile (Q_3) of X are obtained by substituting $p = 0.25$, 0.5, and 0.75 in (61), respectively. The analytical expressions are*

$$\begin{aligned} QF_{Q1} &= g_0\left(\left(\left(2^{\alpha} - 1\right)(0.25) + 1\right)^{(1/\alpha)} - 1\right)^{1/\beta}, \\ QF_{Q2} &= g_0\left(\left(\left(2^{\alpha} - 1\right)(0.50) + 1\right)^{(1/\alpha)} - 1\right)^{1/\beta}, \end{aligned} \quad (62)$$

and

$$QF_{Q3} = g_0\left(\left(\left(2^{\alpha} - 1\right)(0.75) + 1\right)^{(1/\alpha)} - 1\right)^{1/\beta}, \quad (63)$$

respectively.

5.2. Skewness and Kurtosis. Bowley's [41] and Moors's [42] coefficients of skewness and kurtosis can be calculated by the following two equations:

$$\begin{aligned} B_{sk} &= \frac{Q_{0.75} + Q_{0.25} - 2Q_{0.50}}{Q_{0.75} - Q_{0.25}}, \\ M_{kr} &= \frac{Q_{0.375} - Q_{0.125} - Q_{0.625} + Q_{0.875}}{Q_{0.75} - Q_{0.25}}. \end{aligned} \quad (64)$$

Quartiles and octiles based on these descriptive measures provide more robust estimates than the traditional skewness and kurtosis measures. We note that these measures are almost less reactive to outliers and work more effectively for the distributions, deficient in moments. Figure 3 illustrates

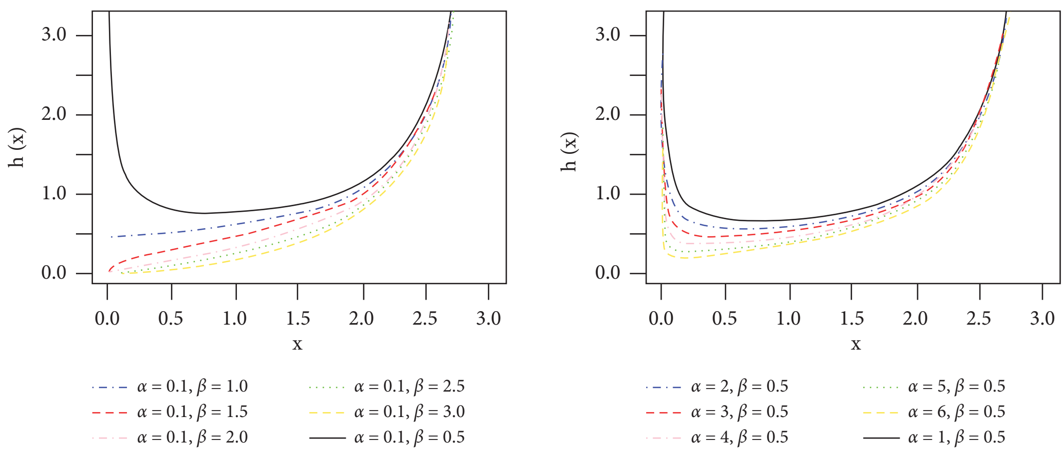

Figure 2: Some curves of the HRF for the EPF distribution.

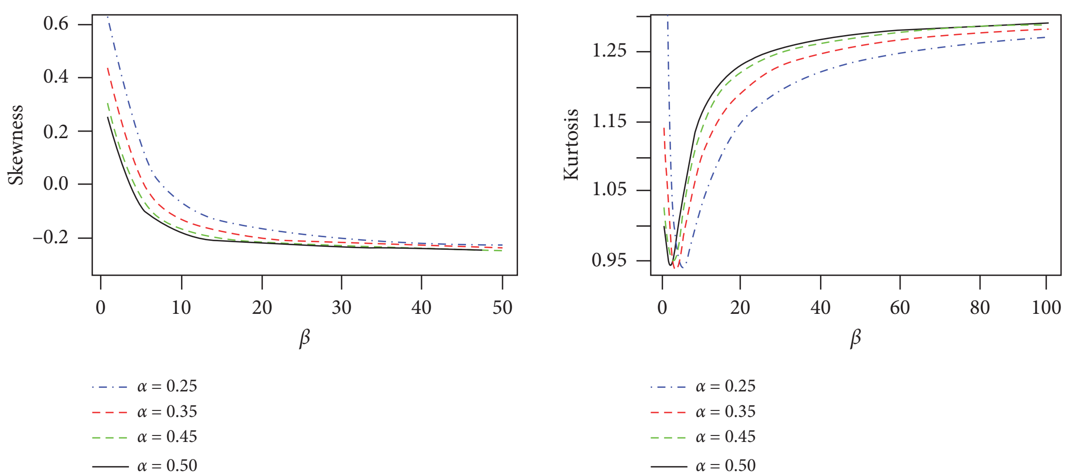

Figure 3: Skewness (left panel) and Kurtosis (right panel) curves for EPF distribution is

the skewness and kurtosis curves for the EPF distribution. We note that skewness and kurtosis are expressed as a function of β. Figure 3 illustrates a positive to negative trend of skewness, and an increasing trend in kurtosis can be observed with the increase of α.

$$I_\delta(X) = \frac{1}{1-\delta}\log\int_0^{g_o} f^\delta(x)dx\ , \delta > 0\, an\ d\, \delta \neq 1. \tag{65}$$

5.3. Entropy Measures. Kurtosis and entropy measures have the same role in comparing the shapes and tail weights of various density functions. The entropy of a random variable X is defined as a measure of uncertainty.

In this section, we have developed numerous well-known entropy measures including Rényi [43], Havrda and Charvat [44], and Mathai and Haubold [45]. For more details, see some promising work of Basit et al. [46], Dey et al. [47], and Ijaz et al. [48].

Theorem 9. *If $X \sim EPF(\alpha,\beta)$, then the Rényi entropy of X is*

$$\frac{1}{1-\delta}\log\left(\frac{\alpha\beta}{(2^\alpha-1)}\right)^\delta\sum_{i=0}^{\infty}\binom{\delta(\alpha-1)}{i}g_0^{-\beta(i+1)} \frac{(g_o)^{\beta i+\delta(\beta-1)+1}}{\beta i+\delta(\beta-1)+1};\, \delta>0\, an\ d\, \delta\neq 1. \tag{66}$$

Proof. The Rényi entropy for X is defined by

Using Equation (10), we can write

$$f^\delta(x) = \left(\frac{\alpha\beta}{(g_0)^\beta(2^\alpha-1)}\right)^\delta x^{\delta(\beta-1)}\left(1+\left(\frac{x}{g_0}\right)^\beta\right)^{\delta(\alpha-1)}. \tag{67}$$

Then, integration $f^\delta(x)$ gives

$$\int_0^{g_o} f^{\delta}(x)dx = \left(\frac{\alpha\beta}{(2^{\alpha}-1)}\right)^{\delta}\sum_{i=0}^{\infty}\binom{\delta(\alpha-1)}{i}(g_o)^{-\beta(i+1)} \frac{(g_o)^{\beta i+\delta(\beta-1)+1}}{\beta i+\delta(\beta-1)+1}. \tag{68}$$

Hence, the Rényi entropy reduces to

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left(\frac{\alpha\beta}{(2^{\alpha}-1)}\right)^{\delta}\sum_{i=0}^{\infty}\binom{\delta(\alpha-1)}{i}g_0^{-\beta(i+1)} \frac{(g_o)^{\beta i+\delta(\beta-1)+1}}{\beta i+\delta(\beta-1)+1}. \tag{69}$$

The expression developed in (68) is quite helpful in the further computation of entropy measures of Havrda and Charvat, and Mathai and Haubold. The final expressions of Havrda and Charvat, and Mathai and Haubold entropy measures are presented in Table 2.

Table 3 presents the results of Rényi, Havrda and Charvat, and Mathai and Haubold entropy measures for some choices of model parameters for ($g_o = 3$), Set-I ($\alpha = 1.1, \beta = 2.1$), Set -II ($\alpha = 2.1, \beta = 2.5$), Set –III ($\alpha = 0.1, \beta = 0.6$), and Set-IV ($\alpha = 0.01, \beta = 1.1$).

A wide range of positive and negative values of entropy measures makes the EPF distribution more flexible and versatile. □

5.4. Distribution of Order Statistics. Let $X_1, \ldots, X_n$ be a random sample of size n and their corresponding order statistics (OS) $X_{(1)} < \ldots < X_{(n)}$ from the EPF distribution. The PDF of the i-th OS is $f_{(i:\,n)}(x) = 1/B(i, n-i+1)!(F(x))^{i-1}(1-F(x))^{n-i}f(x)$, $i = 1, 2, 3, \ldots, n$.

The i-th OS density is obtained by incorporating Equations (6) and (8) in the last equation.

$$f_{(i:\,n)}(x) = \frac{1}{B(i, n-i+1)!}\left(\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{i-1} \left(\frac{2^{\alpha}-\left(1+(x/g_0)^{\beta}\right)^{\alpha}}{2^{\alpha}-1}\right)^{n-i} \frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)}x^{\beta-1}\left(1+(x/g_0)^{\beta}\right)^{\alpha-1}. \tag{70}$$

The minimum and maximum OS densities are obtained, respectively, by substituting $i = 1, n$ in (70).

The i-th OS CDF is defined by

$$G(x) = \sum_{r=i}^{n}\binom{n}{r}(1-F(x))^{n-r}F^{r}(x). \tag{71}$$

The i-th OS CDF of the EPF distribution reduces to

$$G(x) = \sum_{r=1}^{n}\binom{n}{r}\left(\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{r} \left(\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{n-r}. \tag{72}$$

The median i-th OS PDF is

$$f_{(m+1:\,n)}(x) = \frac{(2m+1)!}{(m!)^2}f(x)(F(x))^{m}(1-F(x))^{m}. \tag{73}$$

The X_{m+1} median OS PDF of the EPF distribution has the form:

$$f_{(m+1:\,n)}(x) = \frac{(2m+1)!}{(m!)^2}\frac{\alpha\beta}{(g_0)^{\beta}(2^{\alpha}-1)}x^{\beta-1}\left(1+\left(\frac{x}{g_0}\right)^{\beta}\right)^{\alpha-1} \left(\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{m}\left(\frac{2^{\alpha}-\left(1+(x/g_0)^{\beta}\right)^{\alpha}}{2^{\alpha}-1}\right)^{m}. \tag{74}$$

The i-th and j-th OS joint distribution is defined by

$$f\left(x_i, x_j\right) = C(F(x_i))^{i-1}\left(F\left(x_j\right)-F(x_i)\right)^{j-i-1} \left(1-F\left(x_j\right)\right)^{n-j}f(x_i)f\left(x_j\right). \tag{75}$$

For the i-th and j-th OS joint distribution of the EPF model is as follows:

$$f\left(x_i, x_j\right) = \frac{C(\alpha\beta)^2\left(x_i x_j\right)^{\beta-1}}{(g_0)^{2\beta}(2^{\alpha}-1)^2}\left[\left(1+\left(\frac{x_i}{g_0}\right)^{\beta}\right)\left(1+\left(\frac{x_j}{g_0}\right)^{\beta}\right)\right]^{\alpha-1} \left[\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right]^{i-1}\left[1-\frac{\left(1+(x/g_0)^{\beta}\right)^{\alpha}-1}{2^{\alpha}-1}\right]^{n-j} \left[\frac{\left(1+\left(x_j/g_0\right)^{\beta}\right)^{\alpha}-\left(1+(x_i/g_0)^{\beta}\right)^{\alpha}}{2^{\alpha}-1}\right]^{j-i-1}. \tag{76}$$

5.5. Bivariate and Multivariate Extensions. In this section, we develop the bivariate and multivariate extensions for the EPF distribution by following the Morgenstern family and the Clayton family

The CDF of the bivariate EPF distribution followed by the Clayton family for the random vector (X, Y) is

$$F(V_1, V_2) = (1+\eta(1-Z_1(v_1))(1-Z_2(v_2)))Z_1(v_1)Z_2(v_2), \tag{77}$$

where $|\eta| \le 1$, $Z_1(v_1) = ((1+(x_1/g_0)^{\beta_1})^{\alpha_1} - 1/2^{\alpha_1} - 1)$ and $Z_2(v_2) = ((1+(x_2/g_0)^{\beta_2})^{\alpha_2} - 1/2^{\alpha_2} - 1)$.

The CDF of the bivariate EPF distribution followed by the Morgenstern family for the random vector (V_1, V_2) is defined as

Table 2: The final expressions for Havrda and Charvat, and Mathai and Haubold entropy measures.

Entropy/Support	Measure	Expression
Havrda and Charvat $\omega > 0,\ \omega \neq 1$	$(1/1-\omega)(\int_0^{g_o} f^{\omega}(x)dx - 1)$	$\frac{1}{1-\omega}\left(\left((\frac{\alpha\beta}{(2^{\alpha}-1)})^{\omega}\sum_{i=0}^{\infty}\binom{\omega(\alpha-1)}{i} g_0^{-\beta(i+1)} \times \frac{(g_o)^{\beta i+\omega(\beta-1)+1}}{\beta i+\omega(\beta-1)+1}\right)-1\right)$
Mathai and Haubold $\vartheta > 0, \vartheta \neq 1$	$(1/\vartheta-1)(\int_0^{g_o} f^{2-\vartheta}(x)dx - 1)$	$\frac{1}{\vartheta-1}\left(\left((\frac{\alpha\beta}{(2^{\alpha}-1)})^{2-\vartheta}\sum_{i=0}^{\infty}\binom{(2-\vartheta)(\alpha-1)}{i} g_0^{-\beta(i+1)} \times \frac{(g_o)^{\beta i+(2-\vartheta)(\beta-1)+1}}{\beta i+(2-\vartheta)(\beta-1)+1}\right)-1\right)$

Table 3: Numerical analysis for Rényi, Havrda and Charvat, and Mathai and Haubold entropy measures.

Entropy	Int.	Set -I	Set -II	Set -III	Set-IV
Rényi	$\delta = 1.1$	48.5073	45.5568	42.9999	36.4934
	$\delta = 1.5$	13.2292	12.4246	11.7272	9.9527
	$\delta = 1.7$	10.7094	10.0580	9.4934	8.0570
	$\delta = 1.9$	9.3094	8.7432	8.2525	7.0038
Havrda and Charvat	$\omega = 1.1$	20.8214	20.9130	20.7450	21.0333
	$\omega = 1.5$	4.6657	4.7472	4.4961	4.8391
	$\omega = 1.7$	3.4637	3.5389	3.2307	3.6180
	$\omega = 1.9$	2.7785	2.8473	2.4510	2.9149
Mathai and Haubold	$\beth = 1.1$	−19.0736	−18.9800	−19.1095	−18.8459
	$\beth = 1.5$	−2.7702	−2.6883	−2.7356	−2.5421
	$\beth = 1.7$	−1.3975	−1.3360	−1.3444	−1.2080
	$\beth = 1.9$	−0.4388	−0.4138	−0.4027	−0.3487

$$C(x,y) = \left(x^{-(\tau_1+\tau_2)} + y^{-(\tau_1+\tau_2)} - 1\right)^{-1/(\tau_1+\tau_2)}; \ \tau_1+\tau_2 \geq 0. \tag{78}$$

Let $v_1 \sim$ EPF (α_1, β_1), and $v_2 \sim$ EPF (α_2, β_2). Then, we set $x = Z_1(v_1) = ((1+(x_1/g_0)^{\beta_1})^{\alpha_1} - 1/2^{\alpha_1} - 1)$, and $y = Z_2(v_2) = ((1+(x_2/g_0)^{\beta_2})^{\alpha_2} - 1/2^{\alpha_2} - 1)$.

The CDF of the bivariate EPF distribution followed by the Clayton family for the random vector (V_1, V_2) is

$$G(v_1,v_2) = \left(\left(\frac{\left(1+(x_1/g_0)^{\beta_1}\right)^{\alpha_1}-1}{2^{\alpha_1}-1}\right)^{(\tau_1+\tau_2)} + \left(\frac{\left(1+(x_2/g_0)^{\beta_2}\right)^{\alpha_2}-1}{2^{\alpha_2}-1}\right)^{(\tau_1+\tau_2)} - 1\right)^{-(1/(\tau_1+\tau_2))}. \tag{79}$$

A simple n-dimensional extension of the last version for EPF distribution has the form:

$$H(x_1,x_2,x_3,\ldots,x_n) = \left(\sum_{j=1}^{n}\left(\left(1\frac{\left(1+(x_1/g_0)^{\beta_j}\right)^{\alpha_j}-1}{2^{\alpha_j}-1}\right)^{(\tau_1+\tau_2)}\right)+1-n\right)^{-(1/(\tau_1+\tau_2))}. \tag{80}$$

6. Statistical Inference

In this section, we discuss six estimation techniques for the EPF parameters as follows: maximum likelihood estimators (MLEs), maximum product of spacing estimators (MPSEs), percentile estimators (PCEs), Cramér von–Mise distance estimators (CVMEs), Anderson–Darling estimators (ADEs), and right-tail Anderson–Darling estimators (RTADEs).

6.1. Maximum Likelihood Estimators. Let $x_1, \ldots, x_n$ be a random sample of size n from the EPF model; then, the likelihood function of $L(\phi) = \prod_{j=1}^{n} f(x_j)$ is given by

$$L(\phi) = \left(\frac{\alpha\beta}{(g_0)(2^{\alpha}-1)}\right)^n \prod_{j=1}^{n}\left(\frac{x_j}{g_0}\right)^{\beta-1}\prod_{j=1}^{n}\left(1+\left(\frac{x_j}{g_0}\right)^{\beta}\right)^{\alpha-1}. \tag{81}$$

The log $L(\phi) = l(\phi)$ takes the form:

$$l(\phi) = n\log(\alpha) + n\log(\beta) - n\log(g_0) - nlog(2^\alpha - 1) + (\beta - 1)\sum_{j=1}^{n}\log\left(\frac{x_j}{g_0}\right) + (\alpha - 1)\sum_{j=1}^{n} log\left(1 + \left(\frac{x_j}{g_0}\right)^\beta\right). \quad (82)$$

Let $y_j = (x_j/g_0)$. The partial derivatives for the parameters α and β are

$$\frac{\partial l(\phi)}{\partial\alpha} = \frac{n}{\alpha} - \frac{n(2^\alpha\log 2)}{2^\alpha - 1} + \sum_{j=1}^{n}\log\left(1 + y_j^\beta\right), \quad (83)$$

and

$$\frac{\partial l(\phi)}{\partial\beta} = \frac{n}{\beta} + \sum_{j=1}^{n}\log y_j + (\alpha - 1)\sum_{j=1}^{n}\frac{y^\beta\log y_j}{\left(1 + y_j^\beta\right)}. \quad (84)$$

The maximum likelihood estimates ($\widehat{\phi} = \widehat{\alpha, \beta}$) of the EPF parameters can be obtained by maximizing (82) or by solving the above nonlinear equations simultaneously. These nonlinear equations although do not provide an analytical solution for the MLEs and the optimum values of α, and β. Consequently, the Newton–Raphson type algorithm is an appropriate choice to obtain the MLEs.

6.2. Maximum Product of Spacing Estimators. The MPSEs are alternatives to the MLEs, and they are introduced by Cheng and Amin [49, 50]. Let $x_1, \ldots, x_n$ be a uniform spacing of a random sample taken from the EPF distribution is defined by $D_j = F(x_{j:n}) - F(x_{(j-1:n)})$, $j = 1, 2, 3, \ldots, n$, where D_j denotes the uniform spacing, x_j is j-th order statistics, and $(x_{0:n}) = 0$, $F(x_{n+1:n}) = 1$, and $\sum_{j=1}^{n+1}D_j = 1$, and the MPSEs of the EPF parameters are obtained by maximizing

$$P(\alpha,\beta) = \frac{1}{n+1}\sum_{j=1}^{n+1}\log D_j. \quad (85)$$

These estimators can also be obtained by solving

$$\frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{D_j}\left(\nabla_t\left(x_{j:n}\right) - \nabla_t\left(x_{(j-1:n)}\right)\right) = 0, \quad (86)$$

where

$$\nabla_1\left(x_{j:n}\right) = \frac{\partial}{\partial\alpha}F\left(x_{j:n}\right), \quad \nabla_2\left(x_{j:n}\right) = \frac{\partial}{\partial\beta}F\left(x_{j:n}\right). \quad (87)$$

6.3. Percentile Estimators. The percentile method was introduced by Kao [51]. This method allows estimating the unknown parameters if the distribution function has a closed-form expression. Suppose $u_j = j/(n+1)$ be an unbiased estimator of $F(x_{j:n})$. The PCEs of the EPF parameters are obtained by minimizing

$$P(\alpha,\beta) = \sum_{j=1}^{n}\left(x_{j:n} - g_0\left(\left((2^\alpha - 1)u_j + 1\right)^{(1/\alpha)} - 1\right)^{(1/\beta)}\right)^2, \quad (88)$$

with respect to α and β, respectively.

6.4. Cramér von–Mise Estimators. Cramér [52] and Von Mises [53] introduced a relatively less-biased minimum distance estimator called the CVMEs. It can be obtained by making a difference between the estimates of the CDF and empirical CDF. The CVMEs of the EPF parameters are obtained by minimizing

$$C(\alpha,\beta) = \frac{1}{12n} + \sum_{j=1}^{n}\left(F\left(x_{j:n}\right) - \frac{2j-1}{2n}\right), \quad (89)$$

with respect to α and β. Furthermore, the CVMEs follow by solving the nonlinear equations as

$$C(\alpha,\beta) = \frac{1}{12n} + \sum_{j=1}^{n}\left(F\left(x_{j:n}\right) - \frac{2j-1}{2n}\right)\nabla_t\left(x_{j:n}\right) = 0, \quad (90)$$

where $\nabla_t(x_{j:n}) = 0$ for $t = 1, 2$ is defined by (87).

6.5. Anderson–Darling and Right-Tail Anderson–Darling Estimators. Another type of minimum distance estimators is the ADEs. The ADEs of the EPF parameters are obtained by minimizing

$$A(\alpha,\beta) = -n - \frac{1}{n} + \sum_{j=1}^{n}(2j-1)\left(log\left(F\left(x_{j:n}\right)\right) + log\left(S\left(x_{j:n}\right)\right)\right), \quad (91)$$

with respect to α and β, respectively. The ADEs are also obtained by solving the following nonlinear equation as

$$\sum_{j=1}^{n}(2j-1)\left(\frac{\nabla_t\left(x_{j:n}\right)}{F\left(x_{j:n}\right)} + \frac{\nabla_t\left(x_{(n+1-j:n)}\right)}{S\left(x_{n+1-j:n}\right)}\right), \quad (92)$$

where $\nabla_t(x_{j:n}) = 0$ (for $t = 1, 2$) is defined by (87).

The RTADEs of the EPF parameters can be determined by minimizing

$$A(\alpha,\beta) = \frac{n}{2} - 2 + \sum_{j=1}^{n}\left(F\left(x_{j:n}\right)\right) - \frac{1}{n}\sum_{j=1}^{n}(2j-1)log\left(S\left(x_{j:n}\right)\right), \quad (93)$$

with respect to α and β.

7. Simulation Experiment

In this section, we perform a simulation study to assess the behavior of different estimators in estimating the EPF parameters. We generate $N = 1{,}000$ replicates using (61) for several sample sizes $n = 25$, 50, and 100 with different combinations of the parameters. We calculate the average values of the estimates (AEs):

Table 4: The AEs and MSEs for $(\alpha = 1.5 \text{ and } \beta = 2.0)^T$.

n	Par.	Est.	MLEs	MPSEs	PCEs	CVMEs	ADEs	RTADEs
25	$\widehat{\alpha}$	AE MSE	1.5430 (0.0808)	1.5267 (0.0701)	1.5072 (0.0624)	2.1794 (0.0910)	1.5603 (0.0910)	1.5871 (0.1243)
	$\widehat{\beta}$	AE MSE	2.0408 (0.3065)	2.0101 (0.2620)	2.7331 (2.0871)	2.2208 (0.2764)	2.0208 (0.2764)	2.0156 (0.2693)
50	$\widehat{\alpha}$	AE MSE	1.5380 (0.0658)	1.5225 (0.0631)	1.5051 (0.0615)	1.7521 (0.0739)	1.5474 (0.0704)	1.6042 (0.0988)
	$\widehat{\beta}$	AE MSE	2.0080 (0.2592)	2.0020 (0.2522)	2.8602 (2.2141)	2.1503 (0.2278)	2.0027 (0.2530)	2.0026 (0.2529)
100	$\widehat{\alpha}$	AE MSE	1.5048 (0.0609)	1.5037 (0.0610)	1.5001 (0.0575)	1.5407 (0.0086)	1.5053 (0.0610)	1.5179 (0.0597)
	$\widehat{\beta}$	AE MSE	2.0009 (0.2509)	2.0005 (0.2505)	2.6984 (1.6397)	2.0602 (0.0541)	2.0006 (0.2506)	2.0011 (0.2511)

Table 5: The AEs and MSEs for $(\alpha = 2.5 \text{ and } \beta = 1.5)^T$.

n	Par.	Est.	MLEs	MPSEs	PCEs	CVMEs	ADEs	RTADEs
25	$\widehat{\alpha}$	AE MSE	2.5273 (0.6168)	2.5153 (0.5912)	2.5001 (0.5627)	2.7881 (1.6927)	2.5425 (0.6557)	2.5900 (0.8361)
	$\widehat{\beta}$	AE MSE	1.5308 (0.0105)	1.5100 (0.0024)	2.1981 (0.8746)	1.8852 (1.3466)	1.5182 (0.0051)	1.5156 (0.0031)
50	$\widehat{\alpha}$	AE MSE	2.5270 (0.6001)	2.5394 (0.5378)	2.5001 (0.5627)	2.6770 (1.2651)	2.5448 (0.6367)	2.5675 (0.7333)
	$\widehat{\beta}$	AE MSE	1.5188 (0.0055)	1.5047 (0.0008)	2.2336 (0.7808)	1.7453 (0.8573)	1.5088 (0.0024)	1.5088 (0.0025)
100	$\widehat{\alpha}$	AE MSE	2.5249 (0.5734)	2.5197 (0.4719)	2.5000 (0.5626)	2.5033 (0.5659)	2.5241 (0.6150)	2.5281 (0.6267)
	$\widehat{\beta}$	AE MSE	1.5032 (0.0001)	1.5006 (0.0000)	2.2794 (0.7528)	1.5003 (0.0000)	1.5018 (0.0001)	1.5018 (0.0001)

$$AE(\widehat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}\widehat{\varphi}, \quad (94)$$

and the mean square errors (MSEs):

$$MSE(\widehat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\varphi}_i - \varphi\right)^2, \quad (95)$$

where $\varphi = (\alpha, \beta)$.

The selection of the best estimation method will be made having a minimum value of MSEs. The *R* software (DEoptim package) is adopted to obtain the simulation results. The results of AEs and MSEs (in parenthesis) for the MLEs, MPSEs, PCEs, CVMEs, ADEs, and RTADEs are presented in Tables 4–7. It is noted that the AEs tend to their true parameter values, and the MSEs decrease with the increase in the sample size. This evidence is enough to favor that the estimators are unbiased asymptotically. All estimation methods perform efficiently for different combinations.

8. Application in Automobile Engineering

In this section, we analyze automobile engineering data. The data represent the time to failure (10^3 h) of turbocharger of one type of engine discussed by Xu et al. [54]. The observations are as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0. This dataset is analyzed by Afify et al. [55] and Nassar et al. [56].

The EPF distribution is compared with some well-known competitors, namely, the Weibull power function (W-PF) and the zero-truncated Poisson power function (ZTP-PF). Their CDFs are presented in Table 8. The criterion -log-likelihood (-LL), Akaike information criterion (AIC), along with the goodness-of-fit statistics such as Kolmogorov –Smirnov (KS) with its p-value, are adopted. Some descriptive statistics are presented in Table 9. Table 10 presents the estimates and standard errors (SEs) alongside the goodness-of-fit statistics as well. Based on the results in Table 10, we conclude that the EPF distribution provide better fit among all well-established competitors.

Furthermore, the empirical fitted density (i), estimated CDF (ii), probability-probability (PP) (iii), Kaplan–Meier survival (iv), along with the TTT plot (v), and box plot (vi) are illustrated in Figure 4, respectively. All the estimates and numerical results are calculated using the statistical software *R*, package adequacy model developed by Rafael et al. [57].

Table 6: The AEs and MSEs for $(\alpha = 0.8 \text{ and } \beta = 0.5)^T$.

n	Par.	Est.	MLEs	MPSEs	PCEs	CVMEs	ADEs	RTADEs
25	$\hat{\alpha}$	AE MSE	0.8861 (0.5333)	0.8802 (0.5386)	0.8022 (0.3627)	0.8902 (0.6356)	0.8968 (0.5663)	0.9439 (0.6981)
	$\hat{\beta}$	AE MSE	0.5156 (0.0019)	0.5026 (0.0003)	0.5355 (0.0107)	0.5325 (0.0015)	0.5105 (0.0018)	0.5125 (0.0021)
50	$\hat{\alpha}$	AE MSE	0.8229 (0.4093)	0.8267 (0.4108)	0.8026 (0.3531)	0.8377 (0.5548)	0.8238 (0.4121)	0.8348 (0.4369)
	$\hat{\beta}$	AE MSE	0.5083 (0.0007)	0.5036 (0.0002)	0.5074 (0.0016)	0.5234 (0.0007)	0.5049 (0.0005)	0.5039 (0.0004)
100	$\hat{\alpha}$	AE MSE	0.8076 (0.3702)	0.8061 (0.3677)	0.8023 (0.3227)	0.8089 (0.4527)	0.8089 (0.3721)	0.8243 (0.3958)
	$\hat{\beta}$	AE MSE	0.5021 (0.0000)	0.5008 (0.0000)	0.5006 (0.0000)	0.5110 (0.0001)	0.5008 (0.0000)	0.5005 (0.0000)

Table 7: The AEs and MSEs for $(\alpha = 2.5 \text{ and } \beta = 3.5)^T$.

n	Par.	Est.	MLEs	MPSEs	PCEs	CVMEs	ADEs	RTADEs
25	$\hat{\alpha}$	AE MSE	2.5676 (0.2944)	2.5444 (0.2493)	3.1695 (2.0854)	2.7040 (0.7616)	2.5868 (0.3311)	2.6072 (0.3467)
	$\hat{\beta}$	AE MSE	3.5951 (0.3132)	3.5293 (0.1982)	4.6798 (2.7572)	3.5902 (0.3670)	3.5542 (0.2382)	3.5717 (0.2741)
50	$\hat{\alpha}$	AE MSE	2.5186 (0.1813)	2.5082 (0.1685)	2.9012 (1.1522)	2.5475 (0.2712)	2.5375 (0.2088)	2.5907 (0.3245)
	$\hat{\beta}$	AE MSE	3.5207 (0.1824)	3.5049 (0.1645)	3.7910 (1.9981)	3.5154 (0.1948)	3.5075 (0.1674)	3.5087 (0.1688)
100	$\hat{\alpha}$	AE MSE	2.5035 (0.1636)	2.5025 (0.1623)	2.7081 (0.8632)	2.5011 (0.1609)	2.5018 (0.1616)	2.5035 (0.1636)
	$\hat{\beta}$	AE MSE	3.5035 (0.1633)	3.5009 (0.1608)	3.5679 (1.6502)	3.5003 (0.1602)	3.5020 (0.1618)	3.5020 (0.1618)

Table 8: List of some competitive models of CDFs.

Model	CDFs of model	Author (s)
W-PF	$G_I(x) = 1 - e^{-\alpha(x^\beta/_{g0}^\beta - x^\beta)^\gamma}$, $\alpha, \beta, \gamma > 0$, $0 < x < g_0$	Tahir et al. [10]
ZTP-PF	$G_{II}(x) = \frac{1-e^{-\alpha(x/g_0)^\beta}}{1-e^{-\alpha}}$, $\alpha, \beta > 0$, $0 < x < g_0$	Okorie et al. [58]

Table 9: Descriptive statistics for turbocharger data.

Nature	Min.	Q1	Median	Mean	Q3	Max.	Sk.	Kur.
Turbocharger	1.600	5.075	6.500	6.253	7.825	9.000	−0.638	2.641

Table 10: Parameter estimates, SEs, and goodness-of-fit statistics for turbocharger data.

	Parameter estimates (SEs)			Statistics			
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	AIC	KS	p value (KS)
EPF	0.2516 (0.6067)	2.6418 (0.7979)	–	76.9082	157.8166	0.0496	1.0000
W-PF	1.2327 (0.7999)	3.1875 (1.7180)	0.7639 (0.2057)	77.6748	161.4164	0.0515	0.9999
ZTP-PF	2.7724 (0.5035)	3.8728 (0.5176)	–	82.4755	168.9510	0.1253	0.5555

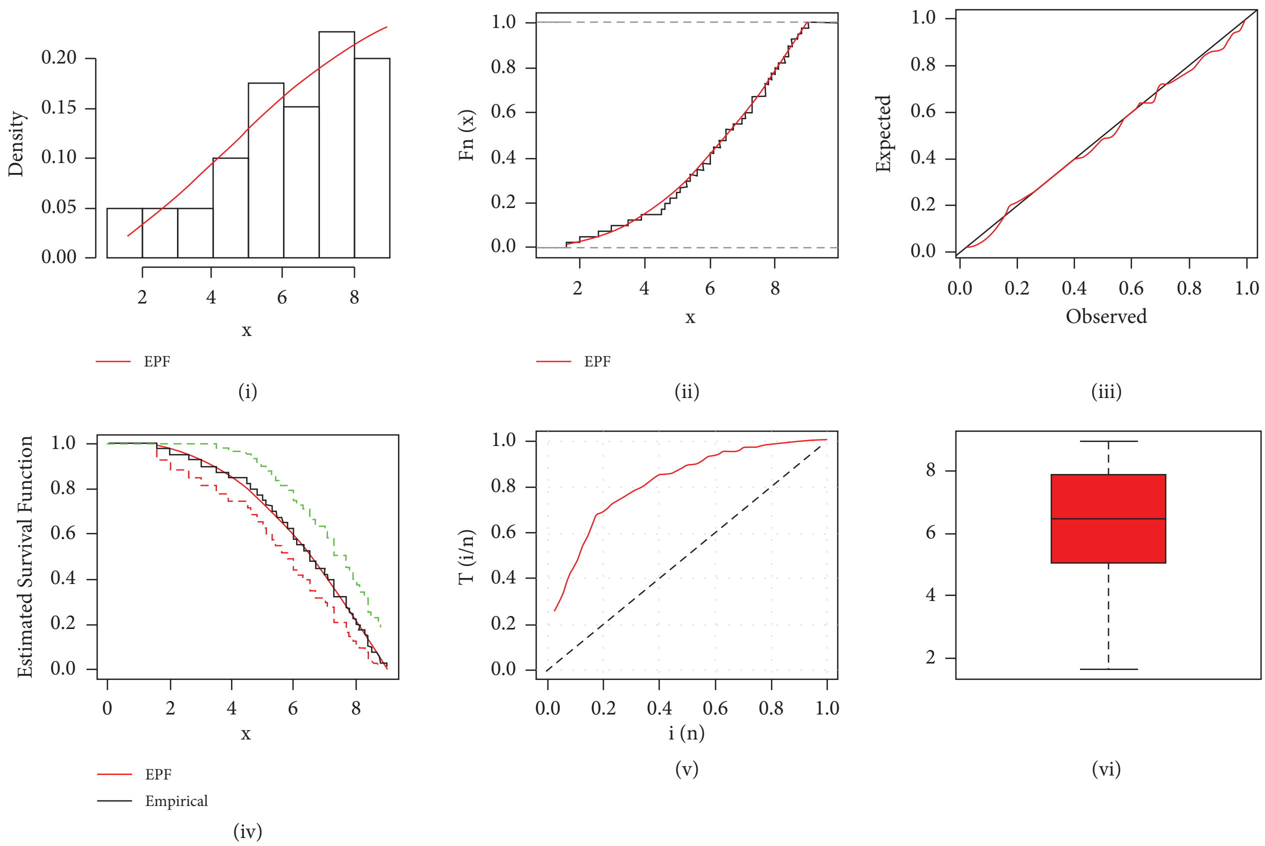

Figure 4: Fitted density (i), estimated CDF (ii), PP plot (iii), Kaplan–Meier survival (iv), TTT plot (v), and box plot (vi) for turbocharger data.

9. Conclusions

In this paper, we develop a new class that generates optimal univariate continuous models called the new exponentiated-G class. A special member of the proposed family called the exponentiated power function (EPF) distribution is studied in detail. Numerous statistical and reliability characteristics are discussed. Furthermore, the EPF distribution has flexible shapes for its density and hazard functions. For the estimation of EPF parameters, we followed the six accredited techniques named, MLEs, MPSEs, PCEs, CVMEs, ADEs, and RTADEs. A simulation experiment is performed to compare the performance of different estimation techniques. Our results show that the estimation techniques perform very well. The applicability of the EPF distribution is addressed using real-life data form the engineering field. The results show that the EPF distribution provides better fit as compared to other well-known competitors.

For some possible future studies, the EPF distribution can be adopted to analyze entropy measures following the works of Siddiqui et al. [59] and Rashid et al. [60].

Data Availability

This work is mainly a methodological development and has been applied on secondary data; but if required, data will be provided.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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