Research Article
Some Results about Weak UP-algebras

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1. Introduction

A few abstract algebras, such as BCK-algebras and BCI-algebras, have their roots in implicational logic, see [1, 2]. Many facets of daily life have benefited substantially from recent advances in artificial intelligence. Logical systems are the fundamental artificial intelligence techniques that aid in decision-making. The evolution and introduction of BCK-algebras and BCI-algebras are driven by the essential axioms of the implicational calculus, see [1, 2]. Taking into account the important linkages between these algebras and related logics, translation techniques have been created to link the theorems and formulae of a logic and the corresponding algebra. As a result, those involved in the fields of artificial intelligence, logical systems, and algebraic structures continue to be interested in the study of abstract algebras (and their generalizations) that have been inspired by logical systems. As a result, the BCH-algebra class, a new class of algebras, has been introduced [3]. Recently, another algebra class, known as gBCH-algebra class, has been introduced and studied in [4, 5]. The significant feature of this new class is its ability to cover all other mentioned classes. In the context of this class, further study through introduction of generalized d-algebras and isomorphism theorems in generalized d-algebras has been developed [6]. For more on such other algebras such as SU-algebra [7], BCC algebra [8] and the significance of these algebras we refer to [9–11]. Although the immediate applications played vital role in getting attention to the above mentioned algebras, however, the development of theoretical aspects of these algebras can not be ignored. Particularly, when they provide different (or a generalized) framework. Consequently, the study on these algebras continued and recently the notion of a KU-algebra [12] was introduced and studied further in [13]. Later, a generalized class of KU-algebras, known as UP-algebras [14] was defined and investigated. The range of the notions studied for these algebras is very vast, such as: the ideals and congruences [12], cubic KU-ideals [13], (a, β)-fuzzy KU-Ideals [15], interval-valued fuzzy KU ideals and (θ, δ)-fuzzy KU-ideals [16, 17], anti fuzzy and intuitionistic fuzzy KU-ideals [18, 19] in KU-algebras. In UP-algebras, the fuzzy sets and the generalized fuzzy sets were introduced and studied in [20, 21]. Moreover, the derivations of UP-algebras have been studied in [22, 23]. In 2019, the isomorphism theorems
for UP-algebras were proved [24] and the results were extended later on. For more on the recent related study/applications of these algebras in multiple directions, we refer [25, 26] and references therein. Keeping the above motivation in view, we introduce a general class of algebras, known as weak UP-algebras (WUP-algebras) in such a way that the classes of KU-algebras and UP-algebras become the subclasses of the class of WUP-algebras. Further, we give necessary examples to show that the UP-algebra (consequently) class is a proper sub-class of the WUP-algebra class. Consequently, all investigations of this paper are valid for the KU-algebra class and the UP-algebra class. In particular, we study some fundamental aspects including maximal elements, branches and the subalgebra consisting of maximal elements of a WUP-algebra. We also define regular congruences on a WUP-algebra and study the corresponding quotient algebras. We prove that the congruence \( q \) generated from the maximal elements of a WUP-algebra \( \chi \) is regular and the corresponding quotient algebra \( \chi /q \) is isomorphic to \( \text{Max}(\chi) \), the subalgebra consisting of maximal elements of \( \chi \).

2. Weak UP-Algebras (WUP-Algebras)

In this section, we introduce the notion of a Weak UP-algebra and prove that the classes of KU-algebras and UP-algebras are proper subclasses of the class of WUP-algebras. Before proceeding further, we recall the notions of a KU-algebra [12] and an UP-algebra [14].

By an algebra \( (\chi, \ldots ,0) \) of type \( (2,0) \), we mean a non-empty set \( \chi \) together with a binary operation "\( \cdot \)" and a distinguished element 0 satisfying some specified axioms. Further, for \( x, y \), we shall write \( xy \) and \((\ldots)(\ldots)\) will mean "\( \cdot \)" between the bracketed expressions.

**Definition 1 (KU-algebra).** An algebra \( \chi = (\chi, \ldots ,0) \) of type \( (2,0) \) is called a KU-algebra if it satisfies the following conditions for any \( x, y, z \in \chi \):

\[
(KU - 1) (xy) ((xz)(yz)) = 0, \\
(KU - 2) 0x = x, \\
(KU - 3)x0 = 0, \\
(KU - 4) xy = 0 \text{ and } yx = 0 \text{ imply } x = y.
\]

**Definition 2 (UP-algebra).** An algebra \( \chi = (\chi, \ldots ,0) \) of type \( (2,0) \) is called a UP-algebra if it satisfies the following conditions for any \( x, y, z \in \chi \):

\[
(UP - 1) (yz) ((xy)(xz)) = 0, \\
(UP - 2) 0x = x, \\
(UP - 3)x0 = 0, \\
(UP - 4) xy = 0 \text{ and } yx = 0 \text{ imply } x = y.
\]

It is shown in [14] that every KU-algebra is an UP-algebra. Moreover, it is described in [14] that the algebra of the following Example 1 is an UP-algebra but not a KU-algebra. Consequently, the class of UP-algebras is wider than the class of KU-algebras.

**Example 1 (see [14]).** Let \( \chi = \{0,1,2,3,4\} \) and the binary operation defined as (see Table 1):

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We now give our notion of a weak UP-algebra (WUP-algebra).

**Definition 3 (Weak UP-algebra).** An algebra \( \chi = (\chi, \ldots ,0) \) of type \( (2,0) \) is called a weak UP-algebra (WUP-algebra) if it satisfies the following conditions for any \( x, y, z \in \chi \):

\[
(WUP - 1) (yz) ((xy)(xz)) = 0, \\
(WUP - 2) 0x = x, \\
(WUP - 3) ((x)0)y = 0 \text{ implies } (x)0 = y, \\
(WUP - 4) xy = 0 \text{ and } yx = 0 \text{ imply } x = y.
\]

**Definition 4 (Subalgebra of a WUP-algebra).** A subset \( \mathcal{Y} \) of a WUP-algebra \( \chi \) is called a subalgebra of \( \chi \) if \( \mathcal{Y} \) is closed under the operation of \( \chi \).

Now, we prove the following proposition.

**Proposition 1.** Every UP-algebra \( \chi \) is a WUP-algebra.

**Proof.** It is sufficient to prove WUP-3. Let \( x, y \in \chi \) be such that \( ((x)0)y = 0 \). The condition UP-3 gives \( x0 = 0 \) and hence \( ((x)0)y = 0 \text{ becomes } 0y = 0 \). Consequently, by UP-2 \( y = 0 \). On the other hand, \( ((x)0)0 = (0)0 = 00 = 0 \) and WUP-3 gives \( (x)0 = 0 \). Thus, \( y = (x)0 \). Hence \( \chi \) is a WUP-algebra.

On a WUP-algebra \( \chi = (\chi, \ldots ,0) \), we define the binary relation \( \leq \) by: \( x \leq y \text{ if and only if } xy = x/y \). The following example shows that the converse of Proposition 1 is not true, in general.

**Example 2.** Let \( \chi = \{0,1,2\} \) and the binary operation be defined as in the following table (see Table 2):

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Routine calculations show that \( \chi \) is a WUP-algebra but not a UP-algebra because \( 20 = 2 \neq 0 \).

**Example 3.** Let \( \chi = \{0,a,b,c,d,e,f,g,h\} \) and the binary operation be defined as in the following table (see Table 3):

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**Remark 1.** From Proposition 1 and Examples 2 and 3, we deduce that every UP-algebra is a WUP-algebra but converse is not true. Thus the class of WUP-algebras contains both classes of KU-algebras and UP-algebras. Consequently, we feel that the investigation of the properties of WUP-algebras will add some important results to the mathematical literature.

Now, we prove some important properties of WUP-algebras.
Table 2: A WUP-algebra which is not a UP-algebra.

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Theorem 1. For a WUP-algebra $\chi$ and for any $x, y, z$ in $\chi$, we have:

1. $xx = 0$,
2. $xy = 0$ and $yz = 0$ imply $xz = 0$,
3. $xy = 0$ implies $(zx)(zy) = 0$,
4. $xy = 0$ implies $(yz)(xz) = 0$,
5. $x((y0)(y)(xy)) = 0$.

Proof. The proofs of the parts 1–4 of this theorem are exactly the same as in Proposition 1.7, parts 1–4 (page 39 of [14]). So, we only prove (5). For this, consider $x((y0)(y)(xy)) = (0x)((y0)(y)(xy)) = 0$ (by WUP-1).

Remark 2

1. In Theorem 1 (5), if we substitute $x$ for $y$, then we get $x((x0)(x))(x) = 0$, that is, $x((x0)x) = 0$ which gives $x \leq (x0)0$ for all $x \in \chi$.
2. In an UP-algebra, $y0 = 0$, so Theorem 1 (5) gives $x((0)(y)(x)) = 0$ or $x(y)(x) = 0$, which yields proposition 1.7 (5) of [14].
3. In the sequel, we denote the element $(x0)0$ of a WUP-algebra $\chi$ by $x0$.
4. From Theorem 1, we conclude that the relation $\leq$ on a WUP-algebra $\chi$ is reflexive as well as transitive.

Definition 5 (UP-Part). Let $\chi$ be a WUP-algebra. The UP-part of $\chi$, denoted by $UPP(\chi)$, is defined as $\{y : y \in \chi \text{ such that } y0 = 0\}$.

Example 4. In WUP-algebras of Examples 2-3, their UP-parts are $\{0, 1\}$ and $\{0, a, b, c\}$, respectively.

Definition 6 (Maximal Element). Let $\chi$ be a WUP-algebra. The element $x0 = (x0)0$ corresponding to $x \in \chi$ is called a maximal element of $\chi$. We also call it the maximal element corresponding to $x$ and denote it by $max(x)$. That is $max(x) = x0$.

We denote the set of all maximal elements of a WUP-algebra by $Max(\chi)$. That is, $Max(\chi) = \{x0 : x0 = (x0)0, x \in \chi\}$.

Now, we define the notion of a branch of a WUP-algebra.

Definition 7 (Branch of a WUP-algebra). Let $x0$ be a maximal element of a WUP-algebra $\chi$. Then the branch of $\chi$ determined by $x0$, denoted by $B(x0)$, is the subset $\{y : y \in \chi \text{ and } y0 = x0\}$ of $\chi$.

Remark 3. We note that $B(0) = \{y : y \in \chi \text{ and } y0 = 0\} = UPP(\chi)$.

Example 5. In Example 2, the elements 0, 2 are maximal elements and the branches are $B(0) = \{0, 1\}$ and $B(2) = \{2\}$, whereas in Example 3, 0, c, g, h are maximal elements and the branches are $B(0) = \{0, a, b, c\} = UPP(\chi), B(2) = \{c, d\}$, $B(g) = \{g, f\}$ and $B(h) = \{h\}$.

In the following theorem, we prove a sufficient condition for two elements $x$ and $y$ of a WUP-algebra to have same corresponding maximal elements.

Theorem 2. Let $\chi$ be a WUP-algebra. If $x, y \in \chi$ are such that $x \leq y$, then $x0 = y0$.

Proof. Since $x \leq y$, so by applying Theorem 1 (4) twice we get $(x0)0 \leq (y0)0$. That is, $x0 \leq y0$, which gives $x0y0 = 0$.

Now, by WUP-3, we get $x0 = y0$.

Proposition 2. Let $\chi$ be a WUP-algebra. If $x0 \in Max(\chi)$, then $x0 = (x0)0$.

Proof. Let $x0 \in Max(\chi)$, we consider $x0(x0)0 = (x0)0 \leq (x0)0(x0)(x0)0 = 0$, by WUP-1. Thus $x0(x0)0 = 0$, which by WUP-3 gives $x0 = (x0)0$.

Theorem 3. Let $\chi$ be a WUP-algebra. If $x, y \in \chi$ then $(xy)y0 = x0y0$ and $Max(\chi)$ is a subalgebra of $\chi$.

Proof. Let $x0, y0 \in Max(\chi)$. Then there exist $x, y \in \chi$ such that $x \leq x0$ and $y \leq y0$. By using Theorem 1 (3) and (4), we get $xy \leq xy0$ and $x0y0 \leq xy0$. Using Theorem 2, we get $(xy)y0 = (xy)y00$ and $(x0)y0 = (x0)y00$. Thus $(xy)y0 = (x0)y00$. Since $xy \leq xy0$ and $(xy)y0 = (x0)y00$, therefore $xy \leq (x0)y00$. Now, by Theorem 2 and Proposition 2, we get $(xy)y00 = ((x0)y0)y00 = x0y00$. Hence $(xy)y0 = x0y0$. Since $(xy)y0 \in Max(\chi)$, so $x0y0 \in Max(\chi)$. Thus Max(\chi) is a subalgebra of $\chi$.

Theorem 4. Let $\chi$ be a WUP-algebra. Then for all $x, y \in \chi$, $xy \in B(0) = UPP(\chi)$ if and only if $x$ and $y$ belong to the same branch of $\chi$. 
Proof. Let \( x, y \in B(x_0) \). Then \( x \leq x_0 \) and \( y \leq x_0 \). So by Theorem 1 (3), we get \( xy \leq xx_0 = 0 \). Thus \( xy \leq 0 \), which implies \( xy \in B(0) \). Conversely, let \( x, y \in \chi \) and \( xy \in B(0) \). Thus \( xy \leq 0 \). By Theorem 2, we get \((xy)_0 = 0_0 = (00)_0 = 0 = 0_0 \). This gives \( x_0y_0 = 0 \), which by WUP-3 gives \( x_0 = y_0 \). So \( x, y \) belong to the same branch of \( \chi \). □

**Proposition 3.** Let \( \chi \) be a WUP-algebra. Let \( x \in B(x_0), \ y \in B(y_0) \). Then \( xy \in B(x_0y_0) \).

**Proof.** Clearly, \((xy)_0 \in M(\chi)\). Also \( xy \leq (xy)_0 = x_0y_0 \). Since \( \text{Max}(\chi) \) is a subalgebra of \( \chi \), so \( x_0, y_0 \in \text{Max}(\chi) \) implies \( x_0y_0 \in \text{Max}(\chi) \). Hence, \( xy \leq x_0y_0 \) gives \( xy \in B(x_0y_0) \). □

**Proposition 4.** Let \( \chi \) be a WUP-algebra. Then its UPP-part \( B(0) \) is a subalgebra of \( \chi \).

**Proof.** Let \( x, y \in B(0) \). So by above proposition, we get \( xy \in B(00) = B(0) \). Hence \( B(0) = \text{UPP}(\chi) \) is a subalgebra of \( \chi \).

In the next theorem, we show that the collection of branches produced by all the maximal elements in a WUP-algebra \( \chi \) partitions \( \chi \).

**Theorem 5.** Let \( \chi \) be a WUP-algebra. Let \( M(\chi) \) be the collection of all maximal elements of \( \chi \). Then

(a) \( \chi = \bigcup_{x \in M(\chi)} B(x_0) \).

(b) \( B(x_0) \cap B(y_0) = \emptyset \) for any \( x_0, y_0 \in M(\chi) \) and \( x_0 \neq y_0 \).

**Proof**

(a) Obviously \( B(x_0) \) is a non-empty subset of \( \chi \) because \( x_0 \leq x_0 \) gives \( x_0 \in B(x_0) \). Consequently, \( \cup_{x \in M(\chi)} B(x_0) \subseteq \chi \). Also for each \( x \in \chi \), there is \( x_0 = ((x0)0) \) such that \( x \leq x_0 \), so \( x \in B(x_0) \subseteq \cup_{x \in M(\chi)} B(x_0) \). Hence \( \chi \subseteq \cup_{x \in M(\chi)} B(x_0) \). Combining, we get \( \chi = \bigcup_{x \in M(\chi)} B(x_0) \).

(b) Let \( x_0, y_0 \in M(\chi) \) and \( x_0 \neq y_0 \). Let \( z \in B(x_0) \cap B(y_0) \). So, \( z \leq x_0 \) and \( z \leq y_0 \). That is \( z_0 = 0 \) and \( z_0y_0 = 0 \). Now, WUP-1 gives \((x_0y_0)((z_0x_0)(z_0y_0)) = 0 \). So \((x_0y_0)(00) = 0 \). That is, \((x_0y_0)0 = 0 \). So, \((x_0y_0)0 = 0 \). Now, WUP-3 gives \((x_0y_0)_0 = 0 \). That is, \( x_0y_0 = 0 \). Again, WUP-3 gives \( x_0 = y_0 \), a contradiction. Thus, \( B(x_0) \cap B(y_0) = \emptyset \). □

Remark 4

(1) Every WUP-algebra \( \chi \) is the union of disjoint branches of \( \chi \), determined by its maximal elements.

(2) Every \( x \in \chi \) belongs to a unique branch \( B(x_0) \) determined by its corresponding maximal element \( x_0 \). That is, for every \( x \in \chi \), there is a unique maximal element \( x_0 \) such that \( x \leq x_0 \).

3. Regular Congruences on WUP-Algebras

In this section, we discuss regular congruences on WUP-algebras and the corresponding quotient WUP-algebras.

**Definition 8.** Let \( \chi \) be a WUP-algebra. A relation \( q \) on \( \chi \) is called a congruence on \( \chi \) if for any \( x, y, z, p, r \in \chi \), the following conditions hold:

1. \( xqy \),
2. \( xyq \) implies \( yqx \),
3. \( xyq, yqz \) imply \( xqz \),
4. \( xqy, yqr \) imply \( xqyqr \).

**Definition 9 (Regular Congruence).** A congruence \( q \) on a WUP-algebra \( \chi \) is called a regular congruence if (5) \( xyq0 \) implies \( xqy, y \in \chi \).

**Example 6.** Let \( \chi \) be a WUP-algebra. We define a relation \( q \) by: \( x, y \in \chi \), \( xqy \) if and only if \( x_0 = y_0 \). Then the relation \( q \) is a regular congruence on \( \chi \).

**Verification.**

(1) Let \( x \in \chi \), since \( x_0 = x_0 \) therefore \( xqx \).

(2) Let \( x, y \in \chi \) and \( xyq \), therefore \( x_0 = y_0 \). So \( y_0 = x_0 \). Thus \( yq\).

(3) Let \( x, y, z \in \chi \), \( xqy \) and \( yqz \). Then \( x_0 = y_0 \) and \( y_0 = z_0 \). Hence \( x_0 = z_0 \) and hence \( xqz \).

(4) Let \( x, y, p, r \in \chi \) with \( xqy \) and \( yqr \). Hence \( x_0 = p_0 \) and \( y_0 = r_0 \). Now, consider \((xy)_0 = x_0y_0 = p_0r_0 = (pr)_0\). So, \( xqyqr \). Thus \( q \) is a congruence on \( \chi \). We, now show that it is a regular congruence.

(5) Let \( x, y \in \chi \) such that \( xqy0 \). Thus \((xy)_0 = (00)_0 = 0 \), which gives \( x_0y_0 = 0 \). Now, by WUP-3, we get \( x_0 = y_0 \). This gives \( xqy \). Hence \( q \) is a regular congruence of \( \chi \).

**Definition 10.** Let \( \chi \) be a WUP-algebra and let \( q \) be a congruence on \( \chi \). Let \( x \in \chi \). We define:

\[ [x]_q = \{ y : y \in \chi \text{ and } yqx \}. \]  

(2)

and call \([x]_q\), the class determined by \( x \). Then the set

\[ \chi/q = \{ [x]_q : x \in \chi \}. \]  

(3)

is called the quotient set of \( \chi \) with respect to the congruence \( q \).

**Theorem 6.** Let \( \chi \) be a WUP-algebra and \( q \) a regular congruence on \( \chi \). Let \( \chi/q \) be the quotient set of \( \chi \). Let the binary operation \( \ast \) in \( \chi/q \) be defined as:

\[ [x]_q \ast [y]_q = [xy]_q, \]

(4)

for all \([x]_q, [y]_q \in \chi/q\). Then \((\chi/q, \ast, [0]_q)\) is a WUP-algebra.
Proof. We first show that the operation $*$ defined by (1) is well defined. Let $x, y, p, r \in \chi$ and $[x]_q = [p]_q$ and $[y]_q = [r]_q$. By property of equivalence relation $q$, we have $xqy$ and $yrq$. Since $q$ is a congruence so $xyqypr$. Thus $[x]_q = [p]_q$ and $[y]_q = [r]_q$.

Now, we verify WUP-1, WUP-2, WUP-3 and WUP-4.

(1) We, now consider $(\{ [y]_q^* \} \ast ([x]_q \ast [y]_q)) \ast (\{ [x]_q \ast [y]_q \}) = ([y]_q \ast (\{ [y]_q^* \} \ast ([x]_q^* \ast [x]_q))) \ast [y]_q^* \ast ([x]_q^* \ast [x]_q) = [y]_q \ast [y]_q = [0]_q$.

(2) Let $[x]_q, [0]_q \in \chi/q$. So $[0]_q^* \ast [x]_q = [0x]_q = [x]_q$.

(3) Let $(\{ [x]_q^* \} \ast [0]_q) \ast [0]_q \ast [y]_q = [0]_q$. Then $[(x000)0]_q = [0]_q$. So $[(x000)y]_q$. That is, $x_0y0q0$. Since $q$ is a regular congruence, so $x_0qy$. Hence $[x]_q = [y]_q$. Since $[(x000)y]_q = [0]_q$. Thus, $([x]_q^* \ast [0]_q) \ast [y]_q = [0]_q$ implies $(\{ [x]_q^* \} \ast [0]_q) \ast [0]_q = [y]_q^* \ast [y]_q$.

(4) Let $[x]_q^* \ast [y]_q = [0]_q$ and $[y]_q^* \ast [y]_q = [0]_q$. Thus $[x]_q \ast [y]_q = [0]_q$ and $[y]_q \ast [y]_q = [0]_q$. This implies $xyq0$. Since $q$ is a regular congruence on $\chi$, so $xy$. Thus, $[x]_q = [y]_q$.

Remark 5. We know that, for a WUP-algebra $\chi$, the relation $q$ defined on $\chi$ by: $xy$ if and only if $x_0 = y_0$, that is, if and only if $x, y$ belong to the same branch of $\chi$, is a regular congruence on $\chi$. Thus from Theorem 6, we have the following corollary.

Corollary 1. Let $\chi$ be a WUP-algebra. Let $q$ be the regular congruence on $\chi$ defined by: $x, y \in \chi, xy$ if and only if $x_0 = y_0$. Then $(\chi/q, * , [0]_q)$ is a WUP-algebra with $*$ defined by (1).

Definition 11. Let $\chi_1, \chi_2$ be WUP-algebras. A mapping $\phi: \chi_1 \rightarrow \chi_2$ is called:

(1) a homomorphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \chi_1$.

(2) an isomorphism if $\phi$ is a homomorphism and a bijection.

(3) the WUP-algebras $\chi_1$ and $\chi_2$ are isomorphic if there is an isomorphism $\psi: \chi_1 \rightarrow \chi_2$.

Theorem 7. Let $\chi$ be a WUP-algebra. Let $q$ be the regular congruence on $\chi$ defined by: $x, y \in \chi, xy$ if and only if $x_0 = y_0$. Let $\chi/q$ be the corresponding quotient algebra. Let $\text{Max}(\chi)$ be the subalgebra of $\chi$ consisting of maximal elements of $\chi$. Then $\chi/q$ is isomorphic to $\text{Max}(\chi)$.

Proof. We define $\phi: \chi/q \rightarrow \text{Max}(\chi)$ by $\phi([x]_q) = x_0$ for all $[x]_q \in \chi/q$.

(a) Firstly, we show that $\phi$ is well-defined. Let $[x]_q = [y]_q$, so $xy$. That is $x, y$ belong to the same branch. So $x_0 = y_0$. Hence, $\phi([x]_q) = \phi([y]_q)$.

(b) Now, we show that $\phi$ is a homomorphism. Let $[x]_q, [y]_q \in \chi/q$. Now, $\phi([x]_q \ast [y]_q) = \phi([xy]_q) = (xy)_0 = x_0y_0 = \phi([x]_q)\phi([y]_q)$. So, $\phi$ is a homomorphism.

(c) To show that $\phi$ is onto, let $x_0 \in \text{Max}(\chi)$. Then there exist $[x]_q \in \chi/q$ such that $\phi([x]_q) = x_0$. Hence $\phi$ is onto.

(d) Now, we show that $\phi$ is one-one. Let $[x]_q, [y]_q \in \chi/q$ be such that $\phi([x]_q) = \phi([y]_q)$. So, $x_0 = y_0$. That is $x$ and $y$ belong to the same branch of $\chi$. Hence $xy$. So, $[x]_q = [y]_q$. Therefore, $\phi$ is one-one.

Summing up we get, $\chi/q$ is isomorphic to $\text{Max}(\chi)$.

Example 7. We consider the WUP-algebra $\chi = \{0, a, b, c, d, e, f, g, h\}$ of Example 3. We note that $\text{Max}(\chi) = \{0, d, f, h\}$. We define the congruence $q$ on $\chi$ by:

\[\begin{align*}
(x, y) &\in q; \\
(x, a) &\in q; \\
(x, c) &\in q; \\
(c, a) &\in q; \\
(c, e) &\in q; \\
(d, d) &\in q; \\
(d, e) &\in q; \\
(g, f) &\in q; \\
h &\in q.
\end{align*}\]

Thus

\[\chi/q = \{[0]_q, [d]_q, [f]_q, [h]_q\}.\]

We define a mapping $\psi: \chi/q \rightarrow \text{Max}(\chi)$ by $\psi([0]_q) = 0, \psi([d]_q) = d, \psi([f]_q) = f, \psi([h]_q) = h$. It is easy to verify that $\psi$ is an isomorphism. So $\chi/q$ is isomorphic to $\text{Max}(\chi)$.

4. Conclusion

We have introduced a new class of algebras, known as Weak UP-algebras. This class is so general that it contains the class of KU-algebras as well as the class of UP-algebras. We have studied some basic properties, maximal elements, branches, regular congruences, quotient algebras of this class of algebras. We have also established an isomorphism between $\chi/q$ and $\text{Max}(\chi)$, where $q$ is the regular congruence defined on $\chi$ with the help of its maximal elements and $\text{Max}(\chi)$ is the subalgebra of $\chi$ consisting of maximal elements of $\chi$.

5. Future Work

It will be interesting to explore other aspects of this wider class of WUP-algebras. In particular, proper commutative WUP-algebras, positive implicational WUP-algebras and
implicative WUP-algebras, their properties and interconnection between them. The study of ideals, filters and different types of graphs on all above mentioned classes of algebras will also be interesting.

**Abbreviations**

WUP-algebras: Interval-valued neutrosophic set

UPP: UP-Part

**Data Availability**

No additional data set is used to support the study.

**Conflicts of Interest**

The authors hereby declare that there is no conflicts of interest regarding the publication of this paper.

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**References**


