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*Research Article*

# Studies of Connected Networks via Fractional Metric Dimension

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Metric dimension is an effective tool to study different distance-based problems in the field of telecommunication, robotics, computer networking, integer programming, chemistry, and electrical networking. In this paper, we study the latest form of metric dimension called fractional metric dimension of some connected networks such as circular diagonal ladder, double sun flower, and double path networks.

## 1. Introduction

The resolving or locating sets were first introduced by Slater, and he called order of the minimum resolving set as location number [1]. By following the idea of locating sets, Melter and Harary defined the term metric dimension (MD). Moreover, they gave a characterization of the MD of trees [2]. Chartrand et al. computed MD of the path and unicyclic networks, and they also characterized all those connected networks of order $q$ having MD 1, $q-1$, and $q-2$. Furthermore, they proposed the solution of integer programming problem (IPP) with the help of MD under certain conditions [3]. Constant MD of cycle-related networks was computed by Murtaza et al. [4]. It was also observed that problem to find MD of a connected network is NP-hard [5]. Locating sets have applications in network discovery [6], coin weighting, robot navigation, pattern recognition [7], and joints in networks [8]. Some interesting results of different invariants of MD can be seen in [9–12], and recently, Dalal et al. computed edge MD of some Toeplitz networks [13].

Currie and Oellermann found integral as well as nonintegral solutions of IPP by using the idea of fractional metric dimension (FMD) [14]. Fahr et al. presented an optimal solution of the IPP by using concept of FMD [15]. Later Arguman and Matthew calculated FMD of some important networks and also defined specific computational criteria to compute FMD of some connected networks [16, 17]. FMD of vertex transitive and hierarchical product of networks was computed by Feng et al. Moreover, they also established bounds of FMD of Cartesian product of networks [18,19]. For the study of FMD of corona product, lexicographic product, and unicyclic and generalized sunlet networks, see [20–22]. Later on Al Khalidi et al. designed computational criteria to compute bounds of FMD of connected networks [23]. Furthermore, Javaid et al. characterized all those connected networks with FMD being exactly 1 [24] and bounds of FMD for metal organic compounds calculated by Moshin et al. [25]. Recently, in 2021, Hassan et al. computed the L-F metric dimension of generalized gear networks in the form of exact value and bounds. Moreover, they also proved that all these networks remain unbounded when their order approaches to infinity [26]. In this dissertation, we have computed exact value of FMD of connected networks such as double sunflower network, circular diagonal network, and double path network.

The article is organized as follows: Section 2 consists of the preliminaries, Section 3 deals with the main results, and Section 4 contains the conclusion.

## 2. Preliminaries

Let $\mathbb{T}$ be a simple undirected network where $V(\mathbb{T}) = \{a_1, a_2, a_3, \ldots, a_q\}$ is the vertex set and

$E(\mathbb{T}) = \{e_1, e_2, e_3, \ldots, e_q\}$ is the edge set, respectively. A network whose vertices are ordered in a way that two vertices are adjacent if and only if they are consecutive in the list is called path network. For any two vertices $\{a, b\} \subseteq V(\mathbb{T})$, the distance between any two vertices $a$ and $b$ denoted by $d(a, b)$ is the number of edges between them. A network is connected if there exists a path between any two vertices. For further study about these preliminary concepts, see [27].

For $\mathbb{Z} = \{x_1, x_2, x_3, \ldots, x_q\} \subseteq V(\mathbb{T})$ and $y \in V(\mathbb{T})$, a representation of the vertex $y$ with respect to $\mathbb{Z}$ is $r(y|\mathbb{Z}) = (r(y, x_1), r(y, x_2), r(y, x_3), \ldots, r(y, x_q))$. If each $y \in V(\mathbb{T})$ possesses unique representation with $\mathbb{Z}$, then set $\mathbb{Z}$ becomes resolving set having $q$ elements of $\mathbb{T}$ and the order of minimum resolving set is called MD of $\mathbb{T}$ defined as $\dim(\mathbb{T}) = \min\{|\mathbb{Z}|\colon \mathbb{Z} \text{ is resolving set of } \mathbb{T}\}$.

A vertex $c \in V(\mathbb{T})$ resolves pair of vertices $(a, b)$ if distance from $a$ to $c$ is not equal to distance from $c$ to $b$. For a pair $\{a, b\} \subseteq V(\mathbb{T})$, the resolving neighbourhood set is defined as $\mathcal{R}\{a, b\} = \{c \in V(\mathbb{T})\colon d(a, c) \neq d(b, c)\}$. A function $\phi\colon V(\mathbb{T}) \longrightarrow [0, 1]$ is called resolving function if $\phi(\mathcal{R}\{a, b\}) \geq 1$ for each $\mathcal{R}(a, b)$ of $\mathbb{T}$, where $\phi(\mathcal{R}\{a, b\}) = \sum_{c \in \mathcal{R}(a,b)} \phi(x)$. Then, FMD is defined as $\dim_f(\mathbb{T}) = \min\backslash\{|\phi|\colon \phi \text{ is minimal resolving function of } \mathbb{T}\backslash\}$.

## 3. Main Results

In this particular section, we present main result to compute exact value of FMD of different connected networks such as circular diagonal ladder networks, double sunflower networks, and double path networks.

*3.1. Fractional Metric Dimension of Double Sunflower Network.* For $1 \leq i \leq q$, the double sunflower $[\mathrm{DSDF}]_q$ is a network of order $3q$ and size $5q$, which is obtained from the sunflower network by adding a new vertex $c_i$ on each edge $a_i a_{i+1}$ and adding edges $b_i c_i$ for each $i$ (see Figure 1) [28].

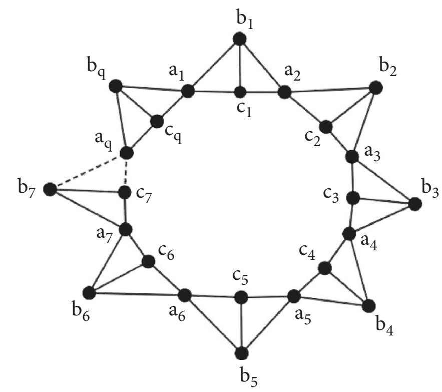


Figure 1: Double sunflower network $[\mathrm{DSF}]_q$.

**Lemma 1.** *Let $[DSF]_q$ with $q \geq 3$ be a double sunflower network. Then,*

*(a) $|\mathcal{R}(b_i c_i)| = 2$ and $|\cup_{i=1}^{q} \mathcal{R}(b_i c_i)| = 3q$.*

*(b) $|\mathcal{R}(b_i c_i)| < |\mathcal{R}(x, y)|$ and $|\mathcal{R}(x, y) \cap \cup_{i=1}^{q} \mathcal{R}(b_i c_i)| > |\mathcal{R}(b_i c_i)|$, where $\mathcal{R}(x, y)$ denotes all other possible resolving neighbourhood sets of $[DSF]_q$.*

*Proof.* Let $a_i$, $b_i$, and $c_i$ be the vertices of $[\mathrm{DSF}]_q$, where $1 \leq i \leq q$, $q + 1 \cong 1 (\bmod q)$, and we have the following.

(a) $\mathcal{R}(b_i c_i) = \{b_i, c_i\}$ with $|\mathcal{R}(b_i c_i)| = 2$ and $|\cup_{i=1}^{q} \mathcal{R}(b_i c_i)| = 3q$.

(b) $\mathcal{R}(a_i b_i) = V[\mathrm{DSF}]_n - \{c_i\}$, $\mathcal{R}(b_i a_{(i+1)}) = V(\mathrm{DSF})_q - \{c_i\}$, $\mathcal{R}(a_i c_i) = V(\mathrm{DSF})_q - \{b_i\}$, $\mathcal{R}(c_i a_{i+1}) = V(\mathrm{DSF})_q - \{b_i\}$, $\mathcal{R}(a_i, c_{i+1}) = V(\mathrm{DSF})_q$, $\mathcal{R}(a_i, b_{i+1}) = V(\mathrm{DSF})_q$, $\mathcal{R}(b_i, b_z) = V(\mathrm{DSF})_q - \{a_{i+z+1/2}\}$, $i \cong 1 (\bmod 2)$ and $i < z$. $\mathcal{R}(b_i, b_u) = V(\mathrm{DSF})_q - \{c_{i+u/2}\}$, $i \cong 1 (\bmod 2)$ and $i < u$. $\mathcal{R}(b_i, c_z) = V[\mathrm{DSF}]_q - \{a_{i+z+1/2}\}$, $i \cong 0 (\bmod 2)$ and $i < z$. $\mathcal{R}(b_i, c_u) = V[\mathrm{DSF}]_q - \{c_{i+u/2}\}$, $i \cong 1 (\bmod 2)$ and $i < z$. $\mathcal{R}(b_i, c_z) = V[\mathrm{DSF}]_q - \{a_{i+u+1/2}\}$, $i \cong 0 (\bmod 2)$ and $i < z$. $\mathcal{R}(b_i, c_u) = V[\mathrm{DSF}]_q - \{a_{i+u+1/2}\}$, $i \cong 0$ $(\bmod 2)$ and $i < z$. $\mathcal{R}(b_i, a_t) = V[\mathrm{DSF}]_q$. $\mathcal{R}(b_i, a_q) = V[\mathrm{DSF}]_q$, $\mathcal{R}(a_i, b_{i+1}) = V[\mathrm{DSF}]_q$, $\mathcal{R}(a_q, c_i) = V[\mathrm{DSF}]_q$.

It can be observed from Table 1 that $|\mathcal{R}(b_i c_i)| < |\mathcal{R}(x, y)|$ and $|\mathcal{R}(x, y) \cap \cup_{i=1}^{q} \mathcal{R}(b_i c_i) > |\mathcal{R}(b_i c_i)|$, where $\mathcal{R}(x, y)$ are the other RN sets. □

**Theorem 1.** *Let $[DSF]_q$ with $q \geq 3$ be a double sunflower network. Then,* $\dim_{Fr}[DSF]_q = q$.

*Proof.* For different vertices of $[\mathrm{DSF}]_q$, we have following cases. □

*Case 1.* For $q = 3$, the possible RN sets are shown in Table 2.

From above RN sets, the RN sets $|\mathcal{R}(b_i c_i)| = 2$, $|\mathcal{R}(b_i c_i)| < |\mathcal{R}(x, y)|$ and $\mathcal{R}(x, y)$ are the other RN sets, where $1 \leq i \leq 3$. Furthermore, $|\cup_{i=1}^{3} \mathcal{R}(b_i c_i)| = 6$ and $|\mathcal{R}(x, y) \cup_{i=1}^{3} \mathcal{R}(b_i c_i)| \geq 2$. Hence, there exists a constant function $\Psi\colon V(\mathrm{DSF})_3 \longrightarrow [0, 1]$ defined by $\Psi(v) = 1/2$, $\forall v \in \cup_{i=1}^{3} \mathcal{R}(b_i c_i)$, which shows that $\Psi$ is a resolving function. In order to show that $\Psi$ is a minimal resolving function, consider $\Psi'\colon V(\mathrm{DSF})_3 \longrightarrow [0, 1]$, where $|\Psi'(v)|$, $< |\Psi(v)|$, and hence $\Psi'(\mathcal{R}(b_i c_i)) < 1$ which means $\Psi'$ is not a resolving function. Consequently,

$$\dim_{Fr}[\mathrm{DSF}]_3 = \sum_{i=1}^{6} \frac{1}{2} = 3. \tag{1}$$

*Case 2.* For $1 \leq i \leq q$ and $q \geq 5$, by Lemma 1, $|\mathcal{R}(b_i c_i)| = 2$, and $|\cup_{i=1}^{q} \mathcal{R}(b_i c_i)| = 2q$ and $|\mathcal{R}(x.y) \cap \cup_{i=1}^{q} \mathcal{R}(b_i c_i)| \geq 2$, $\forall x, y \in V[\mathrm{DSF}]_q$. Hence, there exists a function $\Psi\colon V(\mathrm{DSF})_q \longrightarrow [0, 1]$ defined by $\Psi(v) = 1/2$:

$$\Psi(v) = \begin{cases} \dfrac{1}{2}, & \forall v \in \bigcup\limits_{i=1}^{n} \mathcal{R}(b_i c_i), \\ \\ 0, & \text{otherwsie}, \end{cases} \tag{2}$$

Table 1: Order of each RN set of $[DSF]_q$.

| Order | Comparison |
|---|---|
| $\|\mathscr{R}(a_ib_i)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(b_ia_{i+1})\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(a_ic_i)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(b_ia_{i+1})\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(a_i, c_{i+1})\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(a_i, c_{i+1})\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(a_i, b_{i+1})\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(b_i, a_q)\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(a_i, c_{i+1})\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(b_i, b_z)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(a_q, c_i)\| = 3q$ | $2 < 3q$ |
| $\|\mathscr{R}(b_i, b_u)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(b_i, b_u)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(b_i, c_z)\| = 3q-1$ | $2 < 3q-1$ |
| $\|\mathscr{R}(b_i, a_t)\| = 3q$ | $2 < 3q$ |

Table 2: RN sets and the elements of $[\mathrm{DSF}]_3$.

| RN set | Elements |
|---|---|
| $\mathscr{R}_1 = \mathscr{R}(a_1, a_2)$ | $V[\mathrm{DSF}]_3 - \{a_3, b_1, c_1\}$, |
| $\mathscr{R}_2 = \mathscr{R}(a_1, a_3)$ | $V[\mathrm{DSF}]_3 - \{a_2, b_3, c_3\}$, |
| $\mathscr{R}_3 = \mathscr{R}(a_1, c_2)$ | $V[\mathrm{DSF}]_3$, |
| $\mathscr{R}_4 = \mathscr{R}(a_1, b_2)$ | $V[\mathrm{DSF}]_3$, |
| $\mathscr{R}_5 = \mathscr{R}(b_1, b_2)$ | $V[\mathrm{DSF}]_3 - \{a_2, b_3, c_3\}$, |
| $\mathscr{R}_6 = \mathscr{R}(b_1, b_3)$ | $V[\mathrm{DSF}]_3 - \{a_3, b_1, c_1\}$, |
| $\mathscr{R}_7 = \mathscr{R}(b_1, c_2)$ | $V[\mathrm{DSF}]_3 - \{a_2, b_3, c_3\}$, |
| $\mathscr{R}_8 = \mathscr{R}(b_1, c_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_9 = \mathscr{R}(b_1, a_3)$ | $V[\mathrm{DSF}]_3$, |
| $\mathscr{R}_{10} = \mathscr{R}(b_1, b_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$ |
| $\mathscr{R}_{11} = \mathscr{R}(b_1, b_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_{12} = \mathscr{R}(b_2, b_3)$ | $V[\mathrm{DSF}]_3 - \{a_3, b_1, c_1\}$, |
| $\mathscr{R}_{13} = \mathscr{R}(b_2, c_1)$ | $V[\mathrm{DSF}]_3 - \{a_2, b_3, c_3\}$, |
| $\mathscr{R}_{14} = \mathscr{R}(a_2, a_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_{15} = \mathscr{R}(a_2, c_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_{16} = \mathscr{R}(a_2, b_3)$ | $V[\mathrm{DSF}]_3$, |
| $\mathscr{R}_{17} = \mathscr{R}(a_3, c_1)$ | $V[\mathrm{DSF}]_3$, |
| $\mathscr{R}_{18} = \mathscr{R}(b_3, c_2)$ | $V[\mathrm{DSF}]_3 - \{a_3, b_1, c_2\}$, |
| $\mathscr{R}_{19} = \mathscr{R}(b_3, c_1)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_{20} = \mathscr{R}(c_1, c_3)$ | $V[\mathrm{DSF}]_3 - \{a_1, b_2, c_2\}$, |
| $\mathscr{R}_{21} = \mathscr{R}(c_1, c_2)$ | $V[\mathrm{DSF}]_3 - \{a_2, b_3, c_3\}$, |
| $\mathscr{R}_{22} = \mathscr{R}(a_1b_1)$ | $V[\mathrm{DSF}]_3 - \{c_1\}$, |
| $\mathscr{R}_{23} = \mathscr{R}(a_2b_2)$ | $V[\mathrm{DSF}]_3 - \{c_2\}$, |
| $\mathscr{R}_{24} = \mathscr{R}(a_3b_3)$ | $V[\mathrm{DSF}]_3 - \{c_3\}$, |
| $\mathscr{R}_{25} = \mathscr{R}(b_1a_2)$ | $V[\mathrm{DSF}]_3 - \{c_1\}$, |
| $\mathscr{R}_{26} = \mathscr{R}(b_2a_3)$ | $V[\mathrm{DSF}]_3 - \{c_2\}$, |
| $\mathscr{R}_{27} = \mathscr{R}(b_3a_1)$ | $V[\mathrm{DSF}]_3 - \{c_3\}$, |
| $\mathscr{R}_{28} = \mathscr{R}(a_1c_1)$ | $V[\mathrm{DSF}]_3 - \{b_1\}$, |
| $\mathscr{R}_{29} = \mathscr{R}(a_2c_2)$ | $V[\mathrm{DSF}]_3 - \{b_2\}$, |
| $\mathscr{R}_{30} = \mathscr{R}(a_3c_3)$ | $V[\mathrm{DSF}]_3 - \{b_3\}$, |
| $\mathscr{R}_{31} = \mathscr{R}(c_1a_2)$ | $V[\mathrm{DSF}]_3 - \{b_1\}$, |
| $\mathscr{R}_{32} = \mathscr{R}(c_2a_3)$ | $V[\mathrm{DSF}]_3 - \{b_2\}$, |
| $\mathscr{R}_{33} = \mathscr{R}(c_3a_1)$ | $V[\mathrm{DSF}]_3 - \{b_3\}$, |
| $\mathscr{R}_{34} = \mathscr{R}(b_1c_1)$ | $\{b_1, c_1\}$, |
| $\mathscr{R}_{35} = \mathscr{R}(b_2c_2)$ | $\{b_2, c_2\}$, |
| $\mathscr{R}_{36} = \mathscr{R}(b_3c_3)$ | $\{b_3, c_3\}$. |

such that $\Psi(\mathscr{R}(x, y)) \geq 1$ which shows that $\Psi$ is a resolving function. In order to show that $\Psi$ is a minimal resolving function, consider that there exists another resolving function $\Psi'\colon V(\mathrm{DSF})_q \longrightarrow [0,1]$ such that $|\Psi'(v)| < |\Psi(v)|$, and hence $\Psi'(\mathscr{R}(x, y)) \geq 1$ which means $\Psi$ is not a resolving function. Consequently,

$$\dim_{Fr}[\mathrm{DSF}]_q = \sum_{i=1}^{2q} \frac{1}{2} = q. \tag{3}$$

*3.2. Fractional Metric Dimension of Circular Diagonal Ladder Network.* A circular diagonal ladder $[\mathrm{CDL}]_q$ of size $5q$ and order $2q$ is obtained from a prism network $D_q$ by adding some double crossing edges $a_ib_{i+1}$ and $a_{i+1}b_i$ (see Figure 2) (for more information about these networks, see [28]).

**Lemma 2.** *If $[CDL]_q$ with $q \geq 4$ is a circular diagonal ladder network, then*

*(a) $|\mathscr{R}(a_ib_i)| = 2$ and $|\cup_{i=1}^{q}\mathscr{R}(a_ib_i)| = 2q$.*

*(b) $|\mathscr{R}(a_ib_i)| < |\mathscr{R}(x, y)|$ and $|\mathscr{R}(x, y) \cap \cup_{i=1}^{q}\mathscr{R}(a_ib_i)| \geq |\mathscr{R}(a_ib_i)|$, where $\mathscr{R}(x, y)$ are all the other possible resolving neighbourhood sets.*

*Proof.* Consider inner $(a_i)$ and outer $(b_i)$ vertices of $(\mathrm{CDL})_q$, respectively, where $q+1 \cong 1\,(\mathrm{mod}\,q), u \cong (1\mathrm{mod}\,2)$, $z \cong (0\,\mathrm{mod}\,2)$, and we have

(a) $\mathscr{R}(a_ib_i) = \{a_i, b_i\}$ with $|\mathscr{R}(a_ib_i)| = 2$ and $|\cup_{i=1}^{q}\mathscr{R}(a_ib_i)| = 2q$.

(b) $\mathscr{R}(b_ib_{i+1}) = V([\mathrm{CDL}]_q - \{a_i, a_{i+1}\}$, $\mathscr{R}(a_ia_{(i+1)}) = V(\mathrm{CDL})_q - \{b_i, b_{i+1}\}$, $\mathscr{R}(b_ia_{i-1}) = V[\mathrm{CDL}]_q - \{a_{i-1}, b_{i+1}\}$, $\mathscr{R}(b_ia_{i+1}) = V[\mathrm{CDL}]_q - \{b_{i+1}, a_i\}$, $\mathscr{R}(a_i, a_{i+2}) = V(\mathrm{CDL})_n - \{a_{i+1}, b_{i+1}, a_{n+2i+2/2}, b_{n+2i+2/2}\}$, $\mathscr{R}(b_i, b_t) = V[\mathrm{CDL}]_q$, $\mathscr{R}(a_u, a_{u+z}) = V[\mathrm{CDL}\ ]_q - \{a_{2u+z/2}, b_{2u}+z/2, a_{2u+q/2}, b_{2u+q/2}\}$, $\mathscr{R}(b_u, b_{u+z}) = V(\mathrm{CDL})_q - \{a_{2u+z/2}, b_{2u+z/2}, a_{2u+q/2}, b_{2u+q/2}\}$, $\mathscr{R}(a_u, a_z) = V[\mathrm{CDL}]_q$, $\mathscr{R}(a_1, a_{q-1}) = V[\mathrm{CDL}]_q - \{a_q, b_q, a_{q/2}, b_{q/2}\}$, $\mathscr{R}(b_1, b_{q-1}) = V[\mathrm{CDL}]_q - \{a_q, b_q, a_{q/2}, b_{q/2}\}$.

It can be observed from Table 3 that $|\mathscr{R}(a_ib_i)| < |\mathscr{R}(x, y)|$, $\forall x, y \in V[\mathrm{CDL}]_q$. ☐

**Theorem 2.** *Let $[CDL]_q$ with $q \geq 4$ be a circular diagonal ladder network. Then,*

$$\dim_{Fr}[\mathrm{CDL}]_q = q. \tag{4}$$

*Proof.* For different vertices of $[\mathrm{CDL}]_q$, we have the following cases. ☐

*Case 1.* In this case, we compute FMD of $[\mathrm{CDL}]_q$ with the help of Table 4.

For $1 \leq i \leq 4$, $|\mathscr{R}(a_ib_i)| = 2$ and $|\mathscr{R}(a_ib_i)| < |\mathscr{R}(x, y)|$ where $\mathscr{R}(x, y)$ are the other RN sets of $[\mathrm{CDL}]_4$. Furthermore, $|\cup_{i=1}^{4}\mathscr{R}(a_ib_i)| = V(\mathrm{CDL}_4)$. Hence, we define a function $\Psi\colon V(\mathrm{CDL}_4) \longrightarrow [0,1]$ such that $\Psi(x) = 1/2$ for each $v \in V[\mathrm{CDL}]_4$ and $\Psi(x, y) \geq 1$ $\forall x, yV(\mathrm{CDL}_4)$; therefore, $\Psi$ is a RF. To show that $\Psi$ is a minimal RF, we take

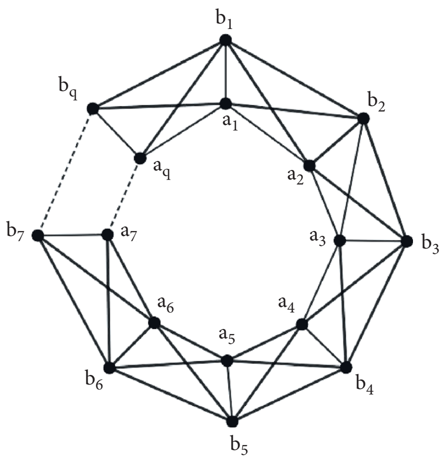


FIGURE 2: Circular diagonal ladder $[\mathrm{CDL}]_q$.

TABLE 3: RN sets and the elements of $[\mathrm{CDL}]_q$.

| Order of each RN set of $[\mathrm{CDL}]_q$ | Comparison |
|---|---|
| $\lvert\mathscr{R}(a_i a_{i+1})\rvert = 2q-2$ | $2 < 2q-2$ |
| $\lvert\mathscr{R}(b_i b_{i+1})\rvert = 2q-2$ | $2 < 2n-2$ |
| $\lvert\mathscr{R}(b_i a_{i-1})\rvert = 2q-2$ | $2 < 2q-2$ |
| $\lvert\mathscr{R}(b_i a_{i+1})\rvert = 2q-2$ | $2 < 2q-2$ |
| $\lvert\mathscr{R}(a_u, a_{u+z})\rvert = 2q-4$ | $2 < 2q-4$ |
| $\lvert\mathscr{R}(b_u, b_{u+z})\rvert = 2q-1$ | $2 < 2q-4$ |
| $\lvert\mathscr{R}(a_u, a_z)\rvert = 2q$ | $2 < 2q$ |
| $\lvert\mathscr{R}(a_1, a_{q-1})\rvert = 2q-4$ | $2 < 2q-4$ |
| $\lvert\mathscr{R}(b_1, b_{q-1})\rvert = 2q-4$ | $2 < 2q-4$ |

another RF such that $|\Psi'(x,y)| < |\Psi(x,y)|$; then, $\Psi'(x,y) < 1$. Consequently,

$$\dim_{Fr}[\mathrm{CDL}]_4 = \sum_{i=1}^{8} \frac{1}{2} = 4 \tag{5}$$

*Case 2.* For $q \geq 5$, $|\mathscr{R}(a_i b_i)| = 2$ and $|\mathscr{R}(a_i b_i)| < |\mathscr{R}(x,y)|$ where $\mathscr{R}(x,y)$ are the other RN sets of $[C\,DL]_q$. Furthermore, $|\cup_{i=1}^{2q}\mathscr{R}(a_i b_i)| = V[\mathrm{CDL}]_q$. Hence, we define a function $\Psi$: $V(\mathrm{CDL_q}) \longrightarrow [0,1]$ such that $\Psi(x) = 1/2$ for each $v \in V[\mathrm{CDL}]_q$ and $\Psi(x,y) \geq 1$ $\forall x, y \in V[\mathrm{CDL}]_4$; therefore, $\Psi$ is a RF. To show that $\Psi$ is a minimal RF, we take another RF such that $|\Psi'(x,y)| < |\Psi'(x,y)|$; then, $\Psi'(x,y) < 1$ which shows that $\Psi$ is a minimal resolving function. Consequently,

$$\dim_{Fr}[\mathrm{CDL}]_q = \sum_{i=1}^{2q} \frac{1}{2} = q. \tag{6}$$

*3.3. FMD of Double Path Network.* In this section, our aim is to compute FMD of double path network. The double path network $P_q^2$ is obtained from path network $P_q$ by taking two copies of $P_q$ by taking every vertex $a_i$ in one copy with the open neighbourhood $N(a_i)$ of the correspondence vertex of its second copy. Furthermore, $|V([P_q^2]| = 2q$ and $|E[P_q^2]| = 2q+6$ (for details, see Figure 3) [29].

**Lemma 3.** *Let $P_q^2$ with $q \geq 3$ be a double path network. Then,*

*(a) $|\mathscr{R}(a_i b_i)| = 2$ and $\cup_{i=1}^{q}\mathscr{R}(a_i, b_i) = V[P_q^2]$.*

*(b) $|\mathscr{R}(a_i, b_i)| < |\mathscr{R}(x,y)|$ and $|\mathscr{R}(x,y) \cap \cup_{i=1}^{q}\mathscr{R}(a_i, b_i)| \geq 2$, where $\mathscr{R}(x,y)$ are all the possible resolving neighbourhood sets of $P_q^2$.*

*Proof.* Let $a_i$, $b_i$ be the vertices of $P_q^2$, where $1 \leq i \leq q$, $q \cong 1\,(\mathrm{mod}\,2)$, $j \cong 0\,(\mathrm{mod}\,2)$, $p \cong 0\,(\mathrm{mod}\,2)$, and $q+1 \cong 1\,(\mathrm{mod}\,q)$.

(a) Since $\mathscr{R}(a_i, b_i) = \{a_i, b_i\}$, $|\mathscr{R}(a_i, b_i)| = 2$ and $|\cup_{i=1}^{q}\mathscr{R}(a_i b_i)| = V[P_q^2] = 2q$.

(b) $\mathscr{R}(a_i, b_q) = V(P_q^2) - \{b_{i+q/2}, a_{i+q/2}\}$, $\mathscr{R}(a_i, a_q) = V(P_q^2) - \{b_{i+q/2}, a_{i+q/2}\}$, $\mathscr{R}(b_i, b_q) = V(P_q^2) - \{b_{i+q/2}, a_{i+q/2}\}$, $\mathscr{R}(a_j, a_q) = V(P_q^2) - \{a_{j+p/2}, a_{j+p/2}\}$, $\mathscr{R}(b_j, b_p) = V(P_q^2) - \{b_{j+p/2}, b_{j+p/2}\}$, $\mathscr{R}(a_j, b_p) = V(P_q^2) - \{a_{j+p/2}, b_{j+p/2}\}$. Now with the help of Table 5, we compare the cardinalities of each RN set.

From Table 5, it is observed that $|\mathscr{R}(a_i, b_i)| < |\mathscr{R}(x,y)|$. Since $\cup_{i=1}^{n} R(a_i, b_i) = V[P_q^2]$, $|\mathscr{R}(x,y) \cap \cup_{i=1}^{q}\mathscr{R}(a_i, b_i)| \geq 2$. □

**Theorem 3.** *Let $[P_q^2]$ be a double path network with $q \geq 3$. Then,*

$$\dim_{Fr}\left[P_q^2\right] = q. \tag{7}$$

*Proof.* For different vertices of $P_q^2$, we have the following cases. □

*Case 1.* In this case, we compute all the RN sets of $[P_q^2]$ and their cardinalities with the help of Table 6.

Since $|\mathscr{R}(e)| = |\mathscr{R}(a_1, b_1)| = |\mathscr{R}(a_2, b_2)| = |\mathscr{R}(a_3, b_3)| = |\mathscr{R}(a_1, a_3)| = |\mathscr{R}(b_1, b_3)| = 2$ and $|\mathscr{R}(e)| < |\mathscr{R}(x,y)|$ of $[P_3^2]$, $|\cup_{i=1}^{3}\mathscr{R}(e_i)| = 6$ and $|\mathscr{R}(x,y) \cap \bigcup \mathscr{R}(e)| \geq 2$. Furthermore, the RN sets $\mathscr{R}(e)$ are not pairwise disjoint. Hence, we define a function $\Psi$: $V((P_3^2) \longrightarrow [0,1]$ such that $\Psi(v) = 1/2$ for each $v \in V[P_3^2]$ and $\Psi(x,y) \geq 1$ $\forall x, yV[P_3^2]$; therefore, $\Psi$ is a RF. To show that $\Psi$ is a minimal RF, we take another RF such that $|\Psi'(x,y)| < |\Psi(x,y)|$; then, $|\Psi(x,y)| < 1$. Hence, $\Psi$ is a minimal RF. Consequently,

$$\dim_{Fr}\left[P_3^2\right] = \sum_{i=1}^{6} \frac{1}{2} = 3. \tag{8}$$

*Case 2.* For $q \geq 4$, by Lemma 3, $|\mathscr{R}(a_i, b_i)| = 2$ and $|\cup_{i=1}^{q}\mathscr{R}(a_i, b_i)| = |V[P_q^2]|$ where the elements of $\mathscr{R}(a_i b_i)$ are pairwise disjoint. Hence, we define a constant function $\Psi$: $V((P_q^2) \longrightarrow [0,1]$ such that $\Psi(v) = 1/2$ for each $v \in V[P_3^2]$ and $\Psi(x,y) \geq 1$ $\forall x, yV[P_3^2]$; therefore, $\Psi$ is a RF. To show that $\Psi$ is a minimal RF, we take another RF such that $|\Psi'(x,y)| < |\Psi(x,y)|$; then, $\Psi'(x,y) < 1$ which shows that $\Psi$ is a minimal resolving function, $\forall x, y \in V[P_3^2]$. Consequently,

Table 4: Cardinality of each RN set.

| Cardinality | Comparison |
|---|---|
| $\mathscr{R}_1 = \mathscr{R}(b_1b_2)$ | $V[\mathrm{CDL}]_4 - \{a_1, a_2\}$, |
| $\mathscr{R}_2 = \mathscr{R}(b_2b_3)$ | $V[\mathrm{CDL}]_4 - \{a_2, a_3\}$, |
| $\mathscr{R}_3 = \mathscr{R}(b_3b_4)$ | $V[\mathrm{CDL}]_4 - \{a_3, a_4\}$, |
| $\mathscr{R}_4 = \mathscr{R}(b_4b_1)$ | $V[\mathrm{CDL}]_4 - \{a_4, a_1\}$, |
| $\mathscr{R}_5 = \mathscr{R}(a_1a_2)$ | $V[\mathrm{CDL}]_4 - \{b_1, b_2\}$, |
| $\mathscr{R}_6 = \mathscr{R}(a_2a_3)$ | $V[\mathrm{CDL}]_4 - \{b_2, b_3\}$, |
| $\mathscr{R}_7 = \mathscr{R}(a_3a_4)$ | $V[\mathrm{CDL}]_4 - \{a_1, a_2\}$, |
| $\mathscr{R}_8 = \mathscr{R}(a_4a_1)$ | $V[\mathrm{CDL}]_4 - \{b_4, b_1\}$, |
| $\mathscr{R}_9 = \mathscr{R}(a_1b_2)$ | $V[\mathrm{CDL}]_4 - \{a_2, b_1\}$, |
| $\mathscr{R}_{10} = \mathscr{R}(a_2b_3)$ | $V[\mathrm{CDL}]_4 - \{a_3, b_2\}$, |
| $\mathscr{R}_{11} = \mathscr{R}(a_3b_4)$ | $V[\mathrm{CDL}]_4 - \{a_4, b_3\}$, |
| $\mathscr{R}_{12} = \mathscr{R}(a_4b_1)$ | $V[\mathrm{CDL}]_4 - \{a_4, b_3\}$, |
| $\mathscr{R}_{13} = \mathscr{R}(a_1b_4)$ | $V[\mathrm{CDL}]_4 - \{a_4, b_1\}$, |
| $\mathscr{R}_{14} = \mathscr{R}(b_1a_2)$ | $V[\mathrm{CDL}]_4 - \{b_2, a_1\}$, |
| $\mathscr{R}_{15} = \mathscr{R}(b_2, a_3)$ | $V[\mathrm{CDL}]_4 - \{b_3, a_2\}$, |
| $\mathscr{R}_{16} = \mathscr{R}(b_3, a_4)$ | $V[\mathrm{CDL}]_4 - \{b_4, a_3\}$, |
| $\mathscr{R}_{17} = \mathscr{R}(b_4, a_1)$ | $V[\mathrm{CDL}]_4 - \{b_1, a_4\}$, |
| $\mathscr{R}_{18} = \mathscr{R}(a_1b_1)$ | $\{a_1, b_1\}$, |
| $\mathscr{R}_{19} = \mathscr{R}(a_2b_2)$ | $\{a_2, b_2\}$, |
| $\mathscr{R}_{20} = \mathscr{R}(a_3b_3)$ | $\{a_3, b_3\}$, |
| $\mathscr{R}_{21} = \mathscr{R}(a_4b_4)$ | $\{a_4, b_4\}$, |
| $\mathscr{R}_{22} = \mathscr{R}(a_1, a_3)$ | $\{a_1, a_3, b_1, b_3\}$, |
| $\mathscr{R}_{23} = \mathscr{R}(a_1, b_3)$ | $\{a_1, a_3, b_1, b_3\}$, |
| $\mathscr{R}_{24} = \mathscr{R}(b_1, a_3)$ | $\{a_1, a_3, b_1, b_3\}$, |
| $\mathscr{R}_{25} = \mathscr{R}(b_1, b_3)$ | $\{a_1, a_3, b_1, b_3\}$, |
| $\mathscr{R}_{26} = \mathscr{R}(a_2, a_4)$ | $\{a_2, a_4, b_2, b_4\}$, |
| $\mathscr{R}_{27} = \mathscr{R}(a_2, b_4)$ | $\{a_2, a_4, b_2, b_4\}$. |

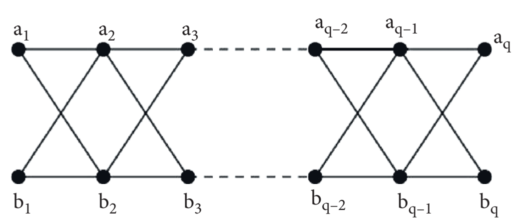


Figure 3: Double path network $[P_q^2]$.

Table 5: The possible resolving neighbourhood sets of $P_q^2$.

| | |
|---|---|
| $\lvert\mathscr{R}(a_i, a_q)\rvert = 2q - 2$ | $2 < 2q - 2$ |
| $\lvert\mathscr{R}(a_i, b_q)\rvert = 2q - 2$ | $2 < 2q - 2$ |
| $\lvert\mathscr{R}(b_i, b_q)\rvert = 2q - 2$ | $2 < 2q - 2$ |
| $\lvert\mathscr{R}(a_j, a_p)\rvert = 2q - 2$ | $2 < 2q - 2$ |
| $\lvert\mathscr{R}(b_j, b_p)\rvert = 2q - 2$ | $2 < 2q - 2$ |

Table 6: Cardinality of each RN set.

| Cardinality RN sets of $[P_q^2]$ | Comparison Elements |
|---|---|
| $\mathscr{R}_1 = \mathscr{R}(a_1, b_1)$ | $\{a_1, b_1\}$, |
| $\mathscr{R}_2 = \mathscr{R}(a_3, b_3)$ | $\{a_3, b_3\}$, |
| $\mathscr{R}_3 = \mathscr{R}(a_1, a_2)$ | $V[P_3^2]$, |
| $\mathscr{R}_3 = \mathscr{R}(a_1, a_3)$ | $\{a_1, a_3\}$, |
| $\mathscr{R}_5 = \mathscr{R}(a_1 b_2)$ | $V[P_3^2]$, |
| $\mathscr{R}_6 = \mathscr{R}(a_1, b_3)$ | $V[P_3^2] - \{b_2, a_3\}$, |
| $\mathscr{R}_7 = \mathscr{R}(a_2, b_1)$ | $V[P_3^2]$, |
| $\mathscr{R}_8 = \mathscr{R}(a_2, b_2)$ | $\{a_2, b_2\}$, |
| $\mathscr{R}_9 = \mathscr{R}(a_2, b_3)$ | $V[P_3^2]$, |
| $\mathscr{R}_{10} = \mathscr{R}(a_2, a_3)$ | $V[P_3^2]$, |
| $\mathscr{R}_{11} = \mathscr{R}(a_3, b_1)$ | $\{a_3, b_1\}$, |
| $\mathscr{R}_{12} = \mathscr{R}(a_3, b_2)$ | $V[P_3^2]$, |
| $\mathscr{R}_{13} = \mathscr{R}(b_1, b_3)$ | $\{b_1, b_3\}$, |
| $\mathscr{R}_{14} = \mathscr{R}(b_1 b_2)$ | $V[P_3^2]$, |
| $\mathscr{R}_{15} = \mathscr{R}(b_2 b_3)$ | $V[P_3^2]$. |

$$\dim_{Fr}\left[P_q^2\right] = \sum_{i=1}^{2q} \frac{1}{2} = q. \tag{9}$$

## 4. Conclusion

This paper deals with the latest invariant of metric dimension called fractional metric dimension, and we have computed exact value of fractional metric dimension of different connected networks like double sunflower network, circular diagonal ladder network, and double path network. Furthermore, it is also proved that the fractional metric dimension of these networks depends on their order. Now, we close our discussion with the following open problem.

Characterization of the connected works attaining the exact value of fractional metric dimension is still an open problem under the condition that the cardinality of the minimum resolving neighborhood sets is greater than 2.

## Data Availability

The data used to support the findings of this study are included within the article. However, the reader may contact the corresponding author for more details of the data.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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