Research Article

Q-Type Spaces of Harmonic Mappings

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1. Introduction

A complex-valued function $h = u + iv$ of $\xi = x + iy$ is said to be a harmonic mapping of an open subset $\Lambda$ of the complex plane $\mathbb{C}$ provided that its related Laplace equation is satisfied on $\Lambda$. That is,

$$\Delta h = 4h_{\xi \xi} = 0$$

where $h_{\xi \xi}$ stands for the mixed complex second partial derivative of $h$.

A harmonic mapping $h$ in the unit disk admits the canonical decomposition $h = f + g$ where $f$ and $g$ are analytic functions, and $g(0) = 0$. Analytic functions are harmonic mappings because their real and imaginary components are harmonic according to the Cauchy-Riemann equations. We shall assume that all the functions under consideration are defined on $D = \{ \xi \in \mathbb{C} : |\xi| < 1 \}$. In the complex plane $\mathbb{C}$, $T$ be its boundary. Let $A$ denote the normalized Lebesgue area measure on $D$, and $dA(\xi)$ indicate the normalized Lebesgue area measure on $D$.

The family of all holomorphic functions on $D$. $Har(D)$ signifies the family of all harmonic functions on $D$. Let $\sigma_v(\xi)$ be the Möbius transformations $\sigma_v : D \to D$ be defined by for each $v \in D$.

$$\sigma_v(\xi) = \frac{v - \xi}{1 - \bar{v}\xi}, \quad \xi \in D.$$  

It is worth noting that $\sigma_v(\sigma_v(\xi)) = \xi$, and therefore $\sigma_v^{-1} = \sigma_v$. It has the following useful property:

$$(1 - |\sigma_v(\xi)|^2) = \frac{(1 - |\xi|^2)(1 - |v|^2)}{|1 - \bar{v}\xi|^2} = \left(1 - |v|^2\right)|\sigma_v'(\xi)|.$$  

We look at the linear structure of the harmonic equivalents of the $\alpha$-Bloch spaces and the $Q^p$ space in particular. We also apply to the harmonic setting and certain well-known features that apply to the analytic case.

A bounded linear operator $T : X \to Y$ is said to be bounded below if there exists $C > 0$, such that $\|Tx\|_Y \geq C\|x\|_X$ for all $x \in X$.

Remark 1. A closed range operator is a linear operator that is bounded below (see Proposition 6.4 in [1]). If the operator is injective, the opposite is true and then the converse also holds (see Theorem 5.17.2 in [2]).

The $C_\phi$ operator is defined on the space of complex-valued functions $f$ with domain $\Lambda$, given an analytic self-map $\phi$ of $D$, and the composition operator induced by $\phi$ is defined as the operator:

$$C_\phi f = f^\phi, \quad \text{for} \quad f \in H(D).$$  

It is obvious that such an operator preserves harmonic mappings immediately. Assuming that analytic functions are
clearly harmonic, the idea of how to extend the harmonic mappings is interesting. The norm on the larger space fits with the Banach space structures of known spaces of analytic functions $X$. When confined to the elements of $X$, the norm is $X$.

The organization of this work will be as follows: we dedicate Section 2 to study some basic facts about the harmonic spaces and the concept of $C_\phi^p$-spaces. In Section 3, we performed characterization of boundedness and compactness of a composition operator of $C_\phi$ for each of these spaces.

We will use the notation $U = V$ to indicate that the quantities are comparable; that is, there exist two positive constants $C_1, C_2$ that satisfy $C_1V \leq U \leq C_2V$. Similarly, if only the first or second inequality holds, we say $U \geq V$ or $U < V$.

2. Harmonic Spaces

In this section, we present some useful properties and auxiliary results to discuss $Q_{H^p}$.

For $f$ analytic in $D$, the Bloch constant of $f$ is

\[ B(f) = \sup_{\xi \in D} (1 - |\xi|^2) |f'(\xi)| < \infty. \]  

(5)

We remember that an analytic function $f$ belongs to $B^a$, if and only if $B^a(f) = \sup_{\xi \in D} (1 - |\xi|^2)^a |f'(\xi)| < \infty$ (see [3]) with norm

\[ \|f\|_{B^a} = |f'(0)| + B^a(f) < \infty. \]  

(6)

Colonna (see [4]) shows that the Bloch constant of $h = f + \overline{g}$ can be expressed in terms of the moduli of the derivatives of $f$ and $g$ as follows:

\[ b_h^a = \sup_{\xi \in D} (1 - |\xi|^2)^a \left( |h_\zeta(\xi)| + |h_\zeta(\xi)| \right). \]  

(7)

Colonna (see [4]) presented the following theorem.

Theorem 1. With $h$ as above, $h \in B_h$ if and only if $f$ and $g$ are in $B$. Furthermore,

\[ \max\{B(f), B(g)\} \leq B_h \leq B(f) + B(g). \]  

(8)

For $a > 0$, the harmonic $a$-Bloch space $B^a_{H^p}$ is the collection of all $h \in \mathcal{H}ar(D)$ such that $b_h^a < \infty$, where

\[ b_h^a = \sup_{\xi \in D} (1 - |\xi|^2)^a \left( |h_\zeta(\xi)| + |h_\zeta(\xi)| \right). \]  

(9)

The harmonic little $a$-Bloch space $B^a_{H^p}$ is defined as the subspace of $B^a_{H^p}$ consisting of the mappings $h \in \mathcal{H}ar(D)$ such that

\[ \lim_{|\xi| \to 1} (1 - |\xi|^2)^a \left( |h_\zeta(\xi)| + |h_\zeta(\xi)| \right) = 0. \]  

(10)

This space is a continuation of Zhu’s harmonic mappings of the (analytic) $a$-Bloch space $B^a$. Thus, representing $h \in \mathcal{H}ar(D)$ as $f + \overline{g}$ with $f, g \in H(D)$ and $g(0) = 0$, we can see that $h_\zeta = f'$ and $h_\zeta = \overline{g}'$. Hence, $b_h^a = \sup_{\xi \in D} (1 - |\xi|^2)^a \left( |f'(\xi)| + |g'(\xi)| \right)$, and

\[ \frac{1}{2} \left( B^a(f) + B^a(g) \right) \leq \max\{B^a(f), B^a(g)\} \leq B^a(f) + B^a(g). \]  

(11)

Consequently, a harmonic mapping $h$ is associated with $B^0_{H^p}$, if and only if the unique functions $f$ and $g$ are analytic on $D$ and $h = f + \overline{g}$ with $g(0) = 0$ and are in the $B^0$ space (see [2, 5] for additional details on the spaces $B^0_{H^p}$). The space $B^0$ is the classical Bloch space for $a = 1$, and the appropriate harmonic extension will be represented by $B^0_{H^p}$. Automorphisms of the unit disk are of the form

\[ \sigma_r(\xi) = \lambda \left( \frac{\nu - \xi}{1 - \nu \xi} \right) \text{ with } |\lambda| = 1 \text{ and } |\nu| < 1. \]  

(12)

In particular, if $\lambda = 1$, then we obtain the involution automorphism that exchanges 0 and $\nu$.

The Green function of $D$ with pole at $w \in D$ is denoted by $g(\xi, w) = -\log|\sigma_r(\xi)|$. The pseudohyperbolic disk with a (pseudohyperbolic) center $\nu \in D$ and a (pseudohyperbolic) radius $r \in (0, 1]$ is represented by $D(\nu, r) = \{\xi \in D : |\sigma_r(\xi)| < r\}$.

Now, we introduce the concept of $C^p_{H^p}$ space which generalizes the concept of $C^p$ space as follows.

Definition 1. For $p \in [0, \infty)$, a function $h \in \mathcal{H}ar(D)$ is said to be in the class $C^p_{H^p}$ if

\[ Q_p(h) = \left( \sup_{\xi \in D} \left| h(\xi) + h(\overline{\xi}) \right|^p \right)^{1/p} < \infty. \]  

(13)

\[ Q_p^p(h) = \left| h(0) \right| + Q_p(h) < \infty. \]  

(14)

The norm of $C^p_{H^p}$ is identified as

\[ \|h\|_{C^p_{H^p}} = |h(0)| + Q_p(h) < \infty. \]  

(15)

It is obvious that the correspondence is ongoing $h \mapsto \|h\|_{C^p_{H^p}}$ is a norm on $C^p_{H^p}$, which we refer to as the harmonic $C^p_{H^p}$.

Because the Green function is conformally invariant, it can be used in a variety of applications $g(\sigma(\xi), \sigma(\overline{\nu})) = g(\xi, \nu)$ for all $\sigma \in Aut(D)$ and $\xi, \nu \in D$, and each class is almost self-evident. $Q^p_{H^p}$ is conformally invariant in the following sense: if $h \in Q^p_{H^p}$, then $Q^p_{H^p}(h \sigma) = Q^p_{H^p}(h)$ for all $\sigma \in Aut(D)$.

Remark 2. $h_\zeta$ reduces to $h'$ when $h$ is an analytic function, and $h_\zeta = 0$. Thus, the analytic $C^p$ space is the set of analytic functions in the harmonic $Q^p$ space, and the respective $C^p$ norms coincide. However, due to the presence of the mixed product resulting from squaring the quantity $|h_\zeta(\xi)| + |h_\zeta(\xi)|$, this choice of the norm does not allow a modification that may lead to a (complex) inner product in this situation. As a result, we will propose an alternative norm on $C^p_{H^p}$ that solves the problem while also extending the analytic $C^p$ norm. We will see that the above-mentioned
Lemma 1. Let \( p \in [0, \infty) \), and \( h \in \mathcal{H}(D) \). Then, \( h \in \mathcal{Q}_p \), if and only if
\[
\sup_{\xi \in D} \left( \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right)^{1/2} < \infty.
\]
(15)

**proof 1.** Proof. By the inequalities \(-2 \log t \geq 1 - t^2, t \in (0, 1]\); as well as
\[
-\log t \leq 4\left(1 - t^2\right), \quad t \in \left(\frac{1}{4}, 1\right),
\]
(16)
it suffices to show
\[
(Q^p(h))^2 \times \sup_{\xi \in D} \left( \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\sigma_\xi| \right)^2 dA(\xi) \right).
\]
(17)

Because \( \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 d\zeta \) is a nondecreasing function of \( r \in (0, 1) \), one has
\[
\int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 d\zeta < \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\xi|^2 \right) dA(\xi),
\]
which implies
\[
\int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\xi|^2 \right) dA(\xi) \leq (Q^p(h))^2.
\]
(19)

This, together with equation (17), applies to \( h^*\sigma_\xi \), and hence
\[
(Q^p(h)) \approx \sup_{\xi \in D} \left( \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\sigma_\xi| \right)^2 dA(\xi) \right)^{1/2}.
\]
(20)

We use the following lemma due to Songxiao [6].

Lemma 2. Let \( X = Q^p \). Then,

1. On compact sets, every bounded sequence \( (f_n) \) in \( X \) is uniformly bounded
2. For any sequence \( (f_n) \) in \( X \) such that \( \|f_n\| \rightarrow 0, f_n - f_n(0) \rightarrow 0 \) uniformly on compact sets

We state and prove the following corollary.

**Corollary 1.** If \( h \) is the real part and imaginary part of an analytic function \( f \), then
\[
Q^p(h) = Q^p(f).
\]
(21)

**proof 1.** Let \( f = \text{Re}(h) \), then \( h \) can be written as \( h = (1/2)(f + \overline{f}) \). Thus,
\[
Q^p(h) = \left( \sup_{\xi \in D} \int_D \left( \left| f'(\xi) + \frac{1}{2} f'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right) \right)^{1/2},
\]
\[
= \left( \sup_{\xi \in D} \int_D \left| f'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right)^{1/2} = Q^p(f).
\]
(22)

Likewise, if \( f = \text{Im}(h) \), then \( h \) can be written as \( h = (1/2i)(f - (1/2i)f) \). So
\[
Q^p(h) = \left( \sup_{\xi \in D} \int_D \left( \left| f'(\xi) + \frac{1}{2} f'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right) \right)^{1/2},
\]
\[
= \left( \sup_{\xi \in D} \int_D \left| f'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right)^{1/2} = Q^p(f).
\]
(23)

\[\square\]

**Theorem 2.** Let \( 0 < p < \infty \) and \( f, g \in H(D) \). Then, \( f, g \in \mathcal{Q}_p \) if and only if \( f + \overline{g} \in \mathcal{Q}_p \). Moreover, if \( g(0) = 0 \), then
\[
\frac{1}{2} \left( \|f\|_{\mathcal{Q}_p} + \|g\|_{\mathcal{Q}_p} \right) \leq \|f + \overline{g}\|_{\mathcal{Q}_p} \leq \|f\|_{\mathcal{Q}_p} + \|g\|_{\mathcal{Q}_p}.
\]
(24)

The above estimate holds even without the assumption \( g(0) = 0 \), as shown in the proof.

**proof 1.** Assume \( f, g \in \mathcal{Q}_p \) and let \( h = f + \overline{g} \). Then,
\[
f^* = h^* = \overline{h}, \quad f^* = \overline{f}, \quad \text{So, using the inequality}
\]
\[
\left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \leq 4 \left( |h_1(\xi)|^2 + |h_2(\xi)|^2 \right).
\]
(25)

This derives from the fact that for \( a, b \geq 0 \),
\[
\left( \frac{a + b}{2} \right)^2 \leq (\max\{a, b\})^2 = \max\{a^2, b^2\} \leq a^2 + b^2,
\]
(26)

we have
\[
\left( \mathcal{Q}_p^p(h) \right)^2 = \sup_{\xi \in D} \int_D \left( |h_1(\xi)| + |h_2(\xi)| \right)^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \]
\[
\leq 4 \left( \sup_{\xi \in D} \int_D \left| f'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) \right) \]
\[
+ \sup_{\xi \in D} \int_D \left| g'(\xi) \right|^2 \left( 1 - |\sigma_\xi| \right)^p dA(\xi) < \infty.
\]
(27)

Therefore, \( h \in \mathcal{Q}_p \) and
\[
\mathcal{Q}_p(f + \overline{g}) \leq 4\left( \mathcal{Q}_p^p(f)^2 + \mathcal{Q}_p^p(g)^2 \right).
\]
(28)

Using the 2-th root, we have
\[
\| (f + \overline{g})^2 \leq 2(\|f\|^2 + \|g\|^2)^{1/2} < 2(\|f\|^2 + \|g\|^2). 
\]

(29)

Then, we derive the upper estimate using the inequality if \(f(0) + \overline{g}(0)\) holds even without the assumption \(g(0) = 0\).

Conversely, assume \(f + \overline{g} \in \mathcal{Q}_H^p\). Then, observing that
\[
|f'(\xi)|^2 + |g'(\xi)|^2 \leq (|f'(\xi)| + |g'(\xi)|)^2,
\]
we obtain
\[
\sup_{\xi \in \mathbb{D}} \left( |f'(\xi)|^2 (1 - |f_p(\xi)|^2) \right) \leq \left( |f_p(\xi)|^2 + \|g_p(\xi)|^2 \right)^{1/2},
\]
\[
\|f + \overline{g}\|_{\mathcal{Q}_H^p} \leq \|f\|_{\mathcal{Q}_H^p} + \|g\|_{\mathcal{Q}_H^p}.
\]
(30)

Hence, noting \(\lim_{p \to 1} \mathcal{Q}_H^p = \mathcal{Q}_H^1\),
\[
\frac{1}{2} \left( \|f\|_{\mathcal{Q}_H^p} + \|g\|_{\mathcal{Q}_H^p} \right) < \left( \|f\|_{\mathcal{Q}_H^p} + \|g\|_{\mathcal{Q}_H^p} \right)^{1/2}.
\]
(33)

When we combine these two inequalities, we get
\[
\frac{1}{2} \left( \|f\|_{\mathcal{Q}_H^p} + \|g\|_{\mathcal{Q}_H^p} \right) \leq \|f + \overline{g}\|_{\mathcal{Q}_H^p}.
\]
(34)

Since \(g(0) = 0\), \(|f(0)| \leq |f(0) + \overline{g}(0)|\). We deduce that
\[
\frac{1}{2} \left( \|f\|_{\mathcal{Q}_H^p} + \|g\|_{\mathcal{Q}_H^p} \right) \leq \|f + \overline{g}\|_{\mathcal{Q}_H^p},
\]
(35)

and then we get the desired result.

The relationship between the \(\mathcal{Q}_p\) seminorm of a harmonic mapping and the \(\mathcal{Q}_p\) seminorms of the corresponding real and imaginary parts is then examined.

\[\square\]

**Proposition 1.** Let \(h\) be harmonic on \(\mathbb{D}\) and let \(u = \text{Re}(h)\) and \(v = \text{Im}(h)\). Then, \(h \in \mathcal{Q}_H^p\), if and only if \(u, v \in \mathcal{Q}_p\).

Moreover,
\[
\frac{1}{4} \left( \|u\|_{\mathcal{Q}_p} + \|v\|_{\mathcal{Q}_p} \right) \leq \|h\|_{\mathcal{Q}_H^p} \leq \|u\|_{\mathcal{Q}_p} + \|v\|_{\mathcal{Q}_p}.
\]
(36)

**Proof.** Assume that \(\beta, \gamma \in \mathcal{Q}_H^p\) are true. The upper estimate arises immediately from the triangle inequality of the norm due to linearity. Assume \(h \in \mathcal{Q}_H^p\) and notice that
\[
J(\beta, \gamma) = \beta_{x_1} \gamma_y - \gamma_x \beta_y,
\]
(37)
gives us
\[
2|J(\beta, \gamma)| \leq \|\nabla\beta\|^2 + \|\nabla\gamma\|^2,
\]
(38)
where \(\nabla\beta = (\beta_x, \beta_y)\), and \(\nabla\gamma = (\gamma_x, \gamma_y)\).

As a result, we have
\[
(\|\nabla\beta\|^2 + \|\nabla\gamma\|^2)^{1/2} < \sqrt{2}(\|\nabla\beta\|^2 + \|\nabla\gamma\|^2)^{1/2}.
\]

Indeed, when equation (39) is squared, the left-hand side yields
\[
\|\nabla\beta\|^2 + \|\nabla\gamma\|^2 + 2J(\beta, \gamma) \leq \sqrt{2}(\|\nabla\beta\|^2 + \|\nabla\gamma\|^2)^{1/2}.
\]
(40)

As a result, we get by canceling opposite terms, ignoring the last word, and merging like terms. \(2(\|\nabla\beta\|^2 + \|\nabla\gamma\|^2)\), which is the square of the right-hand side of the equation (39).

By first expressing the partials with respect to \(\xi\) and \(\bar{\xi}\) in terms of the partials with respect to \(x\) and \(y\), then computing the modulus, and lastly applying the inequality equation (39), we can now compute
\[
|h_{\xi}| + |h_{\bar{\xi}}| = |\beta_x + i\gamma_y| + |\beta_x + i\gamma_y|
\]
\[
= \frac{1}{2} \left( |\beta_x + y\gamma| + |\beta_x - y\gamma| \right)
\]
\[
+ \frac{1}{2} \left( |\beta_x - y\gamma| + |\beta_x + y\gamma| \right)
\]
\[
= \frac{1}{2} \left( (\beta_x + y\gamma)^2 + (\gamma_x - \beta_y)^2 \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( (\beta_x - y\gamma)^2 + (\gamma_x + \beta_y)^2 \right)^{1/2}
\]
\[
= \frac{1}{2} \left( \|\nabla\beta\|^2 + \|\nabla\gamma\|^2 + 2J(\beta, \gamma) \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( \|\nabla\beta\|^2 + \|\nabla\gamma\|^2 - 2J(\beta, \gamma) \right)^{1/2}
\]
\[
\geq \frac{1}{\sqrt{2}} \left( \|\nabla\beta\|^2 + \|\nabla\gamma\|^2 \right)^{1/2}
\]
(41)

where we applied the inequality in the last step
\[
\|(\xi_1, \xi_2)\| \geq \frac{|\xi_1| + |\xi_2|}{\sqrt{2}} \quad \text{for} \quad \xi_j \in \mathbb{C}, \quad j = 1, 2.
\]
(42)

Multiplying by \((1 - |\xi|^2)^{1/2}\) and taking the supremum overall \(\xi \in \mathbb{D}\), as well as by a real-valued harmonic \(\mathcal{Q}_p\) function \(h\), we get
\[
(\mathcal{Q}_p(h))^2 \geq \frac{1}{2} \sup_{\xi \in \mathbb{D}} \mathcal{Q}_p(\nabla\phi)^2 \mathcal{Q}_p(\nabla\gamma)^2 dA(\xi)
\]
\[
\geq \frac{1}{2} \max \{\mathcal{Q}_p(h)^2, \mathcal{Q}_p(h)^2\} \geq \frac{1}{4} (\mathcal{Q}_p(\phi)^2 + \mathcal{Q}_p(\gamma)^2).
\]
(43)
Using inequality equation (42) once more, we have
\begin{equation}
|h(0)| \geq \frac{1}{N^2} (|\beta(0)| + |\gamma(0)|).
\end{equation}
(44)
When equations (43) and (44) are combined, the result is
\begin{equation}
\|h\|_{\ell^p} \geq \frac{1}{4} (\|f\|_{\ell^p} + |y\|_{\ell^p}),
\end{equation}
showing that \(\beta\) and \(\gamma\) are in \(\ell^p\), and that the lower estimate is correct. \(\Box\)

**Theorem 3.** \(\ell^p_H\) is a Banach space under the harmonic norm.

**proof 1.** We simply need to prove completeness now that we have established that \(\ell^p_H\) is a normed linear space. Assume that \((h_n)\) in \(\ell^p_H\) is a Cauchy sequence. The analytic functions are defined by the lower norm estimate in Theorem 2 for each \(n \in N\), \(f, g, h_n\) are in \(\ell^p\) such that \(g_n(0) = 0\) and \(h_n = f_n + \overline{g}_n\), and the sequences \((f_n)\) and \((g_n)\) are Cauchy. Since \(\ell^p\) is complete, these sequences converge in the \(\ell^p\) norm to some analytic functions \(f\) and \(g\), respectively.

Define \(h = f + \overline{g}\). Then, by the upper estimate in Theorem 2, \(h \in \ell^p_H\) and
\begin{equation}
\|h_n - h\|_{\ell^p} \leq 2 \left(\|f_n - f\|_{\ell^p} + \|g_n - g\|_{\ell^p}\right) \longrightarrow 0,
\end{equation}
as \(n \longrightarrow \infty\). Thus, \(h_n \longrightarrow h\) in \(\ell^p_H\), proving the completeness of \(\ell^p_H\). \(\Box\)

**Theorem 4.** \(\ell^p_H \subset \ell^p_H\), for \(0 < p < \infty\). In addition, there is a constant \(C > 0\) that is purely dependent on \(p\), such that for \(h \in \ell^p_H\),
\begin{equation}
\|h\|_{\ell^p} \leq C\|h\|_{\ell^p_H}.
\end{equation}
(47)

\[\ell^p(H^p) = \sup_{x \in D} \int_D \left(1 - |\varphi,(\xi)|^2\right)^p \left[\left|h_{\phi}(\varphi,(\xi))\right| + \left|h_{\phi}^\prime(\varphi,(\xi))\right|\right]^2 dA(\xi)
\]
\[= \sup_{x \in D} \int_D \left(1 - |\varphi,(\xi)|^2\right)^p \left[\left|h_{\phi}(\varphi,(\xi))\right|^2 + \left|h_{\phi}^\prime(\varphi,(\xi))\right|^2\right] dA(\xi)
\]
\[= \sup_{x \in D} \int_D \left(1 - |w|^2\right)^p \left[\left|h_{\phi}(\varphi,(w))\right|^2 + \left|h_{\phi}^\prime(\varphi,(w))\right|^2\right] |\varphi,(w)|^2 dA(\xi)
\]
\[= \sup_{x \in D} \int_D \left(1 - |w|^2\right)^p \frac{1}{|\varphi,(w)|^2} \left[\left|h_{\phi}(\varphi,(w))\right|^2 + \left|h_{\phi}^\prime(\varphi,(w))\right|^2\right] dA(\xi)
\]
\[= \sup_{x \in D} \int_D \left(1 - |w|^2\right)^p \left[\left|h_{\phi}(\varphi,(w))\right|^2 + \left|h_{\phi}^\prime(\varphi,(w))\right|^2\right] dA(\xi) = \ell^p(h^p),
\]
as desired. \(\Box\)

**Theorem 6.** For \(0 < p < \infty\), there is a constant \(C > 0\) such that for all \(h \in \ell^p_H\) and all \(z \in D\),
\begin{equation}
|h(\xi)| \leq C \left(1 + \left(\frac{1}{2} \log \frac{1 + |\xi|}{1 - |\xi|}\right)^{(1/2)}\right)\|h\|_{\ell^p_H},
\end{equation}
(54)
proof 1. Let $h \in \mathcal{C}_H$ and $z \in \mathbb{D}$ be fixed. Assume $\xi \neq 0$ because the result is plainly valid for $\xi = 0$. The triangle inequality and by Proposition 1 in [8] then show that there exists a positive constant $c_0$ such that

$$|\lambda(\xi)| \leq |f(\xi)| + |g(\xi)|$$

$$\leq c_0 \left( 1 + \left( \frac{1}{2} \log \frac{1 + |\xi|}{1 - |\xi|} \right)^{1/2} \right) \left( \|f\|_{\mathcal{C}_H} + \|g\|_{\mathcal{C}_H} \right)$$

(55)

$$\leq 2c_0 \left( 1 + \left( \frac{1}{2} \log \frac{1 + |\xi|}{1 - |\xi|} \right)^{1/2} \right) \|h\|_{\mathcal{C}_H},$$

where Theorem 2 was used to get the last inequality. Taking $C = 2c_0$ as an example, the following is the outcome.

In a similar way as Lemma 2.9 in [7], we state and prove the following theorem in harmonic $\mathcal{O}_H$ space.

Theorem 7. Let $0 < p < \infty$. Then,

(a) Every bounded sequence $(h_n)$ in $\mathcal{C}_H$ is uniformly bounded on compact subsets of $\mathbb{D}$

(b) For any sequence $(h_n)$ in $\mathcal{C}_H$ such that $\|h_n\|_{\mathcal{C}_H} \to 0$, the sequence $h_n - h_n(0)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$

Proof. To prove (a), let $M = \sup_{\xi \in \mathbb{D}} \|h_n\|_{\mathcal{C}_H}$. Let $C$ be compact in $\mathbb{D}$ and let $r_0 = \max_{\xi \in C} |\xi| \in [0, 1]$. Then, by Theorem 5,

$$\sup_{\xi \in \mathbb{D}} \|h_n\|_{\mathcal{C}_H} \leq c_0 \left( 1 + \left( \frac{1}{2} \log \frac{1 + r_0}{1 - r_0} \right)^{1/2} \right) M.$$

(56)

Therefore, $(h_n)$ in $\mathcal{C}_H$ is uniformly bounded on $C$.

Next, to prove (b), let $C$ be compact in $\mathbb{D}$ and let $r_0$ be as above. Since $\|h_n\|_{\mathcal{C}_H} \to 0$, by the definition of norm in $\mathcal{C}_H$, it follows that $h_n(0) \to 0$ as $n \to \infty$.

By Theorem 6, and the triangle inequality, we get

$$|h_n(\xi)| = |h_n(\xi) - h_n(0)| \leq 2c_0 \left( 1 + \left( \frac{1}{2} \log \frac{1 + r_0}{1 - r_0} \right)^{1/2} \right) \|h_n(\xi) - h_n(0)\|_{\mathcal{C}_H}$$

$$\leq 2c_0 \left( 1 + \left( \frac{1}{2} \log \frac{1 + r_0}{1 - r_0} \right)^{1/2} \right) \left( \|h_n(\xi)\|_{\mathcal{C}_H} + \|h_n(0)\|_{\mathcal{C}_H} \right).$$

Therefore, $h_n - h_n(0) \to 0$ uniformly on $C$. □

3. Boundedness and Compactness

Operator Norm

The following characterization of boundedness and compactness of a composition operator of $C_\alpha$ sending one subclass to another is obtained by connecting the $\mathcal{O}_H$ norm of a harmonic mapping $F = f + \bar{g}$ (such that $g(0) = 0$) with the $\mathcal{O}_H$ norms of the associated analytic functions $f$ and $g$.

Corollary 2. Let $p \in (0, \infty)$ and $f \in H$ satisfy

$$N(f) = \sup_{b \in \mathbb{C}} \int_{|\xi - b| < 1} n(f, \xi) dA \xi < \infty.$$ 

(58)

Then, $f \in \mathcal{O}_H$ if and only if $f \in \mathcal{B}$.

We need the following lemmas in the sequel.

Lemma 3 (see [9]). Let $a \in (0, \infty)$ and suppose that $f(\xi) = \sum_{j=1}^{\infty} |y_j|^a$ belongs to the Hadamard gap class. Then, $f \in \mathcal{B}_a$, if and only if

$$\sup_{|y_j|} |y_j|^a < \infty, \text{ where } N = \{1, 2, 3, \ldots\}. $$

(59)

proof 1. For a large number $\mu \in \mathbb{N}$, choose a gap series

$$h_1(\xi) = \sum_{j=0}^{\infty} \xi^j, \quad z \in \mathbb{D}. $$

(60)

Then, apply Lemma 3 to infer that $(1 - |\xi|)^a|h_1(\xi)| \leq \lambda$ holds for all $\xi \in \mathbb{D}$, where $\lambda$ is a constant. Furthermore, let us verify

$$(1 - |\xi|)^a|h_1(\xi)| \geq \lambda, \quad 1 - \mu^{-k} \leq |z| \leq 1 - \mu^{-k(1/2)}, \quad k \in \mathbb{N}. $$

(61)

Observe that for any $\xi \in \mathbb{D}$,

$$|h_1(\xi)| \geq \mu^k |\xi|^k - \sum_{j=0}^{k-1} \mu^j |\xi|^j - \sum_{j=k+1}^{\infty} \mu^j |\xi|^j = \theta_1 - \theta_2 - \theta_3. $$

(62)

And then, fix a $\xi$ with $|\xi| \in [1 - \mu^{-k}, 1 - \mu^{-k(1/2)}]$, $k \in \mathbb{N}$, and put $x = |\xi|^2$.

Thus,

$$\left(1 - \mu^{-k}\right)^k \leq x \leq \left(1 - \mu^{-k(1/2)}\right)^{k^{1/2}} \mu^{k^{1/2}}. $$

(63)

If $\mu$ is large enough, then for $k \geq 1$ one has

$$\frac{1}{3} \leq x \leq \left(\frac{1}{2}\right)^{\mu^{1/2}}, $$

(64)

and hence $\theta_3 \geq (\mu^k/3)$. Since it is easy to establish

$$\theta_2 \leq \frac{k-1}{\mu^k}, $$

(65)

it remains to deal with the third term $\theta_3$. Noting that

$$|\xi|^k \leq |\xi|^k \left(\mu^k + 1\right), \quad n \geq k + 1, $$

(66)

namely, in $\theta_3$, the quotient of two successive terms is not greater than the ratio of the first two terms, and one finds
that the series of $\theta_1$ is controlled by the geometric series having the same first two terms. Accordingly, equation (65) is applied to produce

$$\theta_1 \leq \mu^{k+1} |\xi|^{\mu+1} \sum_{j=0}^{\infty} \left( |\xi|^{\mu+1} \right)^{j}$$

$$= \frac{\mu^{k+1} |\xi|^{\mu+1}}{1 - \mu |\xi|^{\mu+1}} = \frac{k}{1 - \mu x^{\mu}}$$

$$\leq \frac{\mu}{1 - \mu x^{\mu}} \frac{\mu (1/2)^{\mu/2}}{1 - \mu (1/2)\mu (3/2) - \mu (1/2)}.$$ 

The preceding estimates for $\theta_1$, $\theta_2$, and $\theta_3$ imply

$$\left| (h_1)_{\xi} (x) \right| \geq \frac{\mu}{4} \left( k + (1/2) \right),$$

$$\geq \frac{1}{4 \mu^{(1/2)} (1 - |\xi|)}$$

$$\geq \frac{1}{4 \mu^{(1/2)} (1 - |\xi|)}$$

Similarly, one can show that

$$\left| (h_1)_{\xi} (\xi) \right| \geq \frac{1}{4 \mu^{(1/2)} (1 - |\xi|)}$$

In a completely similar manner, one can prove that if

$$h_2 (\xi) = \sum_{j=0}^{\infty} \theta_j^{(1/2)} \xi^j,$$ 

then $(1 - |\xi|)^2 |(h_2)_{\xi} (\xi)| \leq \lambda$ for all $\xi \in \mathbb{D}$ (owing to Lemma 3) and

$$(1 - |\xi|) |(h_2)_{\xi} (\xi)| \leq \lambda, \quad 1 - \mu^{-\left( k+1 \right)} \leq |\xi| \leq 1 - \xi^{-\left( k+1 \right)}, \quad k \in \mathbb{N},$$

we get

$$\left| (h_2)_{\xi} (\xi) \right| \geq \frac{1}{4 \mu^{(1/2)} (1 - |\xi|)}$$

Similarly, one can show that

$$\left| (h_2)_{\xi} (\xi) \right| \geq \frac{1}{4 \mu^{(1/2)} (1 - |\xi|)}$$

By addition equations (69), (70), (73), and (74), we have

$$\left| (h_1)_{\xi} (\xi) \right| + \left| (h_1)_{\xi} (\xi) \right| + \left| (h_2)_{\xi} (\xi) \right| + \left| (h_2)_{\xi} (\xi) \right| \leq \left( 1 - |\xi| \right)^{-n}, \quad \xi \in \mathbb{D}. \quad (75)$$

Our lemma is therefore proved.

\[ \square \]

**Theorem 8.** Let $\alpha, \rho \in (0, \infty)$ and $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic. Then,

(i) The operator $\mathcal{C}_\phi : \mathcal{B}^{\alpha}_{\mathbb{D}} \rightarrow \mathcal{Q}^{\alpha}_{\mathbb{D}}$ is bounded if and only if

$$\sup_{\xi \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left| \phi' (\xi) \right|^2}{\left( 1 - |\phi (\xi)|^2 \right)^{\frac{n}{2}}} \left( 1 - |\sigma_{\rho}(\xi)|^2 \right)^{\rho} dA (\xi) < \infty. \quad (76)$$

(ii) The operator $\mathcal{C}_\phi : \mathcal{Q}^{\rho}_{\mathbb{D}} \rightarrow \mathcal{Q}^{\rho}_{\mathbb{D}}$ is compact if and only if $\phi \in \mathcal{C}^{1}_{\mathbb{D}}$ and

$$\lim_{r \rightarrow 0} \sup_{\xi \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left| \phi' (\xi) \right|^2}{\left( 1 - |\phi (\xi)|^2 \right)^{\frac{n}{2}}} \left( 1 - |\sigma_{\rho}(\xi)|^2 \right)^{\rho} dA (\xi) = 0. \quad (77)$$

(iii) The operator $\mathcal{C}_\phi : \mathcal{Q}^{\rho}_{\mathbb{D}} \rightarrow \mathcal{B}^{\alpha}_{\mathbb{D}}$, is bounded if and only if

$$\frac{\left| \phi' (\xi) \right|^2}{\left( 1 - |\phi (\xi)|^2 \right)^{\frac{n}{2}}} \left( 1 - |\sigma_{\rho}(\xi)|^2 \right)^{\rho} < \infty. \quad (78)$$

(iv) The operator $\mathcal{C}_\phi : \mathcal{Q}^{\rho}_{\mathbb{D}} \rightarrow \mathcal{B}^{\alpha}_{\mathbb{D}}$ is compact if and only if $\phi \in \mathcal{C}^{1}_{\mathbb{D}}$ and

$$\lim_{r \rightarrow 0} \sup_{\xi \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left| \phi' (\xi) \right|^2}{\left( 1 - |\phi (\xi)|^2 \right)^{\frac{n}{2}}} \left( 1 - |\sigma_{\rho}(\xi)|^2 \right)^{\rho} dA (\xi) = 0. \quad (79)$$

**Proof 1.**

(i) consists of a simple computation and a reformulation of the definition of boundedness from Lemmas 1 and 4.

(ii) Let $\phi \in \mathcal{C}^{1}_{\mathbb{D}}$ and let $f$ and $g$ be analytic on $\mathbb{D}$ such that $g (0) = 0$ and $h = f + g$.

By Lemma 1, we are required to show that if a sequence $\{h_n\} \subset \mathbb{D}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\{C_{\phi} h_n\}$ converges to 0.

For each $r \in (0, 1)$, set $\mathcal{B}_r = (\mathbb{D}/\mathcal{B}_r)$. So $\|h r^\phi \|_{\mathcal{B}_r}$ tends to 0 uniformly on $\mathcal{B}_r$. And hence Lemma 4, for a given $\varepsilon > 0$, there is an integer $N > 1$ such that as $n \geq N$,

$$\sup_{\xi \in \mathbb{D}} \int_{\mathbb{D}} \left| \left( \mathcal{C}_{\phi} h_n \right) (\xi) \right|^2 + \left| \left( \mathcal{C}_{\phi} h_n \right) (\xi) \right|^2 \left( 1 - |\sigma_{\rho}(\xi)|^2 \right)^{\rho} dA (\xi) < \varepsilon\|\phi\|_{\mathcal{B}_r}^2. \quad (80)$$

Meanwhile, one may deduce from equation (70) and the evolution of the derivatives of $\mathcal{B}^\alpha$ functions that for any $\varepsilon > 0$, there exists a $\delta \in (0, 1)$, such that $r \in [\delta, 1)$. 


Thus, for each \( r \in (0, 1) \),
\[
N^{2r} r^{2N-2} \sup_{\nu \in D} \int_D \left| \phi'(\xi) \right|^2 \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(83)

By taking \( r \geq N^{- (a/N - 1)} \), we get
\[
\sup_{\nu \in D} \int_D \left| \phi'(\xi) \right|^2 \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(84)

With equation (76) in mind, we can see that for each \( h \in B_{ge} \) and for every \( \epsilon > 0 \), there is a \( \delta = \delta(h, \epsilon) \) (depending on \( h \) and \( \epsilon \)), such that as \( r \in [\delta, 1) \),
\[
sup_{\nu \in D} \int_D \left| \phi'(\xi) \right|^2 \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(85)

As a result, by equation (76), one has arrived to
\[
T(h, \phi, p, r) = \sup_{\nu \in D} \int_D \left[ \left( \hat{h} \varphi_\nu(\xi) \right)^2 - \left( \hat{h} \varphi_\nu(\xi) \right)^2 \right] \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(86)

In fact, if \( h_\nu(\xi) = h(t\xi) \) for \( h \in B_{ge} \) and \( t \in (0, 1) \), then \( h_\nu \rightarrow h \) uniformly on compact subsets of \( D \) as \( t \rightarrow 1 \). Since \( C_{Q_H} : B_{ge} \rightarrow Q_H^p \) is compact,
\[
sup_{\nu \in D} \left| \varphi_\nu(\xi) \right|^2 \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(87)

Furthermore, from Lemma 4, it yields that for every \( \epsilon > 0 \), there is a \( \nu \in (0, 1) \) to insure
\[
\left\| h^* \phi - h^* \phi^\nu \right\|_{C^0} < \epsilon \quad \text{as } \nu \rightarrow 1 
\]  
Since \( C_{Q_H} \) sends \( B_{ge} \) to a relatively compact subset of \( \mathcal{D}_H^p \), there exists, for every \( \epsilon > 0 \), a finite collection of functions \( h_0, h_2, \ldots, h_N \) in \( B_{ge} \), such that for each \( h \in B_{ge} \), there is a \( k \in \{1, 2, \ldots, N\} \) to guarantee
\[
\sup_{\nu \in D} \int_D \left[ \left( h_\nu \varphi_\nu(\xi) \right)^2 - \left( \hat{h} \varphi_\nu(\xi) \right)^2 \right] \left( 1 - |\sigma(\xi)|^2 \right)^p \chi(r) dA(\xi) < \epsilon. 
\]  
(88)
Now, equation (77) is used to induce a \( \delta = \max_{1 \leq k \leq N} \delta (h_k, \epsilon) \) so that as \( r \in [\delta, 1) \).

Thus,

\[
\sup_{h \in B_{\mathbb{R}^n}} \sup_{\epsilon \in D} \int_{\mathbb{D}} \left( \left| (h^2 \varphi, \xi)_\ell (\xi) \right| + \left| (h \varphi, \xi)_\ell (\xi) \right| \right)^2 - \left| \left| (h^2 \varphi, \xi)_\ell (\xi) \right| + \left| (h \varphi, \xi)_\ell (\xi) \right| \right|^2 \left( 1 - \sigma_\epsilon (\xi) \right)^2 dA (\xi) \leq \epsilon. \tag{89}
\]

By Lemma 4, there are two functions \( h_1, h_2 \in B_{\mathbb{R}^n} \) such that

\[
\left| (h_1)_\ell (\xi) \right| + \left| (h_1)_\ell (\xi) \right| + \left| (h_2)_\ell (\xi) \right| \tag{90}
\]

for all \( \xi \in D \). Thus, equation (78) implies

\[
\sup_{\epsilon \in D} \int_{\mathbb{D}} \left( \left| \phi (\xi) \right|^2 \right) \left( 1 - \sigma_\epsilon (\xi) \right)^2 \chi (r) dA (\xi) < \epsilon, \tag{91}
\]

so that equation (70) follows.

(iii) Assume that \( C_\phi : Q_H^\alpha \longrightarrow \mathcal{B}_H^\alpha \) is bounded. Fix \( \xi \in \mathbb{D} \). Let \( \gamma = \phi (\xi) \) and pick \( h_\gamma (\xi) = -\log (1 - \gamma) \).

Then, \( h_\gamma \in Q_H^\alpha \) with \( \| h_\gamma \| > 1 \), due to Corollary 2. By the boundedness of \( C_\phi \), we have

\[
\| h \| \geq \| \gamma (\xi) \| + \| C_\phi (\xi) \| \geq 1 - \| \xi \| \phi (\xi), \tag{92}
\]

Appealing to these constructions, we obtain

\[
\| C_\phi (\xi) \| \geq (1 - \| \xi \| )^2 \left( \left| (h \varphi, \xi)_\ell (\xi) \right| + \left| (h \varphi, \xi)_\ell (\xi) \right| \right)^2 \left( 1 - \sigma_\epsilon (\xi) \right)^2 dA (\xi) \leq \epsilon.
\]

for \( n \in \mathbb{N} \), so \( C_\phi h_n \) does not converge to \( C_\phi f_0 \) in norm. Hence, \( C_\phi : Q_H^\alpha \longrightarrow \mathcal{B}_H^\alpha \) is not compact. This contradicts the above hypothesis.

On the other hand, let \( \phi \in \mathcal{B}_H^\alpha \) and equation (76) hold. In order to show that \( C_\phi : Q_H^\alpha \longrightarrow \mathcal{B}_H^\alpha \) is compact, it suffices to verify that if \( h_n \) is a bounded sequence in \( Q_H^\alpha \) (i.e., \( \sup_{n \in \mathbb{N}} < \infty \)) and it converges to 0 uniformly on any compact subset of \( \mathbb{D} \), then \( \{ C_\phi h_n \} \) approaches 0. By equation (77), we obtain that for any \( \epsilon > 0 \), there exists a \( \delta \in (0, 1) \), such that whenever \( \| \phi (\xi) \| > \delta \),

\[
\left( 1 - \| \xi \| \right)^n \left( \left| (C_\phi h_n) (\xi) \right| + \left| (C_\phi h_n) (\xi) \right| \right) \leq \epsilon \sup_{m \in \mathbb{N}} \| h_m \| \| C_\phi h_m \| < \epsilon \sup_{m \in \mathbb{N}} \| h_m \| O_H^\alpha. \tag{95}
\]
Yet, if $|\phi(\xi)| \leq \delta$, then $(1 - |\xi|^2)^\alpha \left[ |(h_n)\xi(\phi(\xi)) + (h_n)\tau(\phi(\xi))| \right] \to 0$ owing to the fact $h_n$ tend to 0 uniformly on compact subsets of $D$, and hence,

$$(1 - |\xi|^2)^\alpha \left[ |(h_n)\xi(\phi(\xi)) + (h_n)\tau(\phi(\xi))| \right] \leq \|C_{\phi}h_n\|_{B_{\alpha}} \left( 1 - \delta^2 \right)^{-\alpha} \left( 1 - |\xi|^2 \right)^\alpha \left[ |(h_n)\xi(\phi(\xi)) + (h_n)\tau(\phi(\xi))| \right] \to 0.$$  \hfill (96)

We are done because the prior estimations force $\|C_{\phi}h_n\|_{B_{\alpha}} \to 0$. \hfill □

4. Conclusion

In this article, we studied the boundedness and compactness of composition operators acting on a certain class of harmonic spaces. The author characterizes when the composition operator $C_{\phi}: B_{\alpha}^p \to C_{\alpha}^p$ induced by a holomorphic self-map $\phi$ of $D$ is bounded and compact. Here, $C_{\alpha}^p$ is the collection of all harmonic functions of the unit disk such that

$$Q^p(h) = \left( \sup_{\xi \in D} \int_D \left( |h_\xi(\xi)| + |h_\tau(\xi)| \right)^2 g^p(\xi, \nu) dA(\xi) \right)^{1/2} < \infty.$$ \hfill (97)

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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