Research Article

Some New Results on Trans-Sasakian Manifolds

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In this paper, we classify trans-Sasakian manifolds which are realized as real hypersurfaces in a complex space form. We also investigate trans-Sasakian manifolds whose Reeb vector fields are harmonic-Killing. The above results bring some new characterizations for the property of trans-Sasakian 3-manifolds.

1. Introduction

In differential geometry of almost-contact Riemannian manifolds, the so-called trans-Sasakian manifolds play important roles when studying topology as well as geometry of almost-contact structures. Here, an almost-contact metric manifold $M^{2n+1}$ of dimension $2n+1$, $n \geq 1$, together with its almost-contact metric structure $(\phi, \xi, \eta, g)$, is said to be a trans-Sasakian manifold (it is, often referred to, of type $(\alpha, \beta)$) (see [1–3]) if it satisfies

$$\nabla_X \phi Y = \alpha (g(X, Y) \xi - \eta(Y)X) + \beta (g(\phi X, Y) \xi - \eta(Y)\phi X),$$

for all vector fields $X$ and $Y$, where both $\alpha$ and $\beta$ are smooth functions. The classical Sasakian, Kenmotsu, and cosymplectic manifolds (see [1]) are all its trivial cases.

In general, a trans-Sasakian manifold of type $(\alpha, \beta)$ is said to be proper (see [4–6]) when either $\alpha$ or $\beta$ vanishes identically. Marrero in [7] proved that a trans-Sasakian manifold of dimension greater than 3 must be proper. However, such a property holds not necessarily true for general trans-Sasakian manifolds of dimension three. In the past decade, to determine on what geometric conditions a connected, compact, or complete trans-Sasakian three-manifold is proper has been proposed by Deshmukh in [8] and later considered by many authors (see recent results by De et al. [9–12], Deshmukh et al. [8, 13–19], Wang and Wang and Liu [20, 21], Wang [4, 22, 23], Zhao [5, 6] and Ma and Pei [24].

It is interesting to point out that trans-Sasakian three-manifolds isometrically immersed in the Euclidean four-space $\mathbb{R}^4$ have been studied in [14]. In the present paper, extending Deshmukh’s above results, we consider a trans-Sasakian manifold of an arbitrary dimension immersed in a complex space form realized as a real hypersurface. As an immediate corollary, we also present a new characterization for the property of trans-Sasakian three-manifolds without compactness restriction. On the other hand, Zhao [6] provided a characterization for the property by considering the Reeb vector field of a trans-Sasakian three-manifold being affine Killing. In the present paper, we generalize such a result by weakening the above restriction; namely, we need only to suppose that the Reeb vector field is harmonic-Killing (see its definition in Section 4).

2. Trans-Sasakian Manifolds

Let $(M^{2n+1}, g)$ be a smooth Riemannian manifold of dimension $2n+1$ on which there exist a $(1,1)$-type, $(1,0)$-type, and $(0,1)$-type tensor fields $\phi$, $\xi$, and $\eta$, respectively. According to Blair [1], $M^{2n+1}$ is called an almost-contact metric manifold if
\[ \phi^2 = -1 \ d + \eta \otimes \xi, \]
\[ \eta(\xi) = 1, \]
\[ \eta \circ \phi = 0, \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]

for any vector fields \( X \) and \( Y \). \( \xi \) is said to be the Reeb or structure vector field. An almost-contact metric manifold is said to be normal if \( [\phi, \phi] = -2 \ d \eta \otimes \xi \), where \([\phi, \phi]\) denotes the Nijenhuis tensor of \( \phi \). An almost-contact metric manifold is said to be trans-Sasakian if it satisfies equality (1). A three-dimensional almost-contact metric manifold is trans-Sasakian if and only if it is normal. This is not necessarily true for higher dimension.

A normal almost-contact metric manifold is said to be an \( \alpha \)-Sasakian manifold if \( d\eta = \alpha \Phi \) and \( d\Phi = 0 \), where \( \alpha \) is a nonzero constant. An \( \alpha \)-Sasakian manifold reduces to a Sasakian manifold when \( \alpha = 1 \). A normal almost-contact metric manifold is called a \( \beta \)-Kenmotsu manifold if it satisfies \( d\eta = 0 \) and \( d\Phi = 2\beta \eta \wedge \Phi \), where \( \beta \) is a nonzero constant. A \( \beta \)-Kenmotsu manifold becomes a Kenmotsu manifold when \( \beta = 1 \). A normal almost-contact metric manifold is said to be a cosymplectic manifold if it satisfies \( d\eta = 0 \) and \( d\Phi = 0 \). Obviously, the set of all \( \alpha \)-Sasakian manifolds (resp., \( \beta \)-Kenmotsu) is a proper subset of that of all trans-Sasakian manifolds of type \( (\alpha, 0) \) (resp., \( (0, \beta) \)).

Putting \( Y = \xi \) into (1) and using (2), we have
\[ \nabla_{X}\xi = -\alpha \phi X + \beta (X - \eta(X)\xi), \]

for any vector field \( X \). In this paper, all manifolds are assumed to be connected.

### 3. Trans-Sasakian Manifolds as Real Hypersurfaces in Complex Space Forms

Let \( M^n(c) \) be a complete and simply connected complex space form which is complex analytically isometric to the following:

(i) A complex projective space \( \mathbb{C}P^n(c) \) if \( c > 0 \)
(ii) A complex Euclidean space \( \mathbb{C}^n \) if \( c = 0 \)
(iii) A complex hyperbolic space \( \mathbb{C}H^n(c) \) if \( c < 0 \)

Here, \( c \) is the constant holomorphic sectional curvature. Let \( M \) be a real hypersurface imbedded in a complex space form \( M^n(c) \) and \( N \) be a unit normal vector field of \( M \). We denote by \( \nabla \) the Levi-Civita connection of the metric \( g \) of \( M^n(c) \) and \( f \) the complex structure. Let \( g \) and \( \nabla \) be the induced metric from the ambient space and the Levi-Civita connection of the metric \( g \), respectively. Then, the Gauss and Weingarten formulas are given, respectively, as follows:
\[ \nabla_X Y = \nabla_X Y + g(AX, Y)N, \nabla_X N = -AX, \]

for any vector fields \( X \) and \( Y \), where \( A \) denotes the shape operator of \( M \) in \( M^n(c) \). For any vector field \( X \), we put
\[ JX = \phi X + \eta(X)N, \]
\[ JN = -\xi. \]

One can check that (2) holds and, hence, on real hypersurfaces, there exist natural almost-contact metric structures. If the structure vector field \( \xi \) is principal, that is, \( A\xi = \delta \xi \) at each point, where \( \delta = \eta(A\xi) \), then \( M \) is called a Hopf hypersurface and \( \delta \) is called Hopf principal curvature. Moreover, applying the parallelism of the complex structure (i.e., \( \nabla g = 0 \)) of \( M^n(c) \) and using (4) and (5), we have
\[ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \]

for any vector fields \( X \) and \( Y \), \( Y \). Let \( R \) be the Riemannian curvature tensor of \( M \). As \( M^n(c) \) is of constant holomorphic sectional curvature \( c \), the Gauss equation of \( M \) in \( M^n(c) \) is given by
\[ R(X, Y)Z = \frac{c}{4} [g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \]

for any vector fields \( X \) and \( Y \), \( Y \).

Because an almost-contact metric structure exists on a real hypersurface, then it is very interesting to ask what almost-contact metric structure can be if it is realized as a real hypersurface in complex space forms? Some authors have studied contact, Sasakian, and generalized Sasakian space form structures on real hypersurfaces (see [25–27]).

**Theorem 1.** Let \( M^{2n-1} \) be a trans-Sasakian manifold. Then \( M^{2n-1} \) is realized as a real hypersurface in a complex space form \( M^n(c) \), \( n \geq 2 \), if and only if the following statements are valid:

1. If \( M^n(c) = \mathbb{C}P^n(c) \), \( M^{2n-1} \) is locally congruent to a geodesic hypersphere.
2. If \( M^n(c) = \mathbb{C}H^n(c) \), \( M^{2n-1} \) is locally congruent to
   - (i) a horosphere
   - (ii) a geodesic hypersphere
   - (iii) a tube around a totally geodesic \( \mathbb{C}H^{n-1} \)
3. If \( M^n(c) = \mathbb{C}^n \), \( M^{2n-1} \) is locally congruent to
   - (i) a hyperplane \( \mathbb{R}^{2n-1} \)
   - (ii) a sphere \( \mathbb{S}^{2n-1} \)
   - (iii) a cylinder over a plane curve \( \gamma \times \mathbb{R}^{2n-2} \)

**Proof.** If a real hypersurface \( M^{2n-1} \) in a complex space form \( M^n(c) \) is trans-Sasakian, by definition, from (1) and (6), we get
\[ \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) = \eta(Y)AX - g(AX, Y)\xi, \]

where \( \alpha \) and \( \beta \) are constants.
for any vector fields $X$ and $YX, Y$. In the above equality, setting $Y = \xi$ gives

$$AX = \eta(AX)\xi - \beta\phi X + \alpha(\eta(X)\xi - X),$$

(9)

for any vector field $X$. Obviously, it follows that $A\xi = \eta(A\xi)\xi$ and hence, $M^{2n+1}$ is Hopf. Using this in the previous equality, we get

$$AX = (\alpha + \eta(A\xi))\eta(X)\xi - \beta\phi X - \alpha X,$$

(10)

for any vector field $X$. Recall that the shape operator is self-adjoint; it follows directly that $\beta = 0$, and hence,

$$AX = (\alpha + \eta(A\xi))\eta(X)\xi - \alpha X,$$

(11)

for any vector field $X$. Now, the hypersurface is totally $\eta$-umbilical. Conversely, the application of the above equality in (6) implies that the hypersurface is always a trans-Sasakian manifold of type $(\alpha, 0)$. Next, we divide our discussions into two cases.

When the ambient space is $CP^n(c)$ or $CH^n(c)$, following [28, 29], we observe that a totally $\eta$-umbilical real hypersurface satisfying (11) is locally congruent to the following:

(i) A geodesic sphere of radius $r$ in $CP^n$ with $a = -((\sqrt{c}/r)\cot(\sqrt{c}/r/2)$, where $0 < r < (\pi/\sqrt{c})$

(ii) A horosphere in $CH^n$ with $a = -((\sqrt{-c}/r)\cosh(\sqrt{-c}/r/2)$, where $0 < r < \infty$

(iii) A geodesic sphere of radius $r$ in $CH^n$ with $a = -((\sqrt{-c}/r)\cosh(\sqrt{-c}/r/2)$, where $0 < r < \infty$

(iv) A tube of radius $r$ around a totally geodesic complex hyperplane $CH^{n-1}(c)$ in $CH^n(c)$ with $a = -((\sqrt{-c}/r)\tanh(\sqrt{-c}/r/2)$, where $0 < r < \infty$

When the ambient space is $C^n$, from Gauss equations (7) and (11), we see that the hypersurface is pseudo-Einstein, i.e.,

$$QX = \alpha((2n - 3)\alpha - \eta(A\xi))X + (2n - 3)\alpha(\alpha + \eta(A\xi))\eta(X)\xi,$$

(12)

for any vector field $X$, where $Q$ denotes the Ricci operator. The remaining proof follows immediately from Proof of Theorem 1 in [27] (see also [30]).

The converse is easy to check.

In view of Theorem 1, a new characterization for the property of trans-Sasakian 3-manifolds is given. We remark that $\alpha$ in two cases in the proof of Theorem 1 is both constant. \(\square\)

**Corollary 1.** A trans-Sasakian 3-manifold is an $\alpha$-Sasakian manifold if it is realized as a real hypersurface in the complex space form.

As pointed out in the Introduction section, a trans-Sasakian 3-manifold of type $(\alpha, \beta)$ realized as a hypersurface in $\mathbb{R}^4$ is isometric to the Sasakian manifold $S^3(\alpha^3)$ provided that the hypersurface is compact. Such a situation occurs in our Theorem 1 in view of (12) and (11) for $n = 2$ (for more details, see [14, Theorem 2]).

### 4. Harmonic-Killing Reeb Vector Field

From [31], a vector field $V$ on a Riemannian manifold $(M, g)$ is called affine Killing if

$$\nabla_V V = 0,$$

(13)

where $\nabla$ denotes the Levi-Civita connection of the metric $g$ (see also [32]). According to [33, 34], a vector field $V$ on a Riemannian manifold is called harmonic-Killing if each local parameter group of infinitesimal transformations associated to $V$ is a group of harmonic maps. For any harmonic-Killing vector field $V$, from Theorem 2.1 in [33], we have

$$\text{tr}(\nabla_V V) = 0.$$

(14)

By considering the Reeb vector field of trans-Sasakian three-manifolds being affine Killing, Zhao [6] studied the property of trans-Sasakian three-manifolds. In this section, we consider a weaker condition on trans-Sasakian manifolds of arbitrary dimensions.

**Lemma 1.** If the Reeb vector field of trans-Sasakian manifolds $M^{2n+1}$ of type $(\alpha, \beta)$ is harmonic-Killing, then we have

$$\xi(\beta) = -2\beta^2.$$  

Moreover, if $n > 1$, we have $d\beta = -2\beta^2\eta$.

**Proof.** Recall that on any differentiable manifold, there holds (see Yano ([35], pp. 23))

$$(L_V Xg - \nabla_V Xg - \nabla_{[X,Y]}g)(Y, Z) = -g(L_V X)(X, Y, Z)$$

$$-g((L_V Y)(X, Z, Y)).$$

(16)

for any vector fields $X, Y$, and $Z$. Notice that in our case, the Riemannian metric $g$ is parallel and it follows that

$$(\nabla_X L_V g)(Y, Z) = g((L_V Y)(X, Y, Z) + g((L_V Y)(X, Z, Y)).$$

(17)

for any vector fields $X, Y$, and $Z$. Changing the roles of $X, Y$, and $Z$ in the above equality, we obtain

$$(\nabla_Y L_V g)(Z, X) = g((L_V Z)(Y, Z, X)$$

$$+ g((L_V Y)(Y, Z, X)),$$

(18)

$$(\nabla_Z L_V g)(X, Y) = g((L_V X)(Z, X, Y)$$

$$+ g((L_V Y)(Z, Y, X)).$$

(19)

for any vector fields $X, Y$, and $Z$. The addition of (17) with (18) gives an equality; subtracting this equality from (19), with the aid of the symmetry of $L_V V$, we have
harmonic-Killing, from (14) and (20), we have

\[ 2g((L_\xi V)(X,Y),Z) = (\nabla_X L_\xi g)(Y,Z) + (\nabla_Y L_\xi g)(Z,X) - (\nabla_Z L_\xi g)(X,Y), \]

(20)

for any vector fields \(X, Y,\) and \(ZX, Y, Z.\)

From (3), we have

\[ \mathcal{L}_\xi g(Y,Z) = 2\beta(g(Y,Z) - \eta(Y)\eta(Z)), \]

(21)

for any vector fields \(X, Y,\) and \(ZX, Y, Z.\) By a direct calculation, taking the covariant derivative of the above equality, with the aid of (3), we have

\[ (\nabla_X L_\xi g)(Y,Z) = 2X(\beta)(g(Y,Z) - \eta(Y)\eta(Z)) + 2\alpha\beta g(\phi X,Y)\eta(Z) + 2\alpha \xi g(\phi X,Z)\eta(Y) - 2\beta^2 g(X,Y)\eta(Z) - 2\beta^2 g(X,Z)\eta(Y) + 4\beta^2 g(\eta(X)\eta(Y),Z), \]

(22)

for any vector fields \(X, Y,\) and \(ZX, Y, Z.\)

We consider a local orthonormal frame \(\{e_1, \ldots, e_{2n+1}\}\) of the tangent space at each point. By a direct calculation, from (22), we have

\[ \sum_{i=1}^{2n+1} (\nabla_i L_\xi g)(e_i, Z) = 2Z(\beta) - 2\xi(\beta)\eta(Z) - 4n\beta^2 \eta(Z), \]

(23)

\[ \sum_{i=1}^{2n+1} (\nabla_i L_\xi g)(e_i, e_i) = 4nZ(\beta), \]

(24)

for any vector field \(Z.\)

If the Reeb vector field of a trans-Sasakian manifold is harmonic-Killing, from (14) and (20), we have

\[ \sum_{i=1}^{2n+1} (\nabla_i L_\xi g)(e_i, Z) + \sum_{i=1}^{2n+1} (\nabla_i L_\xi g)(Z, e_i) = \sum_{i=1}^{2n+1} (\nabla_i L_\xi g)(e_i, e_i), \]

(25)

for any vector field \(Z,\) which is simplified by using (23) and (24) yielding

\[ (n - 1)D\beta = -\xi(\beta) + 2n\beta^2 \xi, \]

(26)

where \(D\) denotes the gradient operator. Obviously, taking the inner product of the above equality with \(\xi\) implies \(\xi(\beta) = -2\beta^2.\) Moreover, if \(n > 1,\) substituting \(\xi(\beta) = -2\beta^2\) into (26) gives \(D\beta = -2\beta^2 \xi.\)

Lemma 2 (see [36]). If on a Riemannian manifold \(M\) there exists a Killing vector field \(\xi\) of constant length satisfying

\[ k^2(\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi) = g(Y,\xi)X - g(X,\xi)Y, \]

(27)

for a nonzero constant \(k\) and any vector fields \(X\) and \(Y,\) then \(M\) is homothetic to a Sasakian manifold.

Based on the above two lemmas, one of our main results is given.

**Theorem 2.** If the Reeb vector field of a compact and simply connected trans-Sasakian 3-manifold of type \((\alpha, \beta)\) is harmonic-Killing, then the manifold is homothetic to a Sasakian 3-manifold.

**Proof.** Taking into account (15), we have

\[ \text{div}^3 \xi = 3\beta^2 \xi(\beta) + \beta^2 \text{div} \xi = -4\beta^4, \]

(28)

where we have used (3). As the manifold is assumed to be compact, applying Stokes’ theorem on the above equality yields \(\beta = 0.\) Moreover, now from (21), we observe that \(\xi\) is Killing of constant length one. We also claim that \(\alpha\) is a constant and such an assertion is the same with the proof of Theorem 3.1 in [18]. If the constant \(\alpha = 0,\) the manifold is cosymplectic. However, this is impossible. In fact, if \(\alpha = 0,\) with the help of (3), we see that

\[ d\eta(X,Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0, \]

(29)

for any vector fields \(X\) and \(Y.\) Then, \(\eta\) is closed. Since the manifold is assumed to be simply connected, then \(\eta\) is exact; i.e., there exists a smooth function \(f\) on the manifold such that \(\eta = df.\) Consequently, \(\xi = Df\) and there exists a point on the manifold on which \(Df\) vanishes, where we have used the compactness of the manifold. However, as seen in Section 2, \(\xi\) is always a unit vector field, contradicting the above statement. Thus, we conclude that \(\alpha\) is a nonzero constant. Finally, by (1), it is easy to check that (27) is valid. In fact, now the manifold is isometric to a three-sphere \(S^3(\alpha^2)\) (see [19]). This completes the proof.

Theorem 2 is an extension of Corollary 3.7.1 in [6].

In Lemma 1, we have obtained a property, i.e., \(d\beta \land \eta = 0.\) In fact, such an equality is just one of the requirements when defining a local conformal cosymplectic manifold in the sense of Olszak (for more details, see [37]).

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

**Authors’ Contributions**

All authors contributed equally and significantly in writing this article. All authors have read and approved the final manuscript.

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