Hindawi
Journal of Mathematics
Volume 2022, Article ID 1206804, 6 pages
https://doi.org/10.1155/2022/1206804

Research Article

Some Results about Weak UP-algebras

Muhammad Anwar Chaudhry,[1] Asfand Fahad,[2] Muhammad Imran Qureshi,[3] and Urwa Riasat[1]

[1]*Department of Mathematics and Statistics, Institute of Southern Punjab, Multan 60000, Pakistan*
[2]*Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Multan, Pakistan*
[3]*Department of Mathematics, COMSATS University Islamabad Vehari Campus, Vehari 61100, Pakistan*

Correspondence should be addressed to Asfand Fahad; asfandfahad1@yahoo.com

Received 28 April 2022; Accepted 17 August 2022; Published 16 September 2022

Academic Editor: Musavarah Sarwar

We introduce a new class of algebras, weak UP-algebras, briefly written as WUP-algebras, as a wider class than the classes of UP-algebras and KU-algebras. We investigate fundamental aspects including maximal elements, branches and the subalgebra consisting of maximal elements of a WUP-algebra. We also study regular congruences on a WUP-algebra as well as the corresponding quotient WUP-algebras. We prove that the congruence generated from the branches of a WUP-algebra is regular and the corresponding quotient algebra χ/q is isomorphic to the WUP-algebra $\mathrm{Max}(\chi)$, the subalgebra of χ consisting of its maximal elements.

1. Introduction

A few abstract algebras, such as BCK-algebras and BCI-algebras, have their roots in implicational logic, see [1, 2]. Many facets of daily life have benefited substantially from recent advances in artificial intelligence. Logical systems are the fundamental artificial intelligence techniques that aid in decision-making. The evolution and introduction of BCK-algebras and BCI-algebras are driven by the essential axioms of the implicational calculus, see [1, 2]. Taking into account the important linkages between these algebras and related logics, translation techniques have been created to link the theorems and formulae of a logic and the corresponding algebra. As a result, those involved in the fields of artificial intelligence, logical systems, and algebraic structures continue to be interested in the study of abstract algebras (and their generalizations) that have been inspired by logical systems. As a result, the BCH-algebra class, a new class of algebras, has been introduced [3]. Recently, another algebra class, known as gBCH-algebra class, has been introduced and studied in [4, 5]. The significant feature of this new class is its ability to cover all other mentioned classes. In the context of this class, further study through introduction of generalized d-algebras and isomorphism theorems in generalized d-algebras has been developed [6]. For more on such other algebras such as SU-algebra [7], BCC algebra [8] and the significance of these algebras we refer to [9–11]. Although the immediate applications played vital role in getting attention to the above mentioned algebras, however, the development of theoretical aspects of these algebras can not be ignored. Particularly, when they provide different (or a generalized) framework. Consequently, the study on these algebras continued and recently the notion of a KU-algebra [12] was introduced and studied further in [13]. Later, a generalized class of KU-algebras, known as UP-algebras [14] was defined and investigated. The range of the notions studied for these algebras is very vast, such as: the ideals and congruences [12], cubic KU-ideals [13], (α, β)-fuzzy KU-Ideals [15], interval-valued fuzzy KU ideals and (θ, δ)-fuzzy KU-ideals [16, 17], anti fuzzy and intuitionistic fuzzy KU-ideals [18, 19] in KU-algebras. In UP-algebras, the fuzzy sets and the generalized fuzzy sets were introduced and studied in [20, 21]. Moreover, the derivations of UP-algebras have been studied in [22, 23]. In 2019, the isomorphism theorems for UP-algebras were proved [24] and the results were extended later on. For more on the recent related study/applications of these algebras in multiple directions, we refer [25, 26] and references therein. Keeping the above motivation in view, we introduce a general class of algebras, known as weak UP-algebras (WUP-algebras) in such a way that the classes of KU-algebras and UP-algebras become the subclasses of the class of WUP-algebras. Further, we give necessary examples to show that the UP-algebra (consequently) class is a proper sub-class of the WUP-algebra class. Consequently, all investigations of this paper are valid for the KU-algebra class and the UP-algebra class. In particular, we study some fundamental aspects including maximal elements, branches and the subalgebra consisting of maximal elements of a WUP-algebra. We also define regular congruences on a WUP-algebra and study the corresponding quotient algebras. We prove that the congruence q generated from the maximal elements of a WUP-algebra χ is regular and the corresponding quotient algebra χ/q is isomorphic to $\mathrm{Max}(\chi)$, the subalgebra consisting of maximal elements of χ.

2. Weak UP-Algebras (WUP-Algebras)

In this section, we introduce the notion of a Weak UP-algebra and prove that the classes of KU-algebras and UP-algebras are proper subclasses of the class of WUP-algebras. Before proceeding further, we recall the notions of a KU-algebra [12] and an UP-algebra [14].

By an algebra $(\chi, ., 0)$ of type $(2, 0)$, we mean a nonempty set χ together with a binary operation "." and a distinguished element 0 satisfying some specified axioms. Further, for $x.y$, we shall write xy and $(.,.)(.,.)$ will mean "." between the bracketed expressions.

Defintion 1 (KU-algebra). An algebra $\chi = (\chi, ., 0)$ of type $(2, 0)$ is called a KU-algebra if it satisfies the following conditions for any $x, y, z \in \chi$:

$(KU-1)\,(yx)((xz)(yz)) = 0$,
$(KU-2)0x = x$,
$(KU-3)x0 = 0$,
$(KU-4)$ xy = 0 and yx = 0 imply $x = y$.

Defintion 2 (UP-algebra). An algebra $\chi = (\chi, ., 0)$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following conditions for any $x, y, z \in \chi$:

$(UP-1)\,(yz)((xy)(xz)) = 0$,
$(UP-2)0x = x$,
$(UP-3)x0 = 0$,
$(UP-4)$ xy = 0 and yx = 0 imply $x = y$.

It is shown in [14] that every KU-algebra is an UP-algebra. Moreover, it is described in [14] that the algebra of the following Example 1 is an UP-algebra but not a KU-algebra. Consequently, the class of UP-algebras is wider than the class of KU-algebras.

Example 1 (see [14]). *Let*$\chi = \{0, 1, 2, 3, 4\}$*and the binary operation defined as* (see Table1):

Table 1: UP-algebra which is not a KU-algebra.

.	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	2
4	0	2	2	4	0.

We now give our notion of a weak UP-algebra (WUP-algebra).

Defintion 3 (Weak UP-algebra). An algebra $\chi = (\chi, ., 0)$ of type $(2, 0)$ is called a weak UP-algebra (WUP-algebra) if it satisfies the following conditions for any $x, y, z \in \chi$:

$(WUP-1)\,(yz)((xy)(xz)) = 0$,
$(WUP-2)0x = x$,
$(WUP-3)\,((x0)0)y = 0$ implies $(x0)0 = y$,
$(WUP-4)$ xy = 0 and yx = 0 imply $x = y$.

Defintion 4 (Subalgebra of a WUP-algebra). A subset $\mathbb{Y}$ of a WUP-algebra χ is called a subalgebra of χ if $\mathbb{Y}$ is closed under the operation of χ.

Now, we prove the following proposition.

Proposition 1. *Every UP-algebra χ is a WUP-algebra.*

Proof. It is sufficient to prove WUP-3. Let $x, y \in \chi$ be such that $((x0)0)y = 0$. The condition UP-3 gives $x0 = 0$ and hence $((x0)0)y = 0$ becomes $0y = 0$. Consequently, by UP-2 $y = 0$. On the other hand, $((x0)0)0 = (00)0 = 00 = 0$ and WUP-3 gives $(x0)0 = 0$. Thus, $y = (x0)0$. Hence χ is a WUP-algebra.

On a WUP-algebra $\chi = (\chi, ., 0)$, we define the binary relation $\leq$ by: $x \leq y$ if and only if $xy = 0$. The following example shows that the converse of Proposition 1 is not true, in general. □

Example 2. Let $\chi = \{0, 1, 2\}$ and the binary operation be defined as in the following table (see Table 2):

Routine calculations show that χ is a WUP-algebra but not a UP-algebra because $20 = 2 \neq 0$.

Example 3. Let $\chi = \{0, a, b, c, d, e, f, g, h\}$ and the binary operation be defined as in the following table (see Table 3):

Routine calculations show that χ is a WUP-algebra but not a UP-algebra because $g0 = f \neq 0$.

Remark 1. From Proposition 1 and Examples 2 and 3, we deduce that every UP-algebra is a WUP-algebra but converse is not true. Thus the class of WUP-algebras contains both classes of KU-algebras and UP-algebras. Consequently, we feel that the investigation of the properties of WUP-algebras will add some important results to the mathematical literature.

Now, we prove some important properties of WUP-algebras.

Table 2: A WUP-algebra which is not a UP-algebra.

.	0	1	2
0	0	1	2
1	0	0	2
2	2	2	0

Table 3: A WUP-algebra which is not a UP-algebra.

.	0	a	b	c	d	e	f	g	h
0	0	a	b	c	d	e	f	G	h
a	0	0	b	b	d	e	f	F	h
b	0	a	0	a	d	d	f	g	h
c	0	0	0	0	d	d	f	F	h
d	d	d	E	e	0	b	h	H	f
e	d	d	d	d	0	0	h	H	h
f	f	g	F	g	h	h	0	A	h
g	f	f	F	f	h	h	0	0	d
h	h	h	H	h	F	f	d	d	0

Theorem 1. *For a WUP-algebra χ and for any x, y, z in χ, we have:*

(1) $xx = 0$,

(2) $xy = 0$ *and* $yz = 0$ *imply* $xz = 0$,

(3) $xy = 0$ *implies* $(zx)(zy) = 0$,

(4) $xy = 0$ *implies* $(yz)(xz) = 0$,

(5) $x((y0)(yx)) = 0$.

Proof. The proofs of the parts $1-4$ of this theorem are exactly the same as in Proposition 1.7, parts $1-4$ (page 39 of [14]). So, we only prove (5). For this, consider $x((y0)(yx)) = (0x)((y0)(yx)) = 0$ (by WUP-1). □

Remark 2

(1) In Theorem 1 (5), if we substitute x for y, then we get $x((x0)(xx)) = 0$, that is, $x((x0)0) = 0$ which gives $x \leq (x0)0$ for all $x \in \chi$.

(2) In an UP-algebra, $y0 = 0$, so Theorem 1 (5) gives $x(0(yx)) = 0$ or $x(yx) = 0$, which yields proposition 1.7 (5) of [14].

(3) In the sequel, we denote the element $(x0)0$ of a WUP-algebra χ by x_0.

(4) From Theorem 1, we conclude that the relation $\leq$ on a WUP-algebra χ is reflexive as well as transitive.

Defintion 5 (UP-Part). Let χ be a WUP-algebra. The UP-part of χ, denoted by $UPP(\chi)$, is defined as $\{y\colon y \in \chi \text{such that} y0 = 0\}$.

Example 4. In WUP-algebras of Examples 2-3, their UP-parts are $\{0, 1\}$ and $\{0, a, b, c\}$, respectively.

Defintion 6 (Maximal Element). Let χ be a WUP-algebra. The element $x_0 = (x0)0$ corresponding to $x \in \chi$ is called a maximal element of χ. We also call it the maximal element corresponding to x and denote it by $\max(x)$. That is $\max(x) = x_0$.

We denote the set of all maximal elements of a WUP-algebra by $Max(\chi)$. That is, $Max(\chi) = \{x_0\colon x_0 = (x0)0, x \in \chi\}$.

Now, we define the notion of a branch of a WUP-algebra.

Defintion 7 (Branch of a WUP-algebra). Let x_0 be a maximal element of a WUP-algebra χ. Then the branch of χ determined by x_0, denoted by $B(x_0)$, is the subset $\{y\colon y \in \chi \text{and} y \leq x_0\} = \{y\colon y \in \chi \text{and} y x_0 = 0\}$ of χ.

Remark 3. We note that

$$B(0) = \{y\colon y \in \chi \text{and} y \leq 0\} = \{y\colon y \in \chi \text{and} y0 = 0\} = UPP(\chi). \quad (1)$$

Example 5. In Example 2, the elements $0, 2$ are maximal elements and the branches are $B(0) = \{0, 1\}$ and $B(2) = \{2\}$, whereas in Example 3, $0, e, g, h$ are maximal elements and the branches are $B(0) = \{0, a, b, c\} = UPP(\chi)$, $B(e) = \{e, d\}$, $B(g) = \{g, f\}$ *and* $B(h) = \{h\}$.

In the following theorem, we prove a sufficient condition for two elements x and y of a WUP-algebra to have same corresponding maximal elements.

Theorem 2. *Let χ be a WUP-algebra. If $x, y \in \chi$ are such that $x \leq y$, then $x_0 = y_0$.*

Proof. Since $x \leq y$, so by applying Theorem 1 (4) twice we get $(x0)0 \leq (y0)0$. That is, $x_0 \leq y_0$, which gives $x_0 y_0 = 0$. Now, by WUP-3, we get $x_0 = y_0$. □

Proposition 2. *Let χ be a WUP-algebra. If $x_0 \in Max(\chi)$, then $x_0 = (x_0)_0$.*

Proof. Let $x_0 \in \mathrm{Max}(\chi)$, we consider $x_0(x_0)_0 = (x_0)((x_0 0)0) = (0x_0)((x_0 0)(x_0 x_0)) = 0$, by WUP-1. Thus $x_0(x_0)_0 = 0$, which by WUP-3 gives $x_0 = (x_0)_0$. □

Theorem 3. *Let χ be a WUP-algebra. If $x, y \in \chi$ then $(xy)_0 = x_0 y_0$ and $Max(\chi)$ is a subalgebra of χ.*

Proof. Let $x_0, y_0 \in \mathrm{Max}(\chi)$. Then there exist $x, y \in \chi$ such that $x \leq x_0$ and $y \leq y_0$. By using Theorem 1 (3) and (4), we get $xy \leq xy_0$ and $x_0 y_0 \leq xy_0$. Using Theorem 2, we get $(xy)_0 = (xy_0)_0$ and $(x_0 y_0)_0 = (xy_0)_0$. Thus $(xy)_0 = (x_0 y_0)_0$. Since $xy \leq (xy)_0$ and $(xy)_0 = (x_0 y_0)_0$, therefore $xy \leq (x_0 y_0)_0$. Now, by Theorem 2 and Proposition 2, we get $(xy)_0 = ((x_0 y_0)_0)_0 = x_0 y_0$. Hence $(xy)_0 = x_0 y_0$. Since $(xy)_0 \in \mathrm{Max}(\chi)$, so $x_0 y_0 \in \mathrm{Max}(\chi)$. Thus $\mathrm{Max}(\chi)$ is a subalgebra of χ. □

Theorem 4. *Let χ be a WUP-algebra. Then for all $x, y \in \chi$, $xy \in B(0) = UPP(\chi)$ if and only if x and y belong to the same branch of χ.*

Proof. Let $x, y \in B(x_0)$. Then $x \leq x_0$ and $y \leq x_0$. So by Theorem 1 (3), we get $xy \leq xx_0 = 0$. Thus $xy \leq 0$, which implies $xy \in B(0)$. Conversely, let $x, y \in \chi$ and $xy \in B(0)$. Thus $xy \leq 0$. By Theorem 2, we get $(xy)_0 = 0_0 = (00)0 = 00 = 0$. This gives $x_0 y_0 = 0$, which by WUP-3 gives $x_0 = y_0$. So x, y belong to the same branch of χ. □

Proposition 3. *Let χ be a WUP-algebra. Let $x \in B(x_0)$, $y \in B(y_0)$. Then $xy \in B(x_0 y_0)$*

Proof. Clearly, $(xy)_0 \in M(\chi)$. Also $xy \leq (xy)_0 = x_0 y_0$. Since $\mathrm{Max}(\chi)$ is a subalgebra of χ, so $x_0, y_0 \in \mathrm{Max}(\chi)$ implies $x_0 y_0 \in Max(\chi)$. Hence, $xy \leq x_0 y_0$ gives $xy \in B(x_0 y_0)$ □ □

Proposition 4. *Let χ be a WUP-algebra. Then its UPP-part $B(0)$ is a subalgebra of χ.*

Proof. Let $x, y \in B(0)$. So by above proposition, we get $xy \in B(00) = B(0)$. Hence $B(0) = UPP(\chi)$ is a subalgebra of χ.

In the next theorem, we show that the collection of branches produced by all the maximal elements in a WUP-algebra χ partitions χ: □

Theorem 5. *Let χ be a WUP-algebra. Let $M(\chi)$ be the collection of all maximal elements of χ. Then*

(a) $\chi = \cup_{x_0 \in M(\chi)} B(x_0)$.
(b) $B(x_0) \cap B(y_0) = \phi$ *for any* $x_0, y_0 \in M(\chi)$ *and* $x_0 \neq y_0$.

Proof

(a) Obviously $B(x_0)$ is a non-empty subset of χ because $x_0 \leq x_0$ gives $x_0 \in B(x_0)$. Consequently, $\cup_{x_0 \in M(\chi)} B(x_0) \subseteq \chi$. Also for each $x \in \chi$, there is $x_0 = ((x0)0)$ such that $x \leq x_0$, so $x \in B(x_0) \subseteq \cup_{x_0 \in M(\chi)} B(x_0)$. Hence $\chi \subseteq \cup_{x_0 \in M(\chi)} B(x_0)$. Combining, we get $\chi = \cup_{x_0 \in M(\chi)} B(x_0)$.
(b) Let $x_0, y_0 \in M(\chi)$ and $x_0 \neq y_0$. Let $z \in B(x_0) \cap B(y_0)$. So, $z \leq x_0$ and $z \leq y_0$. That is $zx_0 = 0$ and $zy_0 = 0$. Now, WUP-1 gives $(x_0 y_0)((zx_0)(zy_0)) = 0$. So, $(x_0 y_0)(00) = 0$. That is, $(x_0 y_0)0 = 0$. So, $(xy)_0 0 = 0$. Now, WUP-3 give $(xy)_0 0 = 0$. That is, $x_0 y_0 = 0$. Again, WUP-3 gives $x_0 = y_0$, a contradiction. Thus, $B(x_0) \cap B(y_0) = \phi$. □

Remark 4

(1) Every WUP-algebra χ is the union of disjoint branches of χ, determined by its maximal elements.
(2) Every $x \in \chi$ belongs to a unique branch $B(x_0)$ determined by its corresponding maximal element x_0. That is, for every $x \in \chi$, there is a unique maximal element x_0 such that $x \leq x_0$.

3. Regular Congruences on WUP-Algebras

In this section, we discuss regular congruences on WUP-algebras and the corresponding quotient WUP-algebras.

Defintion 8. Let χ be a WUP-algebra. A relation q on χ is called a congruence on χ if for any $x, y, z, p, r \in \chi$:, the following conditions hold:

(1) xqx,
(2) xqy implies yqx,
(3) xqy, yqz imply xqz
(4) xqp, yqr imply $xyqpr$.

Defintion 9 (Regular Congruence). A congruence q on a WUP-algebra χ is called a regular congruence if (5)$xyq0$ implies xqy, $x, y \in \chi$.

Example 6. Let χ be a WUP-algebra. We define a relation q by: $x, y \in \chi$, xqy if and only if $x_0 = y_0$. Then the relation q is a regular congruence on χ.

Verification.

(1) Let $x \in \chi$, since $x_0 = x_0$ therefore xqx.
(2) Let $x, y \in \chi$ and xqy, therefore $x_0 = y_0$. So $y_0 = x_0$. Thus yqx.
(3) Let $x, y, z \in \chi$, xqy and yqz. Then $x_0 = y_0$ and $y_0 = z_0$. Thus $x_0 = z_0$ and hence xqz.
(4) Let $x, y, p, r \in \chi$ with xqp and yqr. Hence $x_0 = p_0$ and $y_0 = r_0$. Now, consider $(xy)_0 = x_0 y_0 = p_0 r_0 = (pr)_0$. So, $xyqpr$. Thus q is a congruence on χ. We, now show that it is a regular congruence.
(5) Let $x, y \in \chi$ such that $xyq0$. Thus $(xy)_0 = (0)_0 = 0$, which gives $x_0 y_0 = 0$. Now, by WUP-3, we get $x_0 = y_0$. This gives xqy. Hence q is a regular congruence of χ.

Defintion 10. Let χ be a WUP-algebra and let q be a congruence on χ. Let $x \in \chi$. We define:

$$[x]_q = \{y\colon\ y \in \chi \text{and} yqx\}. \tag{2}$$

and call $[x]_q$, the class determined by x. Then the set

$$\chi/q = \left\{[x]_q\colon\ x \in \chi\right\}. \tag{3}$$

is called the quotient set of χ with respect to the congruence q.

Theorem 6. *Let χ be a WUP-algebra and q a regular congruence on χ. Let χ/q be the quotient set of χ. Let the binary operation $*$ in χ/q be defined as:*

$$[x]_q * [y]_q = [xy]_q, \tag{4}$$

for all $[x]_q, [y]_q \in \chi/q$. Then $(\chi/q, *, [0]_q)$ is a WUP-algebra.

Proof. We first show that the operation * defined by (1) is well defined. Let $x, y, p, r \in \chi$ and $[x]_q = [p]_q$ and $[y]_q = [r]_q$. By property of equivalence relation q, we have xqp and yqr. Since q is a congruence so $xyqpr$. Thus $[xy]_q = [pr]_q$. That is $[x]_q{}^*[y]_q = [p]_q{}^*[r]_q$.

Now, we verify WUP-1, WUP-2, WUP-3 and WUP-4.

(1) We, now consider $([y]_q{}^*[z]_q)^*(([x]_q{}^*[y]_q)^*([x]_q{}^*[z]_q)) = [yz]_q{}^*([xy]_q{}^*[xz]_q) = [(yz)(((xy)((xz)))))]_q = [0]_q$.

(2) Let $[x]_q, [0]_q \in \chi/q$, so $[0]_q{}^*[x]_q = [0x]_q = [x]_q$.

(3) Let $(([x]_q{}^*[0]_q)^*[0]_q)^*[y]_q = [0]_q$. Then $[((x0)0)y]_q = [0]_q$. So $((x0)0)yq0$. That is, x_0yq0. Since q is a regular congruence, so x_0qy. Hence $[x_0]_q = [y]_q$. So $[(x0)0]_q = [y]_q$. Thus, $(([x]_q{}^*[0]_q)^*[0]_q)^*[y]_q = [0]_q$ implies $([x]_q{}^*[0]_q)^*[0]_q = [y]_q$.

(4) Let $[x]_q{}^*[y]_q = [0]_q$ and $[y]_q{}^*[x]_q = [0]_q$. Thus $[xy]_q = [0]_q$ and $[yx]_q = 0$. This implies $xyq0$. Since q is a regular congruence on χ, so xqy. Thus, $[x]_q = [y]_q$. □

Remark 5. We know that, for a WUP = algebra χ, the relation q defined on χ by: xqy if and only if $x_0 = y_0$, that is, if and only if x, y belong to the same branch of χ, is a regular congruence on χ. Thus from Theorem 6, we have the following corollary.

Corollary 1. *Let χ be a WUP-algebra. Let q be the regular congruence on χ defined by: $x, y \in \chi$, xqy if and only if $x_0 = y_0$. Then $(\chi/q, {}^*, [0]_q)$ is a WUP-algebra with * defined by (1).*

Defintion 11. Let χ_1, χ_2 be WUP-algebras. A mapping $\phi\colon \chi_1 \longrightarrow \chi_2$ is called:

(1) a homomorphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \chi_1$.

(2) an isomorphism if ϕ is a homomorphism and a bijection.

(3) the WUP-algebras χ_1 and χ_2 are isomorphic if there is an isomorphism $\psi\colon \chi_1 \longrightarrow \chi_2$.

Theorem 7. *Let χ be a WUP-algebra. Let q be the regular congruence on χ defined by: $x, y \in \chi$, xqy if and only if $x_0 = y_0$. Let χ/q be the corresponding quotient algebra. Let $Max(\chi)$ be the subalgebra of χ consisting of maximal elements of χ. Then χ/q is isomorphic to $Max(\chi)$.*

Proof. We define $\phi\colon \chi/q \longrightarrow \mathrm{Max}(\chi)$ by $\phi([x]_q) = x_0$ for all $[x]_q \in \chi/q$.

(a) Firstly, we show that ϕ is well-defined. Let $[x]_q = [y]_q$, so xqy. That is x, y belong to the same branch. So $x_0 = y_0$. Hence, $\phi([x]_q) = \phi([y]_q)$.

(b) Now, we show that ϕ is homomorphism. Let $[x]_q, [y]_q \in \chi/q$. Now, $\phi([x]_q[y]\ \ q) = \phi([xy]_q) = (xy)_0 = x_0y_0 = \phi([x]_q)\phi([y]_q)$. So, ϕ is a homomorphism.

(c) To show that ϕ is onto, let $x_0 \in \mathrm{Max}(\chi)$. Then there exist $[x_0]_q \in \chi/q$ such that $\phi([x_0]_q) = x_0$. Hence ϕ is onto.

(d) Now, we show that ϕ is one-one. Let $[x]_q, [y]_q \in \chi/q$ be such that $\phi([x]_q) = \phi([y]_q)$. So, $x_0 = y_0$. That is x and y belong to the same branch of χ. Hence xqy. So, $[x]_q = [y]_q$. Therefore, ϕ is one-one.

Summing up we get, χ/q is isomorphic to $\mathrm{Max}(\chi)$. □

Example 7. We consider the WUP-algebra $\chi = \{0, a, b, c, d, e, f, g, h\}$ of Example 3. We note that $Max(\chi) = \{0, d, f, h\}$. We define the congruence q on χ by: $x, y \in \chi$ and xqy if and only if x, y belong to the same branch of χ. So

$$\begin{aligned} q = \{&(0,0), (0,b), (0,a), (0,c), (b,0), (a,0), (c,0), (b,a), (a,b), (b,c), (c,b),\} \\ &(c,a), (a,c), (e,e), (d,d), (e,d), (d,e), (g,f), (f,g), (h,h)\}. \end{aligned} \tag{5}$$

Thus

$$\chi/q = \left\{[0]_q, [d]_q, [f]_q, [h]_q\right\}. \tag{6}$$

We define a mapping $\psi\colon \chi/q \longrightarrow \mathrm{Max}(\chi)$ by $\psi([0]_q) = 0, \psi([d]_q) = d, \psi([f]_q) = f, \psi([h]_q) = h$. It is easy to verify that ψ is an isomorphism. So χ/q is isomorphic to $\mathrm{Max}(\chi)$.

4. Conclusion

We have introduced a new class of algebras, known as Weak UP-algebras. This class is so general that it contains the class of KU-algebras as well as the class of UP-algebras. We have studied some basic properties, maximal elements, branches, regular congruences, quotient algebras of this class of algebras. We have also established an isomorphism between χ/q and $\mathrm{Max}(\chi)$, where q is the regular congruence defined on χ with the help of its maximal elements and $\mathrm{Max}(\chi)$ is the subalgebra of χ consisting of maximal elements of χ.

5. Future Work

It will be interesting to explore other aspects of this wider class of WUP-algebras. In particular, proper commutative WUP-algebras, positive implicative WUP-algebras and implicative WUP-algebras, their properties and interconnection between them. The study of ideals, filters and different types of graphs on all above mentioned classes of algebras will also be interesting.

Abbreviations

WUP-algebras:	Interval-valued neutrosophic set
UPP:	UP-Part

Data Availability

No additional data set is used to support the study.

Conflicts of Interest

The authors hereby declare that there is no conflicts of interest regarding the publication of this paper.

Acknowledgments

All the authors are thankful to their respective institutes for providing the excellent research facilities.

References

[1] Y. Imai and K. Iseki, "On axiom systems of propositional calculi, XlV," *Proceedings of the Japan Academy*, vol. 2, pp. 19–22, 1966.

[2] K. Iseki, "An algebra related with a Propositional Calculus," *Proceedings of the Japan Academy Series A: Mathematical Sciences*, vol. 42, no. 1, pp. 26–29, 1966.

[3] Q. P. Hu and X. Li, "On BCH-algebras," *Mathematics seminar notes*, vol. 11, pp. 313–320, 1983.

[4] M. A. Chaudhry, A. Fahad, and M. Arshad, "Some results about generalized BCH-algebras," *International Journal of Algebra*, vol. 11, no. 5, pp. 231–246, 2017.

[5] M. Anwar Chaudhry, A. Fahad, Y. Rao, M. Imran Qureshi, and S. Gulzar, "Branchwise solid generalized BCH-algebras," *AIMS Mathematics*, vol. 5, no. 3, pp. 2424–2432, 2020.

[6] M. A. Chaudhry, M. I. Qureshi, A. Fahad, and M. S. Bashir, "Isomorphism theorems in generalized -algebras," *Journal of Prime Research in Mathematics*, vol. 17, no. 2, pp. 149–158, 2021.

[7] S. Keawrahun and U. Leerawat, "On isomorphisms of SU-algebra," *Scientia Magna*, vol. 7, no. 2, pp. 39–44, 2011.

[8] W. A. Dudek, "On BCC-algebras," *Logique et. Analyse*, vol. 129-130, pp. 103–111, 1990.

[9] W. M. Bunder, "BCK and related algebras and their corresponding logics," *The Journal of non-classical logic*, vol. 7, pp. 15–24, 1984.

[10] W. A. Dudek, "Algebras inspired by logics," *Matematychni Studii*, vol. 14, pp. 3–18, 2000.

[11] W. A. Dudek, "Algebraic characterization of the equivalential calculus," *Rivista di Matematica Pura ed Applicata*, vol. 17, pp. 107–112, 1996.

[12] C. Prabpayak and U. Leerawat, "On ideals and congruences in KU-algebras," *Scientia Magna*, vol. 5, no. 1, pp. 54–57, 2009.

[13] M. Akram, N. Yaqoob, and M. Gulistan, "Cubic KU-algebras," *International Journal of Pure and Applied Mathematics*, vol. 89, no. 5, pp. 659–665, 2013.

[14] A. Iampan, "A new branch of logical algebra: UP-algebras," *Journal of Algebra and Related Topics*, vol. 5, no. 1, pp. 35–54, 2017.

[15] M. Gulistan, M. Shahzad, and S. Ahmed, "On (α, β) -fuzzy KU-ideals of KU-algebras," *Afrika Matematika*, vol. 26, no. 3-4, pp. 651–661, 2015.

[16] M. Akram, N. Yaqoob, and J. Kavikumar, "Interval-valued (θ, S)-fuzzy KU-ideals of KU-algebras," *Afrika Matematika*, vol. 92, no. 3, pp. 335–349, 2014.

[17] S. M. Mostafa, M. A. A. Naby, and O. R. Elgendy, "Interval-valued fuzzy KUideals in KU-algebras," *International Mathematical Forum*, vol. 6, no. 64, pp. 3151–3159, 2011.

[18] S. M. Mostafa, M. A. Abdel Naby, and M. M. M. Yousef, "Anti-fuzzy KU-ideals of KU-algebras," *International Journal of Algebra and Statistics*, vol. 1, no. 1, pp. 92–99, 2012.

[19] S. M. Mostafa, M. A. A. Naby, and O. R. Elgendy, "Intuitionistic fuzzy KU-ideals in KU-algebras," *International Journal of Mathematical Sciences & Applications*, vol. 1, no. 3, pp. 1379–1384, 2011.

[20] N. Dokhamdang, A. Kesorn, and A. Iampan, "Generalized fuzzy sets in UP-algebras," *Annals of Fuzzy Mathematics and Informatics*, vol. 16, no. 2, pp. 171–190, 2018.

[21] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, "Fuzzy sets in UP-algebras," *Annals of Fuzzy Mathematics and Informatics*, vol. 12, pp. 739–756, 2016.

[22] A. Iampan, "Derivation of UP-algebras by means of UP-endomorphisms," *Journal of Algebraic Structures and their Applications*, vol. 3, no. 2, pp. 1–20, 2016.

[23] K. Sawika, R. Intasan, A. Kaewwasri, and A. Iampan, "Derivation of UP-algebras," *Korean Journal of Mathematics*, vol. 24, no. 3, pp. 345–367, 2016.

[24] A. Iampan, "The UP-Isomorphism theorems for UP-algebras," *Discussiones Mathematicae General Algebra and Applications*, vol. 39, no. 1, pp. 113–123, 2019.

[25] S. Yuphaphin, P. Kankaew, N. Lapo, R. Chinram, and A. Iampan, "Picture fuzzy sets in UP-algebras by means of a special type," *The Journal of Mathematics and Computer Science*, vol. 25, no. 01, pp. 37–72, 2021.

[26] N. Lapo, S. Yuphaphin, P. Kankaew, R. Chinram, and A. Iampan, "Interval-valued picture fuzzy sets in UP-algebras by means of a special type," *Afrika Matematika*, vol. 33, pp. 55–43, 2022.