Research Article

Existence and Monotone Iterative Approximation of Solutions for Neutral Differential Equations with Generalized Fractional Derivatives

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We study the existence and monotone iterative approximation of mild solutions of fractional-order neutral differential equations involving a generalized fractional derivative of order $0 < \alpha < 1$ which can be reduced to Riemann–Liouville or Hadamard fractional derivatives. The existence of mild solutions is obtained via fixed point techniques in a partially ordered space. The approach is constructive and can be applied numerically. In particular, we construct a monotone sequence of functions converging to a solution which is illustrated by a numerical example.

1. Introduction

Differential equations with fractional order, commonly referred to as fractional differential equations, have important role for describing the dynamics of economic models [1], finance [2], engineering [3], and scientific systems [4–8]. In recent years, various types of the fractional derivatives are defined via fractional integrals and were studied including Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard, Hilfer, Hilfer–Hadamard, Caputo-type, and Liouville fractional derivatives [9–14].

Fractional calculus has triggered several research interests in various aspects including existence and uniqueness of solutions, stability and dynamical analysis, numerical methods for solutions, and modelling. Some authors investigated the nonlocal fractional operator based on a generalized Mittag-Leffler kernel [15, 16]. Dynamics of the fractional differential system is also the main research focus in many applications, for example, HIV modelling [8, 17, 18], COVID-19 modelling [7], and integrator circuit model [19].

In many applications, differential equations involve functions and derivatives that contain unknown functions at a shifted time. This class of differential equation is called a delay differential equation and is commonly used to describe systems when the rate of change depends on the stages in the previous times. Delay differential equations have become important for mathematical models of phenomena in physical, engineering, and biomedical sciences. Especially, many authors have studied epidemic models with time delays [20–22].

One special type of delay differential equation is the neutral differential equation which has been extensively studied by many researchers. Agarwal and Bahuguna [23] studied the existence and solution of the first-order neutral differential equation of Sobolev type with a nonlocal history condition:
\[
\frac{d}{dt} \left[ Bu(t) + f(t, u(t - \tau_i)) \right] + Au(t) = g(t, u(t), u(t - \tau_2)), 0 < t \leq T
\]

where \( \tau = \max\{\tau_1, \tau_2\}, \tau_i > 0, i = 1, 2, T < \infty, \phi \in \rho_{0} := C([-\tau, 0], X) \) in a real Banach space \( X \). The results were proved by using Schauder’s fixed point theorem. Agarwal et al. [24] studied the existence of fractional neural functional differential equations:

\[
C^{\alpha}D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in (t_0, +\infty), t_0 \geq 0 \quad (2)
\]

\( x_{t_0} = \phi, \)

subject to the initial condition

\( x(t) = \xi(t), \quad t \in [k_0, t_0], \quad (6) \)

with \( \alpha \in (0, 1), \rho > 0, 0 \leq \beta \leq 1 \) and \( \gamma = \alpha + \beta(1 - \alpha), \) where \( f \in C([k_0, T] \times \mathbb{R}, \mathbb{R}), g \in C([k_0, T] \times \mathbb{R}, \mathbb{R}), k_0 = \inf_{t \in (t_0, T)} (t - \tau(t)), \tau : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+} \) and the initial \( \xi : [k_0, t_0] \rightarrow \mathbb{R} \). Here, \( \rho \mathcal{D}_{k_0}^{\alpha, \beta} \) is the standard generalized fractional derivative of order \( \alpha \in (0, 1) \).

The main contribution in this paper is that we consider the constructive approach for the existence of solutions for generalized fractional differential equations which can be applied numerically. This generalized fractional derivative can be reduced into Riemann–Liouville fractional derivative, Caputo fractional derivative, and Hadamard fractional derivative as follows. If \( \rho \rightarrow 1 \) and \( \beta = 0, \) our existence result reduces to the existence of mild solutions with respect to the Riemann–Liouville fractional derivative. On the other hand, if \( \rho \rightarrow 1 \) and \( \beta = 1, \) our result implies the existence of mild solution when the Caputo fractional derivative is considered. In addition, if \( \rho \rightarrow 0^{+} \) and \( \beta = 0, \) we obtain the existence of mild solutions for the problem with Hadamard fractional derivative. Furthermore, based on the monotone iterative approach, our result gives a constructive method to approximate the mild solutions. By constructing a monotone iterative sequence of functions from upper or lower solutions, we obtain a sequence of functions converging to the mild solutions.

This paper can be outlined as follows. We introduce some preliminary background in Section 2. We present the existence and monotone iterative approximation result for mild solutions of fractional-order neutral differential equation with delay in the sense of generalized fractional derivative for the Riemann–Liouville and the Hadamard derivatives in Section 3. Finally, in Section 4, we give an example to demonstrate the existence of mild solution of IVP (5).
2. Preliminary Results for Neutral Differential Equations with Generalized Fractional Derivatives

Definition 1 (see [28]). Let $a$ and $b$ be real numbers with $0 < a < b < \infty$. Let $\Omega = [a, b]$ be finite interval on the non-negative numbers $\mathbb{R}^+$. For $\rho > 0$ and $0 \leq \gamma < 1$, we define the space of functions as follows.

$$\mathcal{C}_{\gamma-\rho}[a, b] = \left\{ \varphi: (a, b] \rightarrow \mathbb{R}: \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} \varphi(t) \in C[a, b] \right\}, \quad 0 \leq \gamma < 1,$$

equipped with the norm

$$\|\varphi\|_{C_{\gamma-\rho}} = \max_{t \in \Omega} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} \varphi(t),$$

and $C_{0,\rho}[a, b] = C[a, b]$.

Definition 2. Let $\alpha$ and $\gamma$ be real numbers with $\gamma > 0$ and $f \in \mathcal{L}_1^\gamma(a, b)$, where $\mathcal{L}_1^\gamma(a, b)$ denotes the space of Lebesgue measurable functions. The left-sided generalized fractional integral is given by

$$\rho \mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\gamma-1} f(s) \frac{du}{u^{1-\rho}}, \quad t > a. \quad (11)$$

It should be mentioned that once $\rho = 1$, the integral in (11) becomes the Riemann–Liouville fractional integral. In case that one takes the limit as $\rho \to 0$ in (5), it becomes the Hadamard fractional integral.

It can be seen that the operator $\rho \mathcal{I}_a^\alpha f$ can be expressed as

$$\rho \mathcal{I}_a^\alpha f = \rho \mathcal{I}_a^{\beta(1-\gamma)} \rho \mathcal{I}_a^{\gamma-1} f = \rho \mathcal{I}_a^{\beta(1-\gamma)} \rho \mathcal{I}_a^{\gamma-1} f = \alpha + \beta (1 - \alpha). \quad (12)$$

Definition 3. Let $\alpha$ be a positive real number such that $\alpha \notin \mathbb{N}$ and $n = [\alpha + 1]$, where $[\alpha]$ is the integer part of $\alpha > 0$.

(1) $C[a, b]$ is the space of continuous functions $\varphi$ on $\Omega$ with the norm

$$\|\varphi\|_C = \max_{t \in \Omega} |\varphi(t)|.$$

(2) The weighted space $C_{1-\gamma-\rho}[a, b]$ is the set

$$C_{1-\gamma-\rho}[a, b] = \left\{ \varphi \in C[a, b], \rho \mathcal{I}_a^{\gamma-\rho} \varphi(t) \in C[a, b] \right\}, \quad 0 \leq \gamma < 1,$$

(3) The weighted space $C_{0,\rho}[a, b]$ is the set

$$C_{0,\rho}[a, b] = C[a, b].$$

The left-sided generalized fractional derivative is defined by

$$\rho \mathcal{D}_a^\alpha f(t) = \frac{\varphi^n}{\Gamma(n-\alpha)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\gamma-a-1} f(s) \frac{du}{u^{1-\rho}}, \quad \rho > 0 \text{ and } \varphi = t^{1-\rho}(d/dt).$$

It should be noted that once $\rho = 1$, the derivative in (13) becomes the Riemann–Liouville fractional derivative. Also by taking the limit as $\rho \to 0$, the derivative in (13) becomes the Hadamard fractional derivative.

Definition 4. Let $\alpha$ and $\beta$ be real numbers such that $0 < \alpha < 1$ and $0 \leq \gamma < 1$. If $f \in C_{\gamma+1}[a, b]$ and $\rho \mathcal{I}_a^{\gamma-\rho} f \in C_{\gamma}[a, b]$, then

$$\rho \mathcal{D}_a^\alpha \rho \mathcal{I}_a^{\gamma-\rho} f(t) = f(t) - \frac{\varphi^n}{\Gamma(n-\alpha)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\gamma-a-1} f(s) \frac{du}{u^{1-\rho}},$$

for all $t \in (a, b]$.

Definition 5. The function $x \in C_{1-\gamma}[k_0, T]$ is a mild solution of the initial value problem for the neutral fractional differential equations with generalized fractional derivative (5) if the following integral equation is satisfied:
\[
x(t) = g(t, x(t - \tau(t))) + \frac{1}{\Gamma(y)} \int_{t_0}^{t} \left( \frac{t^n - s^n}{\rho^n} \right)^{y-1} L(s) \frac{ds}{s^{1-p}} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^n - s^n}{\rho^n} \right)^{\alpha-1} f(x(s), x(s - \tau(s))) \frac{ds}{s^{1-p}}, \quad t \in [t_0, T],
\]

and \(x(t) = \xi(t)\) for \(t \in [k_0, t_0]\), where \(L(t_0) = x(t_0) - g(t_0, x(t_0 - \tau(t_0)))\).

Consider the Banach space \(C_{1-}([k_0, T])\) equipped with partially order relation. For any \(x_1, x_2 \in C_{1-}([k_0, T])\), we define the order relation \(x_1 \leq x_2\) if and only if \(x_1(t) \leq x_2(t)\) with respect to \(t\) on \([k_0, T]\). This defines a partial ordering on \(C_{1-}([k_0, T])\). We next outline the preliminary results on partially ordered space. Let \(X = (X, \leq, \|\|)\) be a normed linear space with a partially ordered relation \(\leq\). The space \(X\) is called regular if for any nonconverging sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) such that \(x_n\) converges to \(y\) as \(n \to \infty\), we have \(x_n \leq y\) for all \(n \in \mathbb{N}\). In particular, we see that the space \(C_{1-}([k_0, T])\) is regular.

**Definition 6.** The operator \(\mathcal{T} : X \to X\) is called nondecreasing if for any order \(x_1, x_2 \in X\) have relation \(\leq\) preserved under \(\mathcal{T}\), that is, \(x_1 \leq x_2\) implies \(\mathcal{T}x_1 \leq \mathcal{T}x_2\) for all \(x_1, x_2 \in X\).

**Definition 7 (see [29]).** Let \((X, \leq)\) be a partially ordered set. The operator \(\mathcal{T} : X \to X\) is called partially continuous at \(x\) if for any \(\varepsilon > 0\), there is \(\delta > 0\) such that under the supremum norm, we have \(\|x - \mathcal{T}y\| < \varepsilon\) and \(\|x - y\| < \delta\) for all \(x\) comparable to \(y\) in \(X\). \(\mathcal{T}\) is called partially continuous on \(X\) when the operator \(\mathcal{T}\) is partially continuous at every \(y\) in \(X\).

**Definition 8 (see [29]).** Let \((X, \leq)\) be a partially ordered set. The operator \(\mathcal{T} : X \to X\) is called partially bounded if the set \(\mathcal{T}(\mathcal{C})\) is bounded for all chains \(\mathcal{C}\). In particular, it is called uniformly partially bounded if the set \(\mathcal{T}(\mathcal{C})\) is bounded with the same constant for every chain \(\mathcal{C}\) in \(X\).

**Definition 9 (see [29]).** Let \((X, \leq)\) be a partially ordered set. The operator \(\mathcal{T} : X \to X\) is called partially compact if the set \(\mathcal{T}(\mathcal{C})\) is relatively compact in \(X\) for every chain \(\mathcal{C}\) in \(X\).

**Definition 10 (see [29]).** Let \((X, \leq, \|\|)\). The order relations \(\leq\) and the metric \(d\) induced by the norm \(\|\|\) are said to be compatible if the following condition holds: if \(\{x_n\}_{n \in \mathbb{N}}\) is a subsequence of a monotone sequence \(\{x_n\}_{n \in \mathbb{N}}\) converging to \(y\), then all of the sequence \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(y\).

**Definition 11 (see [29]).** Let \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) be an upper semi-continuous and nondecreasing function. Then, \(\psi\) is called a \(\mathcal{D}\)-function if \(\psi(0) = 0\).

\[
\rho D_{t_0}^{\alpha} \left[ u(t) - g(t, u(t - \tau(t))) \right] \leq f(u(t), u(t - \tau(t))) \quad t \in [t_0, T]
\]

\[
u(t) \leq x(t), \quad t \in [k_0, t_0].
\]
3. Existence and Monotone Iterative Approximation Results

In this section, we present a result on the existence and monotone iterative approximation of a mild solution of fractional-order neutral differential equations involving a generalized fractional derivative.

Theorem 2. Suppose that all seven hypotheses (i)–(vii) are satisfied. Then, IVP (5) has a mild solution $x^* : [k_0, T] \rightarrow \mathbb{R}$ in which $x(t) = \xi(t)$ for $t \in [k_0, t_0]$ and

$$pD_t^\alpha u(t) \in C_{1-\gamma}[t_0, T]$$

(19)

can be monotonically approximated by sequence $x_n, n \in \mathbb{N}$ defined by

$$x_{n+1}(t) = g(t, x_n(t - \tau(t))) + \frac{1}{\Gamma(y)} \int_{t_0}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{y-1} L(t_0) \frac{ds}{s^{1-p}} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{y-1}$$

(20)

where $x_0(t) = u(t)$ is the lower solution and $L(t_0) = x(t_0) - g(t_0, x(t_0 - \tau(t_0)))$.

Proof. We denote $X = C_{1-\gamma}[t_0, T]$ for the partially ordered Banach space. A mild solution to IVP (5) can be considered as the following operator equation:

$$\delta_1 x(t) = g(t, x(t - \tau(t)))$$

$$\delta_2 x(t) = g(t, x(t - \tau(t)))$$

(21)

for $t \in [t_0, T]$. We require to show that the operators $\delta_1$ and $\delta_2$ satisfy all conditions in Theorem 1 which will be divided into 5 parts.

First Step. By Theorem 1, we shall prove that the two operators $\delta_1$ and $\delta_2$ are nondecreasing. For all $x, y \in X$ such that $x \geq y$, by consideration of hypothesis (ii), we obtain

$$\delta_2 x(t) - \delta_2 y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} [f (x(s), x(s - \tau(s))) - f (y(s), y(s - \tau(s))) ] \frac{ds}{s^{1-p}} \geq 0,$$

(22)

for any $x \geq y$ in $X$. Clearly, $\delta_2$ is a nondecreasing operator in $X$.

Second Step. For this step, we will prove that the operator $\delta_1$ satisfies the following properties:

(i) The operator $\delta_1$ is partially bounded on $X$.
(ii) The operator $\delta_1$ is a partially nonlinear $\mathcal{D}$-contraction on $X$.

For this moment, let $x \in X$. By the property of $f$ in hypothesis (iii), we get

$$|\delta_1 x(t)| = |g(t, x(t - \tau(t)))| \leq \|\delta_1 x\| \leq M_g,$$

(25)

for all $t \in [t_0, T]$. Therefore, an operator $\delta_1$ is partially bounded on $X$. Then, we show that the operator $\delta_1$ is $\mathcal{D}$-contraction. For any $x, y \in X$ such that $x \geq y$, we see from assumption (iv) that

$$|\delta_1 x(t) - \delta_1 y(t)| = |g(t, x(t - \tau(t))) - g(t, y(t - \tau(t)))| \leq \phi (|x(t) - y(t)|) \leq \phi (\|x - y\|),$$

(26)
for each \( t \in [k_0, T] \). Hence, we take the supremum norm \( \| \delta_1 x(t) - \delta_1 y(t) \| \leq \phi(\|x - y\|) \) for all \( x, y \in X \) with \( x \geq y \). This implies that \( \delta_1 \) is a partially nonlinear \( \mathcal{D} \)-contraction in the space \( X \).

**Third Step.** We shall show that the operator \( \delta_2 \) satisfies condition (b) of Theorem 1, that is, \( \delta_2 \) is partially continuous on \( X \). We first show that \( \delta_2 \) is partially pointwise convergent. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence in a chain \( \mathcal{C} \) in \( X \) satisfying \( x_n \rightarrow x \) as \( n \rightarrow \infty \). Using the boundedness and continuity of \( f \) from assumptions (i) and (vi) together with the dominated convergence theorem, we get

\[
\lim_{n \to \infty} (\delta_2 x_n)(t) = \lim_{n \to \infty} \frac{1}{\Gamma(y)} \int_{t_0}^{t} \left( \frac{t^\alpha - s^\alpha}{\rho} \right)^{-1} f(x_n(s), x_n(s - \tau(s))) ds
\]

for each \( t \in [t_0, T] \). This implies that \( \delta_2 x_n \) converges to \( \delta_2 x \) pointwise on \( [t_0, T] \). Next, we prove the equicontinuity of \( \{\delta_2 x_n\}_{n \in \mathbb{N}} \) in \( X \). Let \( t_1, t_2 \in [t_0, T] \) with \( t_1 < t_2 \). We have

\[
\lim_{n \to \infty} \frac{1}{\Gamma(y)} \int_{t_0}^{t} \left( \frac{t^\alpha - s^\alpha}{\rho} \right)^{-1} f(x(s), x(s - \tau(s))) ds
\]
\[
\begin{aligned}
&\frac{1}{\Gamma(y)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} L(t_0) \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
&\frac{1}{\Gamma(y)} \int_{t_1}^{t_2} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} L(t_0) \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{y-1} \\
&\frac{1}{\Gamma(y)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{y-2} \right] L(t_0) \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
&\int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{y-1} \right] f(x_n(s), x_n(s - \tau(s))) \frac{ds}{s^{1-\rho}} \\
\leq \frac{L(t_0)}{\Gamma(y)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
&\frac{L(t_0)}{\Gamma(y)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{y-2} \right] \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
&\frac{M_f}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{\alpha-1} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{\alpha-1} \right] \frac{ds}{s^{1-\rho}} \\
\leq \frac{L(t_0)}{\Gamma(y)} \Gamma(y-1) (y-1)^{-1} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
&\frac{L(t_0)}{\Gamma(y)} \Gamma(y-1) (y-1)^{-1} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
\frac{L(t_0)}{\Gamma(y)} \Gamma(y-1) (y-1)^{-1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{y-2} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{y-2} \right] \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\alpha - t_0^\alpha}{\rho} \right)^{y-1} - \left( \frac{t_1^\alpha - t_0^\alpha}{\rho} \right)^{y-1} \\
\frac{p^{1-a}}{\Gamma(\alpha+1)} (t_2 - t_1)^{a} + \frac{M_f}{\rho^{a-1} \Gamma(\alpha)} \int_{t_0}^{t_1} \left( \frac{t_2^\alpha - s^\alpha}{\rho} \right)^{\alpha-1} - \left( \frac{t_1^\alpha - s^\alpha}{\rho} \right)^{\alpha-1} \frac{ds}{s^{1-\rho}} \\
\rightarrow 0,
\end{aligned}
\]
as $t_2 - t_1 \to 0$ uniformly for all $n \in \mathbb{N}$. So, $\delta_2 x_n \to \delta_2 x$ uniformly. Since $\delta_2 x_n$ converges to $\delta_2 x$ pointwise and uniformly, this implies that operator $\delta_2$ is partially continuous on $X$.

**Fourth Step.** We show that $\delta_2$ is partially compact. Let $y \in \delta_2(C)$ where $C$ is a chain in $X$. We have $y = \delta_2(x)$ for some $x \in C$. By condition (vi), we get

$$|y(t)| = |(\delta_2) x(t)| \leq \frac{1}{\Gamma(\gamma)(y - 1)} \int_{t_0}^{t} \left( t^\rho - s^\rho \right)^{\gamma - 2} \frac{L(t_0)}{s^{1-\rho}} \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} + M_f \frac{L(t_0)}{\Gamma(\alpha + 1)} \int_{t_0}^{t} \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} \frac{ds}{s^{1-\rho}}$$

for all $t \in [t_0, T]$. By taking the supremum in the last assertion, we get $\|y(t)\| \leq K$ for all $y \in \delta_2(C)$. Clearly, the operator $\delta_2$ is uniformly bounded on every chain $C$ in $X$. It remains to prove that $\delta_2$ is equicontinuous on every chain $C$ in $X$. Let $y \in \delta_2(C)$ and take $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$. We have

$$|(\delta_2 x)(t_2) - (\delta_2 x)(t_1)| \leq \frac{1}{\Gamma(\gamma)(y - 1)} \int_{t_0}^{t_1} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{L(t_0)}{s^{1-\rho}} \left( \frac{t_2^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} + M_f \frac{L(t_0)}{\Gamma(\alpha + 1)} \int_{t_0}^{t_1} \left( \frac{t_2^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha - 1} \frac{ds}{s^{1-\rho}}$$

$$+ \left| \frac{1}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{L(t_0)}{s^{1-\rho}} \left( \frac{t_2^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} \right|$$

$$+ \left| \frac{1}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \left( \frac{t_1^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{L(t_0)}{s^{1-\rho}} \left( \frac{t_2^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} \frac{ds}{s^{1-\rho}} \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha - 1} f(x(s), x(s - \tau(s))) \frac{ds}{s^{1-\rho}} \right|$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha - 1} \left( \frac{t_1^\rho - s^\rho}{\rho} \right)^{\alpha - 1} f(x(s), x(s - \tau(s))) \frac{ds}{s^{1-\rho}} \leq \frac{L(t_0)}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{ds}{s^{1-\rho}} \left( \frac{t_2^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} + \frac{L(t_0)}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{ds}{s^{1-\rho}} \left( \frac{t_1^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} + \frac{L(t_0)}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{ds}{s^{1-\rho}} \left( \frac{t_1^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1}$$

$$+ \frac{L(t_0)}{\Gamma(\gamma)(y - 1)} \int_{t_1}^{t_2} \left( \frac{t_2^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \left( \frac{t_1^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{ds}{s^{1-\rho}} \left( \frac{t_1^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1}$$
as \( t_2 - t_1 \to 0 \) uniformly for \( y \in \mathcal{S}_2(\mathcal{C}) \). Thus, we get that \( \delta' \) is also relatively compact and we can conclude that the operator \( \delta'' \) is partially compact on every chain \((\mathcal{C})\) in \( X \).

**Fifth Step.** Finally, we prove that there exists an element \( u \in X \) which is a lower solution of IVP (5). By hypothesis (vii), there exist functions \( u \in C[t_0, T] \) and \( u \in C_{1, r}[t_0, T] \) satisfying

\[
\begin{aligned}
&u(t) \leq g(t, u(t - \tau(t))) + \frac{1}{\Gamma(\gamma)\Gamma(\gamma - 1)} \int_{t_0}^{t} \left( \frac{v^\rho - s^\rho}{\rho} \right)^{\gamma - 2} \frac{L(t_0)}{s^{\gamma - 2}} \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\gamma - 1} ds \\
&\quad + \frac{1}{\Gamma(\alpha + 1)} \int_{t_0}^{t} \left( \frac{v^\rho - s^\rho}{\rho} \right)^{\alpha - 1} f(u(s), u(s - \tau(s))) \frac{ds}{s^{\gamma - \rho}},
\end{aligned}
\]

where \( L(t_0) = x(t_0) - g(t_0, x(t_0 - \tau(t_0))) \) for \( t \in [t_0, k] \). This means that \( u \) satisfies \( u \leq \delta'_1 u + \delta'_2 u. \) As a consequence, we conclude that the operator equation \( \delta''_1 x + \delta''_2 x = x \) has a solution since the two operators \( \delta'_1 \) and \( \delta''_2 \) satisfy all conditions in Theorem 1. In addition, the solution of IVP (5) can be monotonically approximated by a sequence \( x_n \) as \( n = 1, 2, \ldots \).

**4. Example**

Consider the neutral fractional differential equation with generalized fractional derivatives

\[
\begin{aligned}
\rho D_0^{\alpha \rho} \left[ x(t) - \frac{\arctan x(t - 0.01)}{2} \right] &= \tanh x(t) - \frac{1}{2e^{x(t - 0.01)}} + 5, \quad t \in [1, 2] \\
x(t) &= 0, \quad t \in [0, 1].
\end{aligned}
\]

Here, \( f(x, y) = \tanh y - \frac{1}{2e^{y}} + 5 \).
and
\[ g(t, x) = \frac{\arctan x}{2}. \] (35)

Let \( \beta = 0; \) we get \( \alpha = \gamma. \) Clearly, both functions \( f \) and \( g \) are continuous and nondecreasing. Considering functions \( f \) and \( g \) in equation (31), we see that
\[ 0 \leq f(x, y) \leq |\tanh x - \frac{1}{2e^x} + 5| \leq M_f = 6, \] (36)

for all \( t \in \mathbb{R}. \) Hence, the functions \( f \) and \( g \) satisfy assumptions (i)–(vi). Finally, for assumption (vii), we claim that a lower solution of equation (24) is given by \( u(t) = 0 \) for all \( t \in [0, 2]. \) We see that \( 0 \leq x(t) \) for all \( t \in [0, 1]. \) Let \( L(t) = x(t) - g(t, x(t - 0.01)) \) for \( t \in \mathbb{R}. \) It can be seen from
\[ 0 \leq g(t, x) \leq \frac{1}{2} (\arctan x - \arctan y) \leq \frac{1}{2} (x - y) \leq \phi(x - y), \] (38)

Next, we verify that the function \( g \) is a \( \mathcal{D} \)-contraction with \( \phi(t) = (1/2)t. \) We have
\[ 0 \leq |g(t, x)| = \frac{1}{2} |\arctan x| \leq M_g = \frac{\pi}{4} \] (37)

for all \( t \in [0, 2]. \) We see that \( 0 \leq x(t) \) for all \( t \in [0, 1]. \) Let \( L(t) = x(t) - g(t, x(t - 0.01)) \) for \( t \in \mathbb{R}. \) It can be seen from
\[ 0 \leq g(t, u(t - \tau(t))) + \frac{1}{\Gamma(y)\Gamma(y - 1)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-2} L(t_0) \frac{ds}{s^{1-p}} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{\alpha-1} f(u(s), u(s - \tau(s))) \frac{ds}{s^{1-\alpha}} \]
\[ = g(t, u(t - 0.01)) + \frac{1}{\Gamma(y)\Gamma(y - 1)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-2} L(1) \frac{ds}{s^{1-p}} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{\alpha-1} f(u(s), u(s - 0.01)) \frac{ds}{s^{1-\alpha}} \]
\[ = g(t, 0) + \frac{1}{\Gamma(y)\Gamma(y - 1)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-2} (-g(1, x(0))) \frac{ds}{s^{1-p}} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{\alpha-1} f(0, 0) \frac{ds}{s^{1-\alpha}} < 4.5 \frac{\rho^{1-\alpha}}{\Gamma(\alpha + 1)} \] (39)

for \( t \in [1, 2]. \) Then, we see that all conditions of Theorem 2 are satisfied. Hence, equation (31) has a solution \( x^*: [0, 2] \to \mathbb{R} \) in which
\[ ^{\text{r}}D^\alpha \left[x(t) - \arctan x(t - 0.01)/2 \right] \in C_{[0, 2]} \]
\[ x(t) = 0, \quad t \in [0, 1], \]
\[ x(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{\alpha-1} f(x_n(s), x_n(s - \tau(s))) \frac{ds}{s^{1-\alpha}} \quad t \in [1, 2] \]
\[ x_{n+1}(t) = g(t, x_n(t - \tau(t))) + \frac{1}{\Gamma(y)\Gamma(y - 1)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-2} L(t_0) \frac{ds}{s^{1-p}} \left( \frac{t^p - t^p_0}{\rho} \right)^{y-1} \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \frac{t^p - t^p_0}{\rho} \right)^{\alpha-1} f(x_n(s), x_n(s - \tau(s))) \frac{ds}{s^{1-\alpha}} \quad t \in [1, 2] \]
\[ x_n(t) = 0, \quad t \in [0, 1], \]
with \( x_0(t) = 0 \) for all \( t \in [1, 2]. \)
We numerically demonstrate the monotone iterative approximation of solutions corresponding to various fractional derivatives including Riemann–Liouville fractional derivative and Hadamard fractional derivative in Figures 1–3.

In Figure 1, we numerically compute a monotone iterative increasing sequence $x_n$ converging to the mild solution of equation (31) for $\beta = 0$ and $\rho \to 1$ which corresponds to the fractional differential equation under the Riemann–Liouville fractional derivative for $\alpha = 0.5$ with time delay $\tau = 0.01$. The lower solution is chosen to be $x_0 = 0$ on $t \in [0, 2]$. It can be seen that sequence $x_n$ increases at $t = 1.01$ and then gradually decreases on $t \in [1, 2]$ which exhibits the behavior of the mild solution.

In Figure 2, we numerically compute a monotone iterative increasing sequence $x_n$ converging to the mild solution of equation (31) for $\beta = 0$ and $\rho = 0.5$ with fractional derivative for $\alpha = 0.5$.

In Figure 3, we numerically compute a monotone iterative increasing sequence $x_n$ converging to the mild solution of equation (31) for $\beta = 0$ and $\rho \to 0^+$ which corresponds to the fractional differential equation under the generalized fractional derivative for $\alpha = 0.5$ with time delay $\tau = 0.01$. The lower solution is chosen to be $x_0 = 0$ on $t \in [0, 2]$. It can be seen that sequence $x_n$ increases rapidly at $t = 1.01$ and then becomes constant on $t \in [1.1, 2]$ which exhibits the behavior of the mild solution.
corresponds to the fractional differential equation under the Hadamard fractional derivative for $\alpha = 0.5$ with time delay $\tau = 0.01$. Similar to Figure 1, the lower solution is chosen to be $x_0 = 0$ on $t \in [0, 2]$. It can be seen that sequence $x_n$ increases at $t = 1.01$ and then gradually decreases on $t \in [1.1, 2]$ which exhibits the behavior of the mild solution.

5. Conclusions

We establish the existence and monotone iterative approximation of mild solutions to the neutral fractional differential equations under a generalized fractional derivative which can be applied numerically. The monotone iterative sequence for the mild solution of equation (31) for $0 < \alpha < 1$ admits the following particular cases. If $\rho \to 1$ and $\beta = 0$, then we have $y = a$ and we obtain the existence and approximation of solution under the Riemann–Liouville fractional derivative. If $\rho \to 0^+$ and $\beta = 0$, then we get the existence and approximation of solution under Hadamard fractional derivative.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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