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Research Article

A New Member of T-X Family with Applications in Different Sectors

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This paper proposes a member of the T-X family that incorporates heavy-tailed distributions, known as "a new exponential-X family of distribution." As a special case, the paper studies a submodel of the proposed class named a "new exponential Weibull (NEx-Wei) distribution." Some mathematical properties including hazard rate function, ordinary moments, moment generating function, and order statistics are discussed. Furthermore, we adopt the method of MLE (maximum likelihood estimation) for estimating its model parameters. A brief Monte Carlo simulation study is conducted to evaluate the performances of the MLEs based on biases and mean square error. Finally, we provide a comprehensive study to illustrate the introduced approach by analyzing three real data sets from different disciplines. The analytical goodness of fit measure of the proposed distribution is compared with other well-known distributions. We hope that the proposed class may produce many more new distributions for fitting monotonic and nonmonotonic data in the field of reliability analysis and survival analysis as well.

1. Introduction

In a number of practical areas such as engineering, biomedical, and actuarial sciences, the observations are generally positive in nature and have a unimodal and hump-shaped distribution. In such scenarios, extreme values form thick right tails, thus, requiring heavy-tailed distributions to model the data. For instance, in engineering, modeling the unusual phenomena associated with the tails of a statistical distribution is of main interest. Earthquakes, floods, hurricanes, tsunamis, and electrical and power outages market risk are some of the examples of such extreme/rare events [1]. In insurance losses, the data are generally recorded on a positive scale, unimodal, hump-shaped, and positively skewed and have a thick right tail [2]. Also, in health service research, medical expenses that cross a given threshold [3] and the length of stay in a hospital generally represent highly skewed and heavily tailed data [4].

All the above-mentioned scenarios and the rate at which they happen are associated with the distribution in terms of shape and the heaviness of its tails. Classical distributions are not suitable for modeling this type of data [5]. Researchers have observed that the use of gamma, exponential, and Weibull models is discouraged in modeling insurance data because of their inefficient results. Consequently, it has been concluded that it is better to use probability distributions having maximum flexibility in order to get higher accuracy in modeling heavy-tailed data than the exponential distribution [6]. To this end, efforts are put on to introduce new "heavy-tailed distributions"; see [7–11].

Distributions where the probabilities on their right tails are greater than the classical exponential models are known as heavy-tailed distributions [12]. For instance, for a cumulative distribution function, we have

$$\lim_{x \longrightarrow \infty} \frac{e^{-px}}{1 - G(x)} = 0, \tag{1}$$

for any $p > 0$; further details are given in [13,14].

The relevant methods proposed in the literature, and mentioned in the references herein, may be very useful in bringing more flexibility to existing distributions. However, they lack flexibility in terms of inference and computations to derive their distributional properties [8]. Another prominent approach relates to the composition of two or more distributions based on predefined weights, which gives an improved fit for heavy-tailed losses [15–18]. It is, therefore, important to introduce a new class of models either from the existing classical distributions or from a new family of distributions to model heavy-tailed data from various fields of life.

Motivated by these concerns, this paper proposes a novel family of heavy-tailed distributions using the T-X technique without adding additional parameters. The suggested method, called "a new exponential-X family of distributions" offers a reliable fit for insurance data.

The remainder of the paper is arranged as follows: Section 2 discusses the proposed method based on the T-X family; see Alzaatreh et al. [19]. Section 3 presents a new exponential Weibull (NEx-Wei) distribution. Some basic mathematical properties of the proposed family are studied in Section 4. Parameters estimation based on the maximum likelihood estimation method is described in Section 5. In the same section, a Mote Carlo simulation study is also conducted. Applications of the proposed family of distributions on data from vehicle insurance loss, engineering, and medicine are illustrated in Section 6. Finally, Section 7 gives the conclusion of the work based on the proposed distribution.

2. Proposed Method

In this section, we introduce a new modified method to obtain a new lifetime distribution. The proposed method is introduced by combining the exponential model having PDF (probability density function) $m(t) = e^{-t}$ with the T-X family proposed by Alzaatreh et al. [19].

Consider a random variable, say T, to be a baseline random variable with PDF $m(t)$, where $T \in [\pi_1, \pi_2]$ for $-\infty \leq \pi_1 < \pi_2 \leq \infty$. Let X be a random variable with CDF (cumulative distribution function) $K(x; \omega)$ depending on the parameter vectorω. Let $W[K(x; \omega)]$ be a function of CDF of y, satisfying the following three conditions.

(i) $W[K(x; \omega)] \in [\pi_1, \pi_2]$,
(ii) $W[K(x; \omega)]$ is differentiable and monotonically increasing,
(iii) $W[K(x; \omega)] \longrightarrow \pi_1$ as $x \longrightarrow -\infty$ and $W[K(x; \omega)] \longrightarrow \pi_2$ as $x \longrightarrow \infty$.

According to the Alzaatreh et al. [19] the CDF of the T-X family method is defined by

$$F_{T-X}(x) = F(x; \omega) = \int_{\pi_1}^{W[K(x;\omega)]} m(t)\mathrm{d}t, \quad x \in \mathbb{R}, \tag{2}$$

where $W[K(x; \omega)]$ satisfies certain conditions presented (I-III). The PDF of T-X distribution, corresponding to equation (1), is given by

$$\begin{aligned} f_{T-X}(x) &= f(x; \omega) \\ &= m\{W[K(x; \omega)]\}\left\{\frac{d}{dx} W[K(x; \omega)]\right\}, \quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

By using the T-X family of distributions, several novel distribution classes have been proposed in the literature. Table 1 provides some $W[K(x; \omega)]$ expressions for some of the widely used members of the T-X family.

Now, by using $m(t) = e^{-t}$ and setting $W[K(x, \omega)] = -\log\{e^{(1-K(x,\omega))} - 1/e - [1 - K(x, \omega)]\}$ in equation (2), we get the CDF of the new Exponential-X family, given by

$$F(x; \omega) = 1 - \left(\frac{e^{[1-K(x,\omega)]} - 1}{e - [1 - K(x, \omega)]}\right), \quad x \in \mathbb{R}, \tag{4}$$

where $K(x, \omega)$ is the CDF of the baseline distribution which may depend on $\omega \in \mathbb{R}$. The PDF of the NEx-X family associated with equation (4) is

$$\begin{aligned} f(x; \omega) = {} & \frac{k(x, \omega)}{(e - [1 - K(x, \omega)])^2} \\ & \left\{(e + K(x, \omega))e^{[1-K(x,\omega)]} - 1\right\}, \quad x \in \mathbb{R}, \end{aligned} \tag{5}$$

where $k(x, \omega) = (\partial K(x, \omega)/\partial x)$.

Similarly, the HF (hazard function) and SF (survival functions) of the NEx-X family are provided by (6) and (7), respectively.

$$h(x; \omega) = \frac{k(x, \omega)}{(e - [1 - K(x, \omega)])\left(e^{[1-K(x,\omega)]} - 1\right)} \left\{(e + K(x, \omega))e^{[1-K(x,\omega)]} - 1\right\}, \tag{6}$$

$$S(x; \omega) = \left(\frac{e^{[1-K(x,\omega)]} - 1}{e - [1 - K(x, \omega)]}\right), \quad x \in \mathbb{R}, \tag{7}$$

The key motivations of the NEx-X family approach are as follows:

(i) A relatively simple approach for extending the available distributions.

Table 1: Some members of T-X family.

S. No.	$W[K(x,\omega)]$	Range of X	T-X family member
1	$K(x,\omega)$	$[0,1]$	Beta-G [20]
2	$-\log[1-K(x,\omega)]$	$(0,\infty)$	Gamma type-1 [21]
3	$-\log[K(x,\omega)]$	$(0,\infty)$	Gamma type-2 [22]
4	$K(x,\omega)/1-K(x,\omega)$	$(0,\infty)$	Gamma type-3 [23]
5	$-\log[1-K(x,\omega)^{\alpha}]$	$(0,\infty)$	Exponentiated T-X family [24]
6	$\log[K(x,\omega)/1-K(x,\omega)]$	$(-\infty,\infty)$	Logistic-G family [25]
7	$\log[-\log(1-K(x,\omega))]$	$(-\infty,\infty)$	The Logistic-X [26]
8	$[-\log(1-K(x,\omega))]/(1-K(x,\omega))$	$(0,\infty)$	New Weibull-X family [27]
9	$-\log(1-K(x,\omega)/e^{K(x,\omega)})$	$(0,\infty)$	Weighted T-X family [28]
10	$-\log(\sigma\overline{K}(x,\omega)/\sigma-K(x,\omega))$	$(0,\infty)$	Exponential T-X family [29]
11	$\log[e^{(1\,K(x,))}-1/e-(1-K(x,\omega))]$	$(0,\infty)$	**New exponential-X family (proposed)**

(ii) To further improve the capabilities of existing distributions.

(iii) To define a special submodel having closed-form CDF, SF, and HF.

(iv) To furnish a better fit for heavy-tailed data.

In addition, the most important motivation is that the proposed method introduces new distributions without inserting extra parameters, which consequently avoids the difficulties of rescaling.

3. Special Submodel of the Proposed Novel Family

In this section of the article, a special submodel based on the proposed family called the NEx-Wei distribution is introduced. Let $K(x;\omega)$ and $k(x;\omega)$ be the corresponding CDF and PDF of the Weibull distribution given by $K(x;\omega) = 1-e^{-\alpha x^{\beta}} x \geq 0,\ \alpha,\beta>0$ and $k(x;\omega) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$, where $\omega = (\alpha,\beta)$. Then the CDF of NEx-Wei model is defined by

$$F(x;\alpha,\beta) = 1-\left(\frac{e^{e^{-\alpha x^{\beta}}}-1}{e-e^{-\alpha x^{\beta}}}\right), \quad x\geq 0,\ \alpha,\beta>0. \tag{8}$$

Expressions for PDF, SF (survival function), and function for HF (hazard rate function) are given in equations (8)–(10), respectively.

$$f(x;\alpha,\beta) = \frac{\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}}{\left(e-e^{-\alpha x^{\beta}}\right)^{2}} \cdot \left\{\left(e+1-e^{-\alpha x^{\beta}}\right)e^{e^{-\alpha x^{\beta}}}-1\right\}, \quad x>0, \tag{9}$$

$$S(x;\alpha,\beta) = \left(\frac{e^{e^{-\alpha x^{\beta}}}-1}{e-e^{-\alpha x^{\beta}}}\right), \quad x>0, \tag{10}$$

$$h(x;\alpha,\beta) = \frac{\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(e^{e^{-\alpha x^{\beta}}}-1\right)^{-1}}{\left(e-e^{-\alpha x^{\beta}}\right)} \cdot \left\{\left(e+1-e^{-\alpha x^{\beta}}\right)e^{e^{-\alpha x^{\beta}}}-1\right\}, \quad x>0. \tag{11}$$

Different shapes for the $f(x;\alpha,\beta)$ of NEx-Wei distribution for various parameter values are sketched in Figure 1.

Figure 2 graphically displays the $h(x;\alpha,\beta)$ of the NEx-Wei model for different combinations of the model parameters. From Figure 2, we can see that the $h(x;\alpha,\beta)$ of the NEx-Wei distribution have six different patterns including (i) increasing, (ii) decreasing, (iii) reverse-J shaped, (iv) unimodal, and (vi) slightly bathtub shaped. Hence, the proposed model is capable and becomes an important model to fit several lifetime data in applied areas, particularly in reliability engineering, biomedical, economics, and finance analysis.

4. Basic Mathematical Properties

This section presents some mathematical properties of the NEx-X family, such as the quantile function and ordinary moments, which can further be used to obtain some important characteristics of the model. In addition to these properties, the moment generating function is also derived.

4.1. Quantile Function. The quantile function (QF), also called inverse distribution function (IDF), is an important statistical terminology used to generate random numbers (RNs). These RNs can be used for simulation purposes to evaluate the performance of the estimators. Later in Section 4, the IDF method has been implemented to generate RNs from the NEx-Wei model. For the proposed model, the QF is given by

$$x_q = Q(u) = F^{-1}(u) = K^{-1}(t), \tag{12}$$

where t is the solution of equation $(1-u)(e-1)+1+(1-u)t-e^{1-t}=0$ and u has the uniform distribution on interval $(0,1)$. The expression can be used to generate RNs from any subcase of the NEx-X family of distributions.

4.2. r^{th} Moment. The r^{th} moment is an important and a useful ST (statistical tool) to obtain certain characteristics and features of a model. These characteristics are known as (i) central tendency: which deals with the mean point of any distribution, (ii) dispersion: which measures the variance of a model, (iii) skewness: which describe the tail behavior of the model, and (iv) kurtosis: which helps in studying the

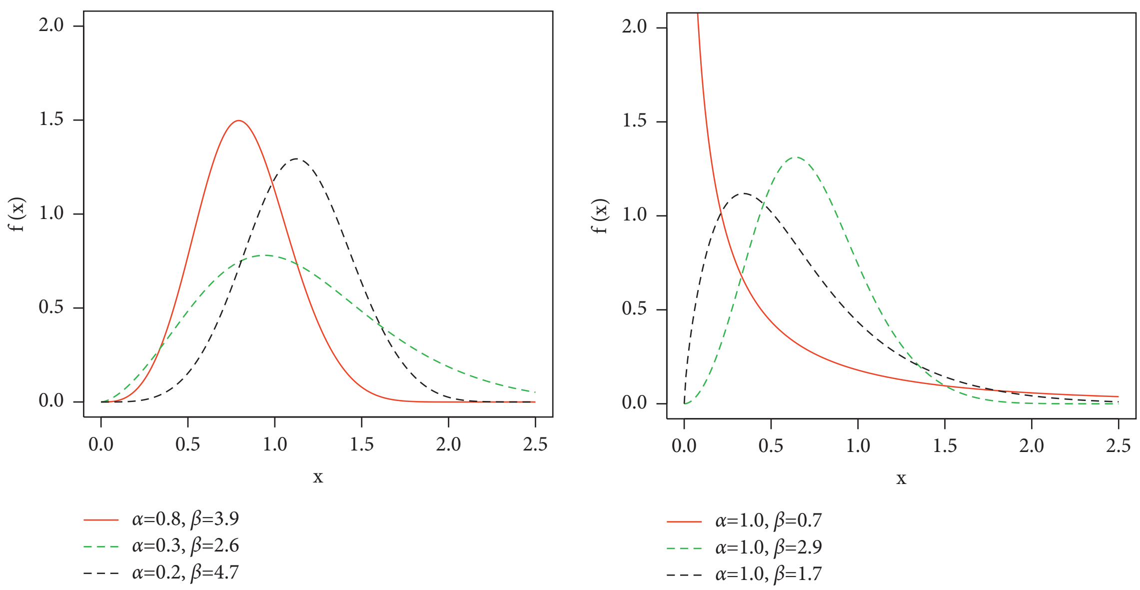

Figure 1: Different plots for the PDF $f(x;\alpha,\beta)$ of the NEx-Wei distribution.

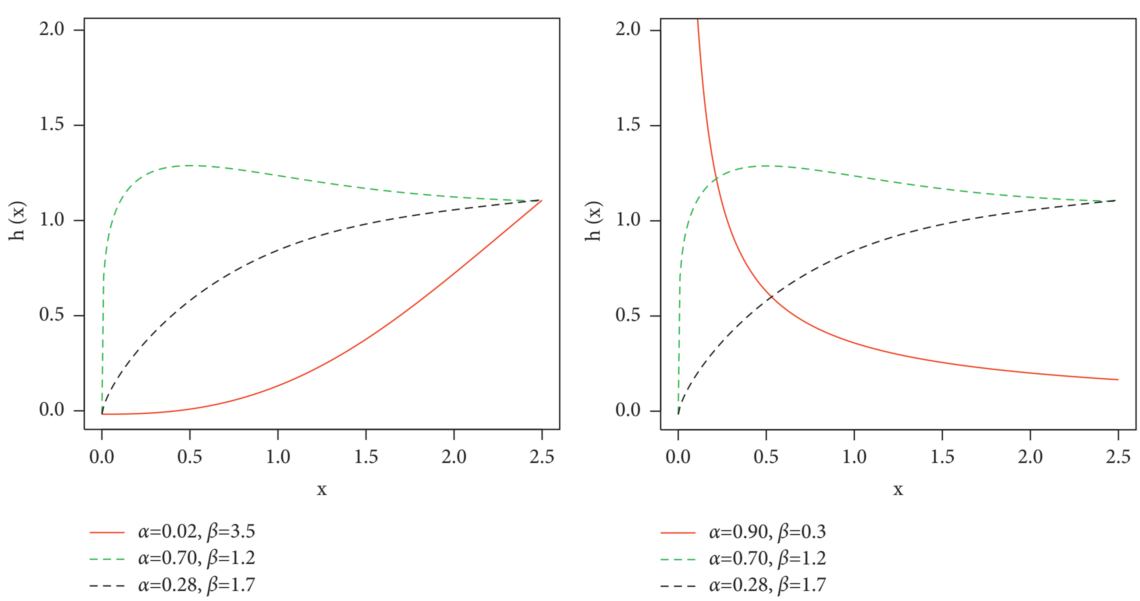

Figure 2: Different plots for the hrf $h(x;\alpha,\beta)$ of the NEx-Wei distribution.

peakedness of the distribution. For the proposed NEx-X family, the r^{th} moment expressed by $\mu_r/$ is derived as

$$\mu_r/ = \int_{-\infty}^{\infty} x^r f(x;\omega)\mathrm{d}x. \tag{13}$$

By (5), we have

$$\begin{aligned}\mu_r/ &= \int_{-\infty}^{\infty} x^r \frac{k(x,\omega)}{(e-[1-K(x,\omega)])^2} \\ &\quad \left\{(e+K(x,\omega))e^{[1-K(x,\omega)]}-1\right\}\mathrm{d}x, \\ \mu_r/ &= \frac{1}{e^2}\int_{-\infty}^{\infty} x^r \frac{k(x,\omega)}{(1-([1-K(x,\omega)]/e))^2} \\ &\quad \left\{(e+K(x,\omega))e^{[1-K(x,\omega)]}-1\right\}\mathrm{d}x.\end{aligned} \tag{14}$$

Using the series expansion

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} n x^{n-1}. \tag{15}$$

When replacing x by $((1-K(x,\omega))/e)$ in (15), we get

$$\frac{1}{(1-((1-K(x,\omega))/e))^2} = \sum_{n=1}^{\infty} n\left(\frac{(1-K(x,\omega))}{e}\right)^{n-1}. \tag{16}$$

Also, using Taylor series representation

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\ldots = \sum_{n=0}^{\infty}\frac{x^i}{i!}. \tag{17}$$

By replacing x by $K(x,\omega)$ in (17), we get

$$e^{(1-K(x,\omega))} = \sum_{i=0}^{\infty} \frac{(1-K(x,\omega))^i}{i!}. \tag{18}$$

By (16) and (18), we get

$$\mu_r' = \sum_{n=1}^{\infty} \frac{n}{e^{n+2}} \int_{-\infty}^{\infty} x^r k(x,\omega)(1-K(x,\omega))^{n-1} \left\{ (e+K(x,\omega)) \sum_{i=0}^{\infty} \frac{(1-K(x,\omega))^i}{i!} - 1 \right\} dx. \tag{19}$$

Furthermore, incorporating the binomial expansion

$$(1-x)^p = \sum_{j=0}^{p} (-1)^j \binom{p}{j} x^j. \tag{20}$$

When replacing x by $K(x,\omega)$ and p by $n-1$ and i, respectively, in (20), we arrive at

$$(1-K(x,\omega))^{n-1} = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} K(x,\omega)^j, \tag{21}$$

$$(1-K(x,\omega))^i = \sum_{l=0}^{i} (-1)^l \binom{i}{l} K(x,\omega)^l. \tag{22}$$

Using (21) and (22), in (19), we obtain

$$\mu_r' = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{i=0}^{\infty}\sum_{l=0}^{i} \frac{n(-1)^{j+l}}{i!e^{n+2}} \binom{i}{l}\binom{n-1}{j} \eta_{r,n,i,j,l} - \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{n(-1)^j}{e^{n+2}} \binom{n-1}{j} \eta_{r,n,j}', \tag{23}$$

where $\eta_{r,n,i,j,l} = \int_{-\infty}^{\infty} x^r k(x,\omega)(e+K(x,\omega))K(x,\omega)^{j+l} dx$ and $\eta_{r,n,j}' = \int_{-\infty}^{\infty} x^r k(x,\omega)K(x,\omega)^j dx$.

Furthermore, a simple general expression for the MGF (moment generating function) of the NEx-X random variable X, say $M_X(t)$, is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'. \tag{24}$$

By using (23) and (24), we get the MGF of the NEx-X family of distributions.

4.3. Order Statistics. In distribution theory, OS (order statistic) has great importance. They make their appearance in the reliability analysis, problems of estimation theory, and life testing in a number of ways. They can characterize the lifetimes of elements or components of a reliability system.

Let $X_1, X_2, \ldots, X_q$ be a random sample of q chosen from NEx-X with CDF and PDF given by (5) and (6), respectively. Then the density function of $g_{r:q}$ is given by

$$g_{r:q}(x) = \frac{1}{B(r,q-r+1)} k(x;\omega)[K(x;\omega)]^{r-1}[1-K(x;\omega)]^{q-r}. \tag{25}$$

We express the 1st order statistic as $X_{1:q} = \min(X_1, X_2, \ldots, X_q)$ and the q^{th} order statistic as $X_{q:q} = \max(X_1, X_2, \ldots, X_q)$. Then, $0 < K(x;\omega) < 1$ for $x > 0$. We may utilize the binomial expansion of $[1-K(x;\omega)]^{q-r}$ as follows:

$$[1-K(x;\omega)]^{q-r} = \sum_{i=0}^{q-r} (-1)^i \left[[1-K(x;\omega)]^{q-r}\right]^i. \tag{26}$$

On using equation (25) into equation (26), we get

$$g_{r:q}(x) = \frac{k(x;\omega)}{B(r,q-r+1)} \sum_{i=0}^{k-r} (-1)^i [K(x;\omega)]^{r+i-1}. \tag{27}$$

Using equations (5) and (6), in equation (27), we obtain the DF (density function) of $g_{r:q}$.

4.4. Residual and Reverse Residual Lifetime. The RL (residual lifetime) of the NEx-X random variable X, expressed by $R_{(X)}(t)$, is derived as

$$R_{(X)}(t) = \frac{S(x+t)}{S(t)},$$

$$R_{(X)}(t) = \frac{(e-[1-K(x+t,\omega)])\left(e^{[1-K(x+t,\omega)]}-1\right)}{\left(e^{1-K(x,\omega)}-1\right)(e-[1-K(x+t,\omega)])}, \quad x \in \mathbb{R}. \tag{28}$$

In addition to the RL, we obtain the RRL (reverse residual lifetime) of the NEx-X distributions denoted by $\overline{R}_{(X)}(t)$. For the NEx-X distributions, the $\overline{R}_{(X)}(t)$ is derived as

$$\overline{R}_{(X)}(t) = \frac{S(x-t)}{S(t)},$$

$$\overline{R}_{(X)}(t) = \frac{(e-[1-K(t,\omega)])\left(e^{[1-K(t,\omega)]}-1\right)}{\left(e^{[1-K(t,\omega)]}-1\right)(e-[1-K(x-t,\omega)])}, \quad x \in \mathbb{R}. \tag{29}$$

5. Estimation and Simulation

This section is divided into two subsections. The first subsection provides a detailed description of the maximum likelihood estimation implemented for estimating the parameters (α, β) of the NEx-Wei model, while the second subsection provides a comprehensive Monte Carlo simulation study for assessing the performance of the MLEs of the proposed method.

5.1. Maximum Likelihood Estimation. Several methods for estimating the parameters of any distribution have been introduced in the literature. The MLE (maximum likelihood estimation) is one of the most frequently used of such methods. This method furnishes estimators with several important properties and can be used in the construction of confidence intervals as well as other tests for checking statistical significance. For further details about MLEs, see [30]. This subsection provides a discussion on the MLEs approach for estimating the model parameters of the NEx-Wei distribution.

Suppose $x_1, x_2, \ldots, x_n$ are the observed values from the pdf given in equation (9) with α andβ as the associated parameters. Corresponding to equation (9), the Log-likelihood function is

$$\begin{aligned} L(x_i;\alpha,\beta) &= n\log(\alpha) + n\log(\beta) + (\beta-1)\sum_{i=1}^{n}\log(x_i) \\ &\quad - \alpha\sum_{i=1}^{n} x_i^{\beta} - 2\sum_{i=1}^{n}\log\left(e - e^{-\alpha x_i^{\beta}}\right) \\ &\quad + \sum_{i=1}^{n}\log\left(\left(e + 1 - e^{-\alpha x_i^{\beta}}\right)e^{e^{-\alpha x_i^{\beta}}} - 1\right). \end{aligned} \tag{30}$$

Taking derivatives of equation (30) with respect to the desired parameters and setting it equal to zero give

$$\begin{aligned} \frac{\partial L(x_i;\alpha,\beta)}{\partial\alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\beta} - 2\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-\alpha x_i^{\beta}}}{\left(e - e^{-\alpha x_i^{\beta}}\right)} \\ &\quad + \sum_{i=1}^{n}\frac{x_i^{\beta}e^{-\alpha x_i^{\beta}}e^{e^{-\alpha x_i^{\beta}}}\left(e^{-\alpha x_i^{\beta}} - e\right)}{\left(\left(e - e^{-\alpha x_i^{\beta}} + 1\right)e^{e^{-\alpha x_i^{\beta}}} - 1\right)} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial L(x_i;\alpha,\beta)}{\partial\beta} &= \frac{n}{\beta} + \beta\sum_{i=1}^{n}\log x_i - \alpha\sum_{i=1}^{n}(\log x_i)x_i^{\beta} \\ &\quad - 2\alpha\sum_{i=1}^{n}\frac{\left((\log x_i)x_i^{\beta}e^{-\alpha x_i^{\beta}}\right)}{\left(e - e^{-\alpha x_i^{\beta}}\right)} \\ &\quad + \alpha\sum_{i=1}^{n}\frac{\left((\log x_i)x_i^{\beta}e^{-\alpha x_i^{\beta}}e^{e^{-\alpha x_i^{\beta}}}\left(e^{-\alpha x_i^{\beta}} - e\right)\right)}{\left(\left(e - e^{-\alpha x_i^{\beta}} + 1\right)e^{e^{-\alpha x_i^{\beta}}} - 1\right)} = 0. \end{aligned} \tag{32}$$

Numerical solutions of (31) and (32) simultaneously yield the MLEs of α and β.

5.2. Simulations. The behaviors of the MLEs of the parameters of the suggested distribution are evaluated in this section based on simulated data. Three sets of parameters of the NEx-Wei model are assessed in the simulation. The process is described below:

(i) With $N = 750$, samples of size $n = 25, 50, 100, \ldots, 750$ are generated from NEx-Wei distribution with parameters α and β.

(ii) Compute MLEs of α and β.

(iii) Calculation of the biases and mean square error (MSE) of the desired model parameters is done by

$$\text{bias}(\hat{\alpha}) = \sum_{i=1}^{500}\left(\hat{\alpha}_i - \alpha\right) \text{ and } MSE(\hat{\alpha}) = \sum_{i=1}^{500}\left(\hat{\alpha}_i - \alpha\right)^2. \tag{33}$$

(iv) Step (iii) is repeated for β.

Simulation results on estimated parameters in terms of MSEs and biases values are provided in Table 2 and also graphically displayed in Figures 3–5. From the simulation results in Table 2, we conclude that the biases for all parameters are positive and the estimated biases and MSEs decrease as the sample size increases.

6. Applications

This section assesses the applicability of the NEx-Wei model in applied areas that include financial, engineering, and medical sciences. In all these areas, the fits of the NEx-Wei model are compared with other familiar distributions.

For checking the goodness of the distributions, we consider different goodness of fits measures in order to examine which competitor provides the best fit to the considered data sets. The goodness of fit measures include CM (Cramer-von-Misses) test statistic, AD (Anderson–Darling) test statistic, KS (Kolmogorov-Smirnov), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), corrected Akaike information criterion (CAIC), and HQIC (Hannan-Quinn Information Criterion) as well as P-values.

In general, a distribution with smaller values for these analytical measures and a greater p-value could be considered a good candidate for the underlying data set. Based on the considered analytical measures, the results reveal that the NEx-Wei distribution produces greater distributional flexibility among all the other applied distributions.

6.1. Application in Vehicle Insurance Loss Data. The first case study is that of insurance, where vehicle insurance losses are considered. The data are taken from the website: http://www.businessandeconomics.mq.edu.au/our-depatments-/Apllied-Finance-and-Acturial-Studies/research/books/GLMs-for-insurance-Data. Some basic measures for the dataset are given by minimum = 1.0, 1st quartile = 23.25, median = 41.60, mean = 55.89, 3rd quartile = 73.20, maximum = 194.00, skewness = 1.253132, kurtosis = 4.08863, variance = 2334.975, and range = 193.00.

Table 2: Simulation results for NEx-Wei distribution for different combination of parameters.

N	par	Set 1: $\alpha = 1.7, \beta = 2.8$			Set 1: $\alpha = 1, \beta = 1.6$			Set 1: $\alpha = 1, \beta = 0.8$		
		MLE	MSEs	Bias	MLE	MSEs	Bias	MLE	MSEs	Bias
25	α	1.9127	0.4856	0.2127	1.1177	0.13861	0.1177	1.0765	0.1105	0.0765
	β	2.9613	0.2651	0.1613	1.6927	0.0888	0.0927	2.0208	0.1321	0.1208
50	α	1.8060	0.1430	0.1060	1.0447	0.0422	0.0447	1.0238	0.0372	0.0238
	β	2.8808	0.1024	0.0808	1.6455	0.0397	0.0455	1.9558	0.0476	0.0558
75	α	1.7600	0.0846	0.0599	1.0215	0.0229	0.0215	1.0260	0.0247	0.0260
	β	2.8483	0.0683	0.0483	1.6302	0.0213	0.0302	1.9357	0.0312	0.0357
100	α	1.7399	0.0647	0.0399	1.0181	0.0166	0.0181	1.0161	0.0165	0.0161
	β	2.8341	0.0451	0.0341	1.6243	0.0153	0.0243	1.9292	0.0226	0.0292
150	α	1.7352	0.0355	0.0352	1.0094	0.0107	0.0094	1.0090	0.0101	0.0090
	β	2.8217	0.0305	0.0217	1.6088	0.0102	0.0088	1.9141	0.0142	0.0141
200	α	1.7175	0.0286	0.0175	1.0141	0.0082	0.0141	1.0149	0.0077	0.0149
	β	2.8144	0.0258	0.0144	1.6090	0.0077	0.0090	1.9163	0.0101	0.0163
250	α	1.7189	0.0262	0.0189	1.0041	0.0063	0.0041	1.0066	0.0057	0.0066
	β	2.8143	0.0189	0.0143	1.6076	0.0059	0.0076	1.9124	0.0086	0.0124
300	α	1.7125	0.0181	0.0125	1.0030	0.0050	0.0030	1.0126	0.0053	0.0126
	β	2.8113	0.0141	0.0113	1.6062	0.0047	0.0066	1.9076	0.0067	0.0076
400	α	1.7166	0.0141	0.0133	1.0068	0.0038	0.0040	1.0039	0.0048	0.0090
	β	2.8102	0.0115	0.0066	1.6032	0.0035	0.0059	1.9066	0.0059	0.0051
500	α	1.7026	0.0100	0.0026	1.0047	0.0027	0.0047	1.0027	0.0028	0.0027
	β	2.8114	0.0089	0.0114	1.6037	0.0028	0.0037	1.9026	0.0042	0.0026
600	α	1.7079	0.0093	0.0079	1.0031	0.0027	0.0031	1.0026	0.0022	0.0026
	β	2.8012	0.0078	0.0012	1.6055	0.0024	0.0055	1.9030	0.0032	0.0030
700	α	1.6995	0.0066	0.0077	1.0039	0.0021	0.0039	1.0046	0.0020	0.0046
	β	2.7974	0.0064	0.0068	1.6046	0.0019	0.0046	1.9035	0.0028	0.0035
750	α	1.7023	0.0065	0.0023	1.0049	0.0021	0.0049	1.0016	0.0020	0.0016
	β	2.8028	0.0055	0.0028	1.6026	0.0019	0.0026	1.8530	0.0027	0.0030

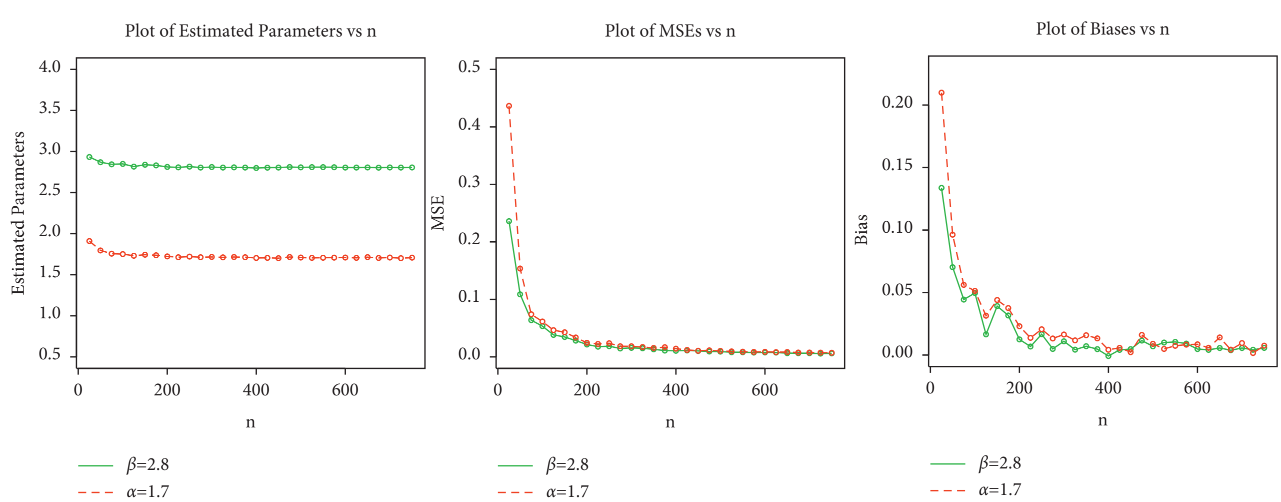

Figure 3: MLEs, MSEs, and biases for $\alpha = 1.7$ and $\beta = 2.8$ of the estimated parameters.

Corresponding to this dataset, the comparison of the NEx-Wei distribution is made with other well-known distributions including APT-Wei (Alpha Power transformed Weibull) [31], Degum [32], Lomax distribution, Burr-XII (B-XII) distribution [33], MO-Wei distribution [34], and Kumaraswamy Weibull (Ku-Wei) distribution [35]. The reason for considering these distributions for comparison purposes is their frequent application in modeling financial and financial risk management problems.

Furthermore, for the analyzed data, the maximum likelihood estimates of the fitted models are presented in Table 3. The numerical values of the analytical measures of

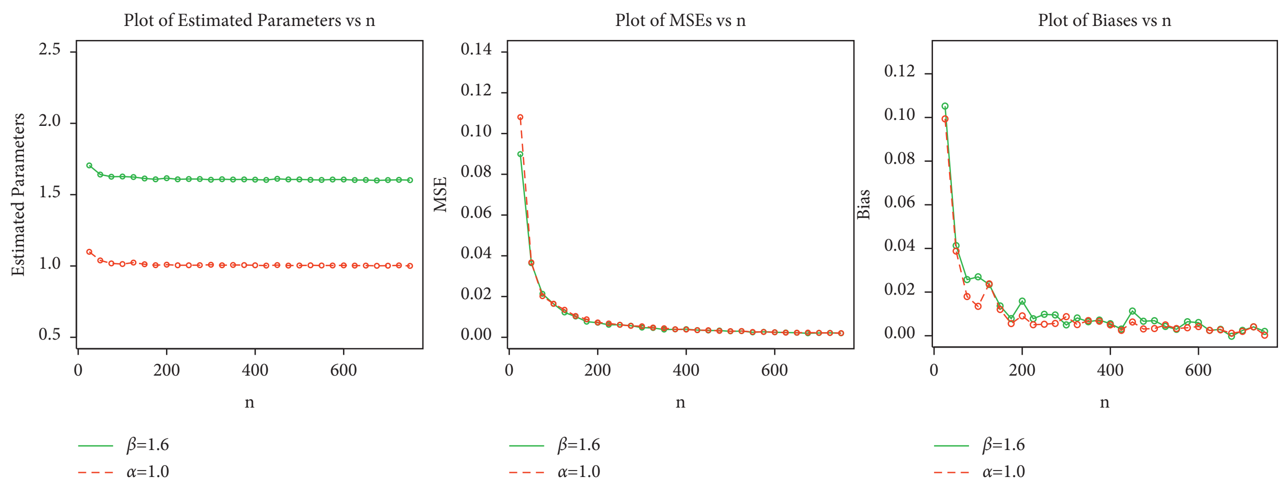

Figure 4: MLEs, MSEs, and biases for $\alpha = 1.0$ and $\beta = 1.6$ of the estimated parameters.

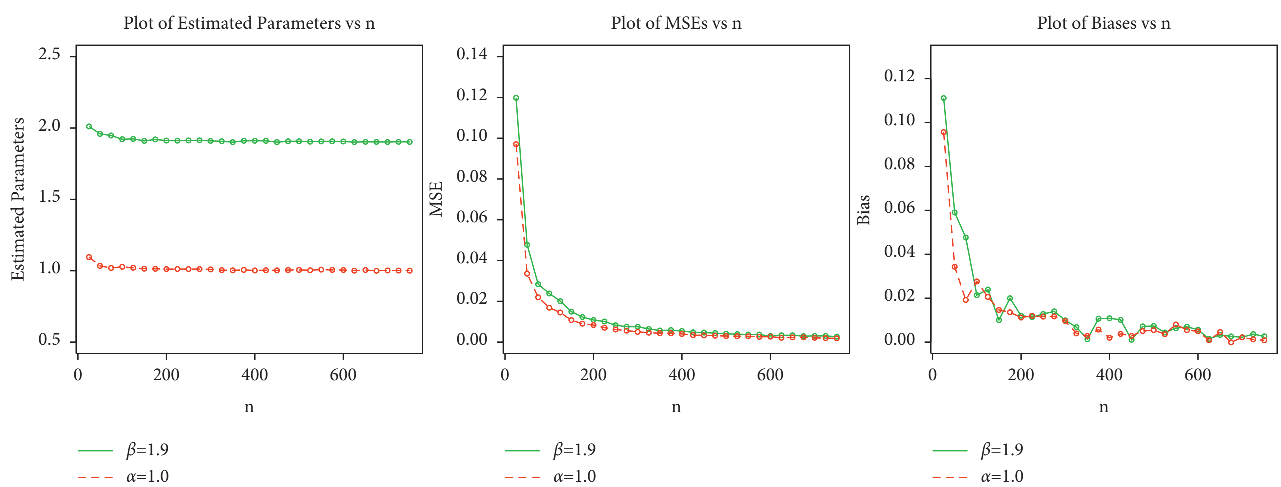

Figure 5: MLEs, MSEs, and biases for $\alpha = 1.0$, and $\beta = 1.9$ of the estimated parameters.

Table 3: The MLEs values of the fitted distributions using vehicle insurance losses data.

Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\sigma}$	$\widehat{a}$	$\widehat{b}$
NEx-Wei	**0.003897**	**1.229428**	—	—	—	—
APT-Wei	0.025821	0.979796	3.444968	—	—	—
B-XII	—	—	—	—	7.860075	0.035724
Ku-Wei	0.316431	0.893541	—	—	2.699737	0.095785
Lomax	4.264217	196.614704	—	—	—	—
Degum	3.041811	87.280010	0.309938	—	—	—
MO-Wei	0.021598	1.013912	—	1.683068	—	—

the fitted models are provided in Tables 4 and 5. For this data, the analytical measures values for the NEx-Wei are AIC = 325.3758, BIC = 328.3072, CAIC = 325.7896, HQIC = 326.3475 CM = 0.03211, AD = 0.1968, KS = 0.0827, and p-value = 0.968.

Based on these analytical measures, the proposed model fits better than the other competing models to the considered data. In the support of the numerical illustration in Tables 4 and 5, the estimated PDF and CDF plots of the NEx-Wei distribution are presented in Figure 6. Moreover, the PP plot and Kaplan-Meier survival plot are presented in Figure 7, whereas Figure 8 shows the box and QQ plots. Obviously, these plots reveal the closer fit of the NEx-Wei model.

6.2. Application in Reliability Engineering. The second case study is from reliability engineering regarding the failure time of cutting layers machine [36]. Basic measures for the

Table 4: The analytical measures of the fitted distributions using vehicle insurance losses data.

Distributions	CM	AD	KS	p-value
NEx-Wei	0.03211	0.19688	0.08272	0.96870
AP-Wei	0.04093	0.20915	0.09147	0.92940
B-XII	0.44285	2.60380	0.37109	0.01837
Ku-Wei	0.05341	0.32749	0.10299	0.85233
Lomax	0.03909	0.24298	0.14514	0.46730
Degum	0.04146	0.23202	0.09372	0.91650
MO-Wei	0.04133	0.13196	0.08940	0.94010

Table 5: The analytical measures of the fitted distributions using vehicle insurance losses data.

Distributions	AIC	BIC	CAIC	HQIC
NEx-Wei	325.37582	328.30722	325.78964	326.34754
APT-Wei	326.65220	331.04941	327.50943	328.10983
B-XII	377.97272	380.90422	378.38654	378.94444
Ku-Wei	330.48661	336.34963	331.96815	332.43435
Lomax	328.10725	331.03854	328.52086	329.07876
Degum	327.27246	331.66965	328.12964	328.73461
MO-Wei	326.62125	331.01846	327.47833	328.07874

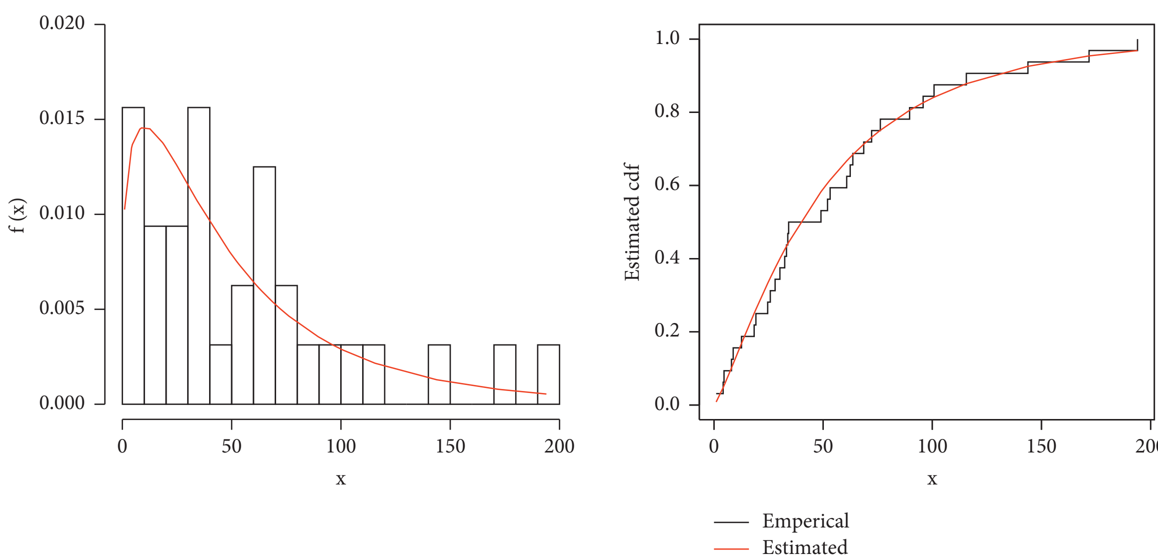

Figure 6: The estimated PDF and CDF plots of the NEx-Wei model for the vehicle insurance losses data.

dataset are given by minimum = 1.0, 1st quartile = 20.75, median = 43.75, mean = 124.10, 3rd quartile = 143.50, maximum = 970.50, skewness = 2.917965, kurtosis = 11.85639, variance = 41601.74, and range = 969.5.

The performance of the proposed model is evaluated by comparing it with other well-known models such as Kumaraswamy Weibull (Ku-Wei) [35], two parameters' Weibull, extended alpha power Weibull (EAP-Wei) [37], Beta Weibull (B-Wei) [38], and new alpha power Weibull (NAP-Wei) [39] models. Furthermore, the Ku-Wei, EAP-Wei, and NAP-Wei models are widely used in the literature for modeling failure time data.

Corresponding to the second data set, the values of MLEs of the parameters are presented in Table 6, whereas the analytical results of the proposed and other competitive models are reported in Tables 7 and 8. For this data, the analytical measures values for the NEx-Wei model are AIC = 331.8761, BIC = 334.6107, CAIC = 332.3376, HQIC = 332.7325, CM = 0.07430, AD = 0.40100, KS = 0.13626, and p-value = 0.6545.

Figure 9 gives the corresponding estimated plots of PDF and CDF. Furthermore, Figure 10 gives the PP and Kaplan-Meier survival plots, whereas Figure 11 shows the box and QQ plots. The results demonstrate, given the positively skewed data (see box plot), that the newly suggested model fits the data better than the other methods.

6.3. Application in Biomedical Science Data. The third case study is from biomedical science, where the dataset consists of forty-four observations reported in [40]. This data set represents the survival time of a group of patients suffering from

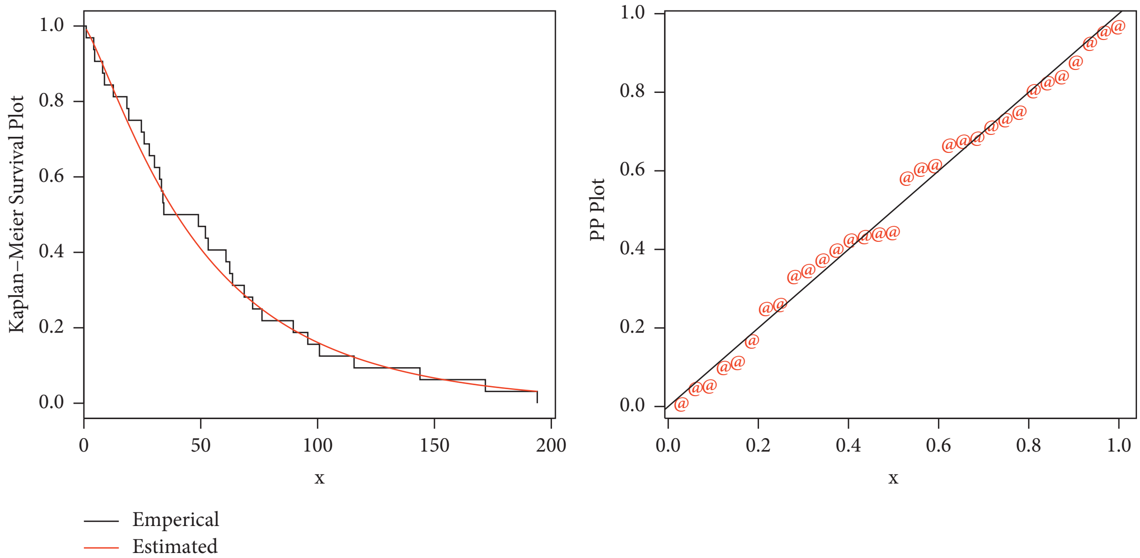

FIGURE 7: The Kaplan-Meier survival and PP plots of the NEx-Wei model for vehicle insurance loss data.

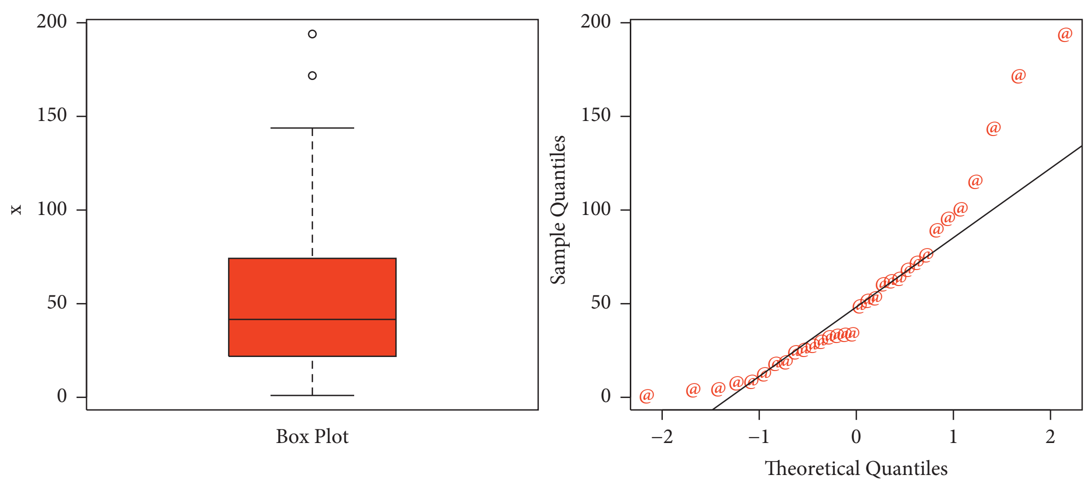

FIGURE 8: The box and QQ plots of the NEx-Wei model for vehicle insurance losses data.

TABLE 6: The MLEs values of the fitted distributions using cutting layers machine data.

Distributions	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}_1$	$\widehat{a}$	$\widehat{b}$
NEx-W	**0.015751**	**0.793979**	—	—	—
Weibull	0.044270	0.686792	—	—	—
Ex-APW	0.091202	0.595231	1.144296	—	—
Ku-W	0.435013	0.515035	—	2.6738437	0.3084275
NAPT-Wei	0.018558	0.767287	0.516289	—	—
BW	0.502332	0.478112	—	2.7972342	0.3442123

head and neck cancer. Some basic measures of head and neck cancer data are given by minimum = 12.20, 1st quartile = 67.21, mean = 223.50, median = 128.50, 3rd quartile = 219.00, maximum = 1776.00, variance = 93287.41, range = 12.20, skewness = 3.38382, and kurtosis = 16.5596.

Corresponding to the third data set, we applied the NEx-Wei model with several other competitive models, namely, the two parameters' classical Weibull, FRL-Wei [41], APT-Wei [31], and MO-Wei [34] distributions.

Furthermore, for the data set, the numerical values of MLEs of the NEx-Wei distribution and other competing model parameters are presented in Table 9. The numerical values of the analytical measures of the fitted models are in Tables 10 and 11. For the dataset, the analytical measures

Table 7: The analytical measures of the fitted distribution using cutting layers machine data.

Distributions	CM	AD	KS	p-value
NEx-Wei	0.07430	0.40101	0.13626	0.65450
Wei	0.08845	0.47593	0.15097	0.52321
Ex-APT-Wei	0.09634	0.51343	0.15185	0.51562
Ku-Wei	0.14565	0.81133	0.14951	0.53522
NAPT-Wei	0.07766	0.41519	0.13828	0.63621
B-Wei	NaN	NaN	0.14572	0.60322

Table 8: The analytical measures of the fitted distribution using cutting layers machine data.

Distributions	AIC	BIC	CAIC	HQIC
NEx-Wei	331.87613	334.61071	332.33761	332.73250
Wei	332.76846	335.50262	333.22952	333.62443
Ex-APT-Wei	335.25339	339.35522	336.21333	336.53790
Ku-Wei	341.86054	347.32915	339.11531	339.24343
NAPT-Wei	334.04573	338.14764	335.00572	335.33032
B-Wei	335.45712	340.92533	334.43216	336.33123

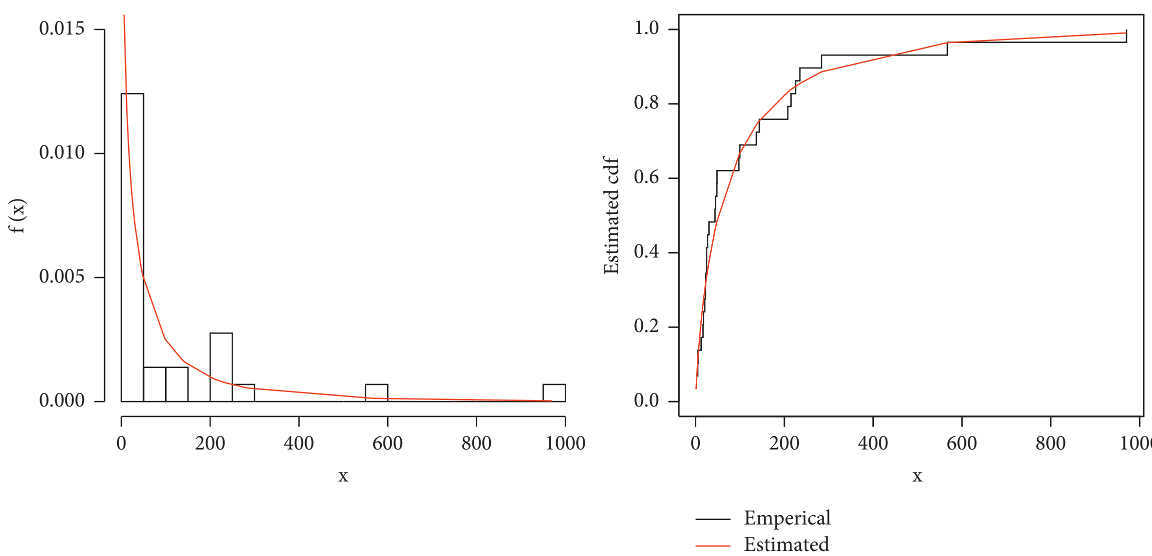

Figure 9: Plots of the estimated PDF and CDF of the NEx-Wei model for failure time of cutting layers machine data.

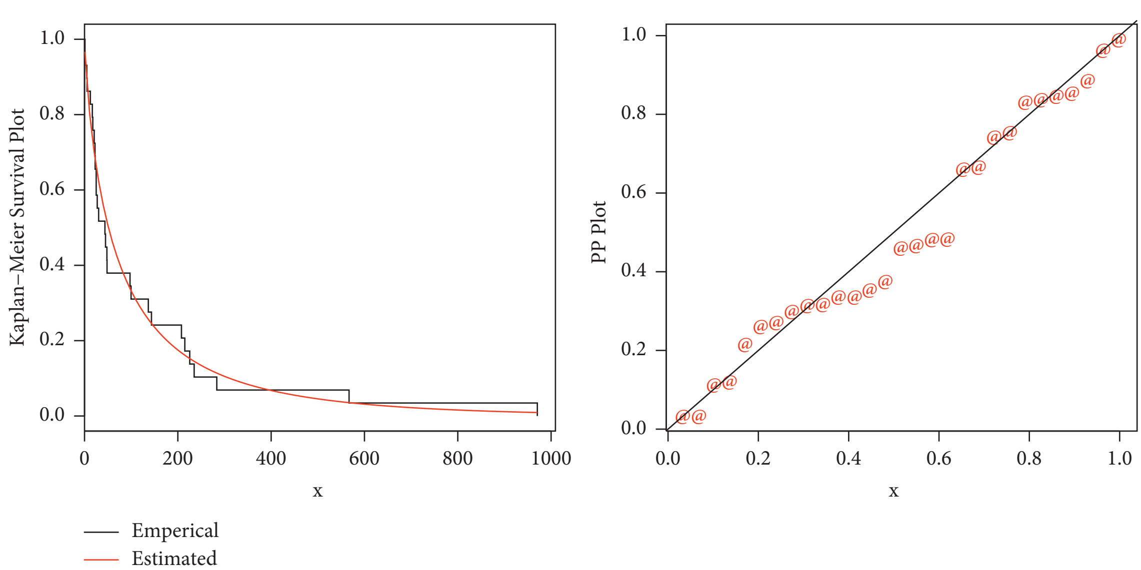

Figure 10: The Kaplan-Meier survival plot and PP plot of the NEx-Wei model for failure time of cutting layers machine data.

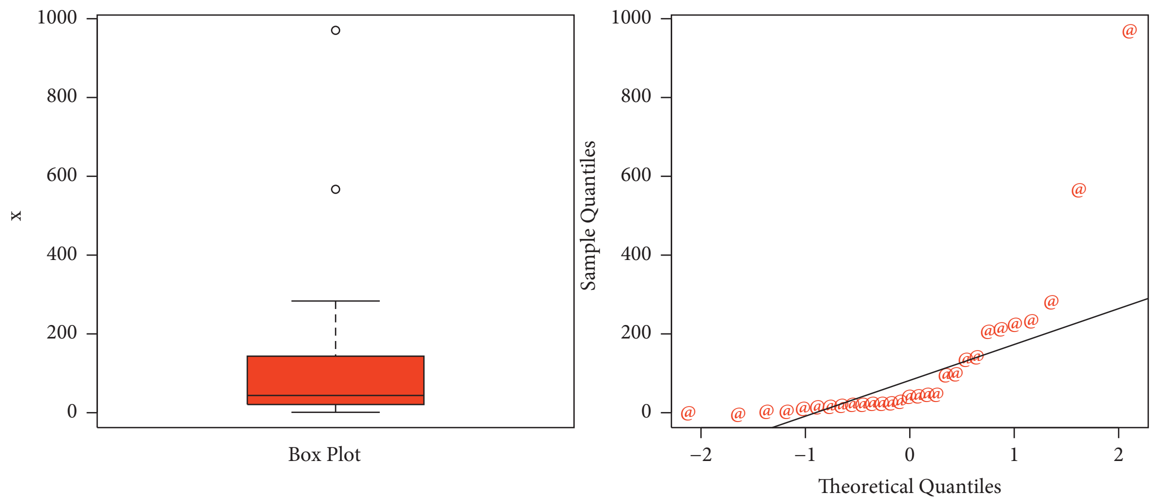

Figure 11: The box and QQ plots of the NEx-Wei model for failure time of cutting layers machine data.

Table 9: The MLEs values of the fitted distributions using head and neck cancer data.

Distributions	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}_1$	$\widehat{\theta}$
NEx-Wei	**0.002456**	**1.022756**	—	—
Wei	0.006771	0.931311	—	—
FRL-Wei	0.028596	0.761846	—	5.721750
APT-Wei	0.003264	0.992700	0.245030	—
MO-Wei	0.003032	1.001188	—	0.507524

Table 10: The analytical measures of the fitted distributions using head and neck cancer data.

Distributions	CM	AD	KS	p-value
NEx-Wei	0.08657	0.51532	0.10061	0.72780
Wei	0.13983	0.81427	0.12612	0.44940
FRL-Wei	0.19103	1.09553	0.13355	0.37890
APT-Wei	0.09338	0.55387	0.10551	0.67230
MO-Wei	0.09492	0.56181	0.11255	0.59330

Table 11: The analytical measures of the fitted distributions using head and neck cancer data.

Distributions	AIC	BIC	CAIC	HQIC
NEx-Wei	565.15680	568.72520	565.44950	566.48012
Wei	567.69411	571.26252	567.98681	569.01751
FRL-Wei	572.88330	578.23593	573.48338	574.86832
APT-Wei	567.77121	573.12381	568.37127	569.75627
MO-Wei	568.20841	573.56123	568.80846	570.19349

values of the NEx-Wei are AIC = 565.1568, BIC = 568.7252, CAIC = 565.4495, HQIC = 566.4801, CM = 0.08657, AD = 0.51532, KS = 0.1006, and p-value = 0.7278.

In the support of the numerical illustration in Tables 10 and 11, the estimated PDF and CDF plots of the NEx-Wei distribution are presented in Figure 12. Moreover, the PP plot and Kaplan-Meier survival plot are presented in Figure 13, whereas Figure 14 shows the box and QQ plots. The results demonstrate, given the positively skewed data (see box plot), that the newly suggested model fits the data closely.

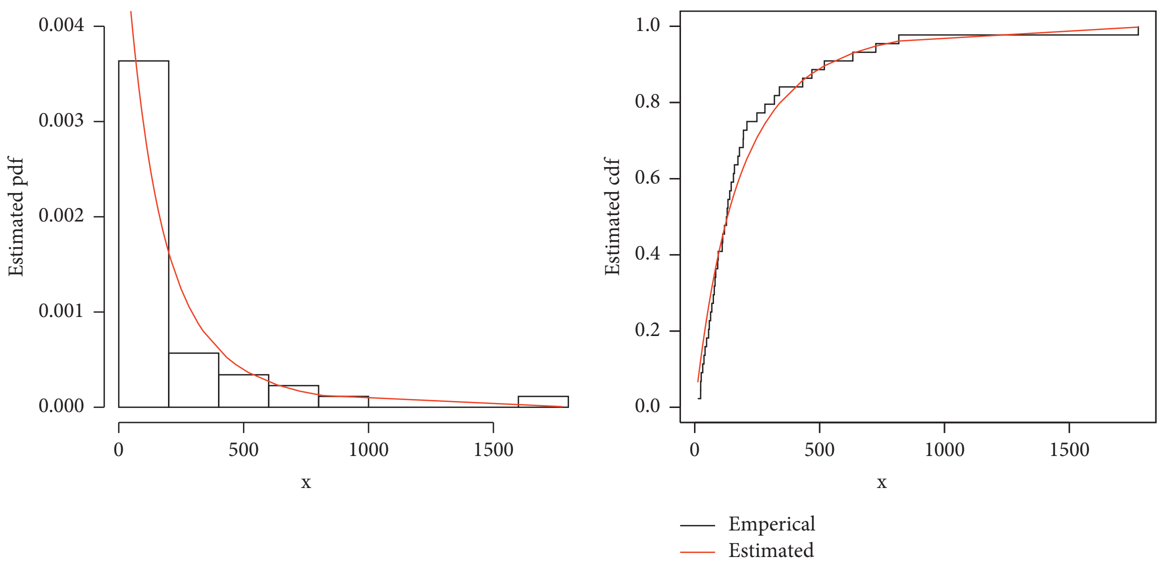

Figure 12: Plots for the estimated PDF and CDF of the NEx-Wei model based on head and neck cancer data.

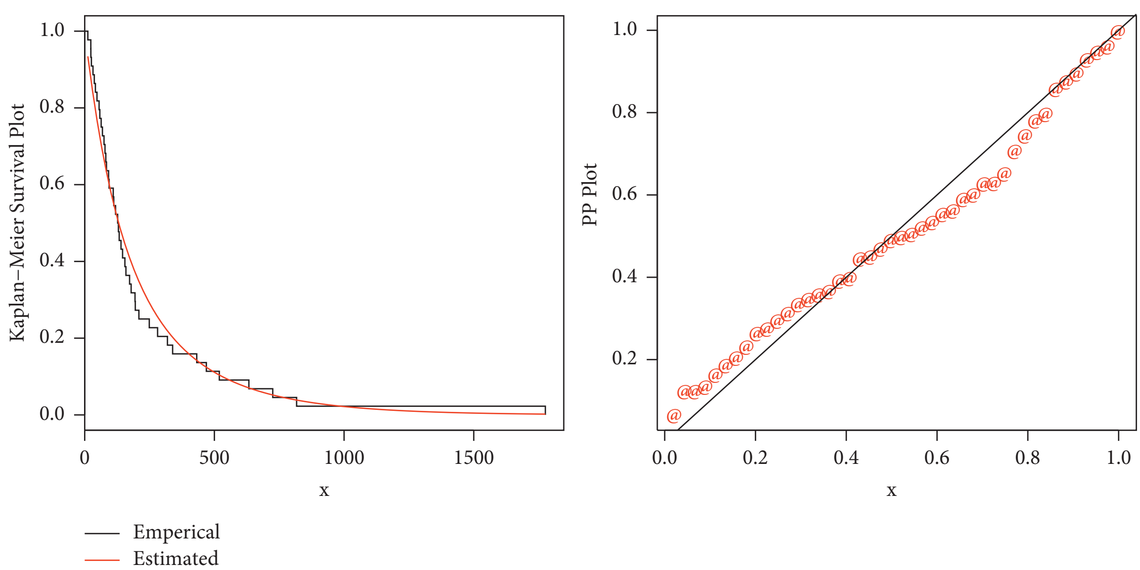

Figure 13: The Kaplan-Meier survival and PP plots of the NEx-Wei model for head and neck cancer data.

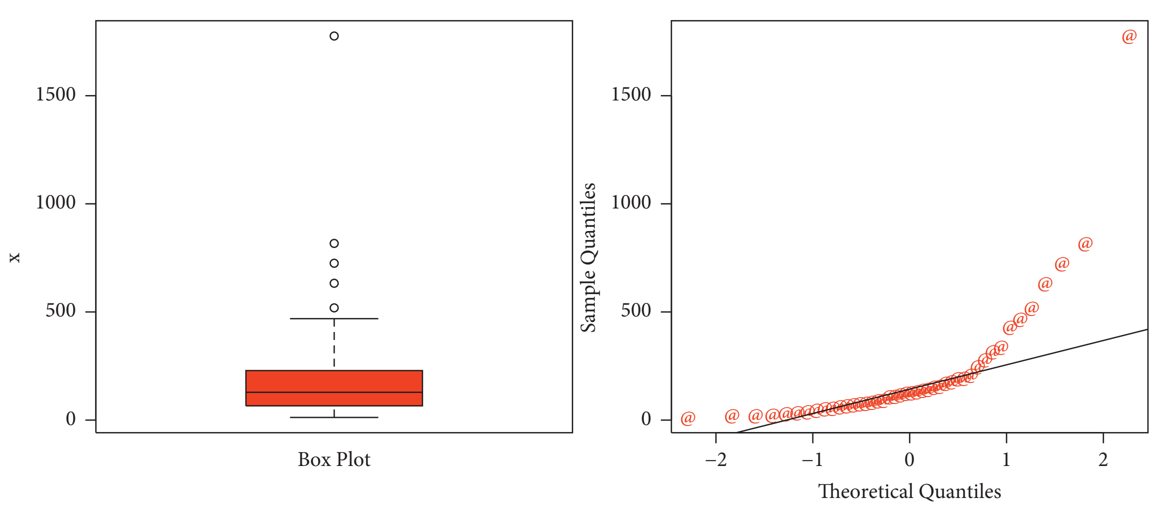

Figure 14: The box and QQ plots of the NEx-Wei model for head and neck cancer data.

7. Conclusion

This article presented the idea of a new family of distribution, called the new exponential-X family or NEx-X. This family of distributions has a wide range of applications without adding additional parameters to the already available distributions. A special submodel of the proposed method called a NEx-Wei (new exponential Weibull) is derived and studied in detail. Besides, general expressions for different statistical properties of the proposed family have been derived including quantile function, moments, moments generating function, and order statistics. MLE (maximum likelihood estimation) method has been used for estimating the unknown parameters, and in addition, a Monte Carlo simulation study is carried out to assess the performance of the proposed model estimators. In the field of reliability engineering, insurance, and medicine, we have analyzed three data sets and the proposed class provides a very good fit for all data sets. We hope that this novel improvement in the theory of the distribution will give more attractive applications in reliability engineering, medical, and other related fields.

Data Availability

The references of the data sets are given within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Supplementary Materials

To replicate the results of the simulation study in Table 2, the simulation codes are provided as a supplementary file. (*Supplementary Materials*)

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