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Research Article

Nonsolvable Groups Whose Degrees of All Proper Subgroups Are the Direct Products of at Most Two Prime Numbers

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Huppert and Manz have determined the nonsolvable groups whose character degrees are products of at most two prime numbers. In this paper, we change the condition from “degrees of a group are products of at most two prime divisors” to “degrees of all proper groups of a group are products of at most two prime divisors” and determine the structure of finite groups with such condition.

1. Introduction

Let

$$n = \prod_{i=1}^{k} p_i^{a_i}, \tag{1}$$

where the p_is are different prime divisors of n, and define

$$\omega(n) = \sum_{i=1}^{n} a_i, \tag{2}$$

the number of prime divisors of n. Assume that all groups are finite in this paper. Let $\operatorname{Irr}(G)$ denote the set of all complex irreducible characters of a group G, and let $\operatorname{Lin}(G)$ be the set of the linear characters of G. Denote by $\operatorname{cd}(G)$ the set of irreducible character degrees of a group G, i.e., $\operatorname{cd}(G) = \{\chi(1)\colon \chi \in \operatorname{Irr}(G)\}$. Usually, a degree means a complex irreducible character degree in this paper. Let

$$\omega(G) = \max_{d \in \operatorname{cd}(G)} \omega(d). \tag{3}$$

The structure of a finite group G with $\omega(G) = 1$ is determined by Isaacs and Passman and Manz; see [1–3], respectively. The influence of Brauer characters with prime-power degrees on the structure of finite groups is considered in [4, 5].

Finite groups G with $\omega(G) = 2$ are determined (see [6, 7]). In particular, if G is nonabelian simple, then G is isomorphic to A_5 or A_7 where A_n is an alternating group of degree n. Recently, Miraali and Robati furthered Huppert's results and identified almost simple groups whose degrees are divisible by at most two primes; see Theorem 3.6 of [8].

Inspired by the works of [6, 8], we change the condition from “$\omega(G) = 2$ for a group G” to “$\omega(H) \leq 2$ for each proper subgroup H of a group G” and will determine the structure of nonsolvable groups whose degrees of all proper subgroups are the direct products of at most two primes. In order to shorten arguments, we give a definition.

Definition 1. Let G be a finite group, and let $\sum G$ be the set of all proper subgroups of G. A group G is called a T-group if $\omega(G) \leq 2$.

By Definition 1, we have the following definition.

Definition 2. A group G is named a TS-group if each $H \in \sum G$ is a T-group and a non-TS-group otherwise. We call an irreducible character $\chi \in \operatorname{Irr}(G)$ a T-character if $\omega(\chi(1)) \leq 2$.

In generality, a T-group does not mean that it is a TS-group.

Example 1. Let $G = A_7$, where A_n is an alternating group of degree n. By pp. 10 of [9], we can have $\mathrm{cd}(A_7) = \{1, 2\cdot 3, 2\cdot 5, 2\cdot 7, 3\cdot 5, 3\cdot 7, 5\cdot 7\}$, so A_7 is a T-group. On the other hand, A_7 has a subgroup isomorphic to $\mathrm{PSL}_2(7)$. We see that $8 \in \mathrm{cd}(\mathrm{PSL}_2(7))$, so there is an irreducible character $\chi \in \mathrm{Irr}(\mathrm{PSL}_2(7))$ with $\omega(\chi(1)) = 3$. Now, $\mathrm{PSL}_2(7)$ is not a T-group and so A_7 is a non-TS-group.

In this paper, we prove the following result.

Theorem 3. *Let G be a nonsolvable TS-group. Then, one of the following holds:*

(1) G is isomorphic to $PSL_2(q)$ where $q = 2rs + 1$ is a prime for some primes r, s (possibly equal)

(2) G is isomorphic to $PSL_2(p^m)$ where $p \in \{2, 3, 5\}$, and m is a prime such that $(p^m - 1)/(\mathrm{gcd}\,(2, q - 1)) = rs^a$ for primes r, s, and $a \in \{0, 1\}$

(3) G is isomorphic to ${}^2B_2(8)$, $PSL_2(2^4)$

(4) G is isomorphic to $S_5 \times A$ with A abelian

(5) G is isomorphic to $A_5 \times A$ with $cd(A) \subseteq \{1, p\}$ for some prime p

(6) G has a normal abelian subgroup M such that $G/M \cong SL_2(5)$

The structure of this paper is formed as follows. In Section 2, some results are given which will be used in the proof of our main theorem. In Section 3, we first give the structure of simple TS-groups and then that of nonsolvable TS-groups.

In a group G, we will use the notation $\max G$ to denote the set of the maximal proper subgroups with respect to subgroup-order divisibility from $\sum G$. Let E_q be an elementary abelian group of order q, denoting the extraspecial group of order q^{1+2m} by $\mathrm{ES}(q^{1+2m})$ or q^{1+2m}. Let C_n be the cyclic group of order n. Let $\mathrm{Mult}(G)$ be the Schur multiplier of a group G. For the other notation and notions, we can refer to [9, 10] for instance.

2. Some Lemmas

In this section, some results about elementary number theory, Frobenius groups, and also subgroup structure of a simple classical Lie group are given.

Lemma 4 [11]. *The only solution of the Diophantine equation $p^m - q^n = 1$ with p and q primes and $m, n > 1$ is $3^2 - 2^3 = 1$.*

Lemma 5 [11, 12]. *With the exceptions of the relations $239^2 - 2\cdot 13^4 = -1$ and $3^5 - 2\cdot 11^2 = 1$, every solution of the equation $p^m - 2q^n = \pm 1$ with p, q prime, $m, n > 1$, has exponents $m = n = 2$; i.e., it comes from a unit $p - q\cdot 2^{1/2}$ of the quadratic field $\mathbb{Q}(2^{1/2})$ for which the coefficients p and q are primes.*

In order to prove our main result, we need some information about certain subgroup structure of a nonabelian simple group.

Lemma 6 (Lemma 2 of [13]). *Let q be a prime power and let n be a positive integer.*

(1) Let $n \geq 8$. Then, A_n has a subgroup A_{n-1}

(2) Let $n \geq 4$, $\varepsilon = \pm$. Then, $PSL_n^{\varepsilon}(q)$ has a subgroup isomorphic to $SL_{n-1}^{\pm}(q)$ or $PSL_{n-1}^{\pm}(q)$, and $SL_n^{\varepsilon}(q)$ has a subgroup of the form $SL_{n-1}^{\varepsilon}(q)$

(3) Let $n \geq 2$. Then, $PSp_{2n}(q)$ has a subgroup $PSp_{2(n-1)}(q)$

(4) Let $n \geq 3$ and q odd. Then, $\Omega_{2n+1}(q)$ contains a subgroup $\Omega_{2n-1}(q)$

(5) Let $n \geq 4$, $\varepsilon = \pm$. Then, $P\Omega_{2n}^{\varepsilon}(q)$ has a subgroup $P\Omega_{2n-2}^{\varepsilon}(q)$ with q odd or $PSp_{2n-2}(q)$ with q even

The following result will be used frequently without reference.

Lemma 7.

(1) A group is a TS-group G if and only if for every $H \in \sum G$, H is a T-subgroup

(2) Let N be a proper subgroup of a TS-group. Then, N is a T-subgroup

(3) Let N be a nontrivial normal subgroup of a TS-group G. Then, G/N is a T-group

Proof. (1) and (2) are obvious by Definition 2

As N is nontrivial, we have that N is a T-group. Assume that G/N is a non-T-group. Then, G/N is a non-TS-group, so G/N has a non-T-group MN/N for certain $M \in \max G$. If $N \leq \Phi(G)$, then $N \leq M$, and $M/N \cap M \cong MN/N$ is a non-T-group. It follows that M is a non-T-group, a contradiction. Now, $N \nleq \Phi(G)$ and let M be a maximal proper subgroup of G with $N \nleq M$. Then, $M < G$, $G = MN$, and so $G/N = MN/N \cong M/M \cap N$ is a non-T-group. It means that M is a non-T-group, a contradiction to the fact $M < G$. □

3. Nonsolvable TS-Groups

In this section, first, we determine the structure of a nonabelian simple TS-group and then that of a nonsolvable TS-group.

It is well-known that a nonabelian simple group is isomorphic to an alternating group A_n, $n \geq 5$, a simple group of Lie type, or a simple sporadic group. So we consider these groups from now on.

Lemma 8. *Let G be an alternating group A_n of degree $n \geq 5$. Assuming that G is a TS-group, then G is isomorphic to A_5 or A_6.*

Proof. An irreducible character of S_n, the symmetric group of degree n, is determined by the partition λ of n, and denote such an irreducible character by χ^λ. Observe that the irreducible characters of A_n are the restrictions of those of S_n to A_n. If $n \geq 14$ and $\lambda = (n-3, 1^3)$, then by Hook's formula, one has

$$\chi^\lambda(1) = \frac{n!}{n \cdot (n-4)! \cdot 3!} = \frac{(n-1)(n-2)(n-3)}{6}. \quad (4)$$

□

See pp. 77 of [14]. Note that for $n \geq 14$, $\lambda = (n-3, 1^3)$ is not self-conjugate, so the character degree of S_n is the same as that of A_n with respect to the partition λ. Hence, $\omega(\chi^\lambda(1)) > 3$, a contradiction.

If $7 \leq n \leq 13$, then by [9] and Lemma 6, we have a subgroup series:

$$\mathrm{PSL}_2(7) < A_7 < A_8 < \cdots < A_{13}. \quad (5)$$

Note that $2^3 \in \mathrm{cd}(\mathrm{PSL}_2(7))$, so $\mathrm{PSL}_2(7)$ is not a T-group; thus, for $13 \geq n \geq 7$, A_n is a non-TS-group.

If $n = 5$, then $\max A_5 = \{A_4, D_{10}, S_3\}$. Observe that $\mathrm{cd}(A_4) = \{1, 3\}$, $\mathrm{cd}(D_{10}) = \{1, 2\}$, and $\mathrm{cd}(S_3) = \{1, 2\}$, so A_5 is a TS-group.

If $n = 6$, then $\max A_6 = \{A_5, 3^2:4, S_4\}$. As $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$, $\mathrm{cd}(3^2:4) = \{1, 2^2\}$, and $\mathrm{cd}(S_4) = \{1, 2, 3\}$, we have that A_6 is also a TS-group.

It follows that G is isomorphic to A_5 or A_6, the desired result.

Note that A_7 is a T-group but a non-TS-group as shown in Example 1.

Lemma 9. *Let G be a nonabelian simple group of classical Lie type. Assuming that G is a TS-group, then G is isomorphic to one of the groups:*

(1) *$\mathrm{PSL}_2(q)$ where $q = 2rs^a + 1$ with $a \in \{0, 1\}$ is a prime for some primes r, s (possibly equal)*

(2) *$\mathrm{PSL}_2(p^m)$ where $p \in \{2, 3, 5\}$, m is a prime and $(p^m - 1)/(\gcd(2, p^m - 1)) = rs^a$ for primes r, s, and $a \in \{0, 1\}$*

(3) *$\mathrm{PSL}_2(2^4)$*

Proof. A simple group of classical Lie type is isomorphic to $\mathrm{PSL}_n(q)$, $n \geq 2$, $\mathrm{PSU}_n(q)$, $n \geq 3$, $\Omega_{2n+1}(q)$, $n \geq 3$, $\mathrm{PSp}_{2n}(q)$, or $\mathrm{P}\Omega_{2n}^{\pm}(q)$ with $n \geq 4$. So these groups are considered in what follows.

Case 1: if G is isomorphic to $\mathrm{PSL}_n(q)$, then G is isomorphic to one of the groups: $\mathrm{PSL}_2(q)$ where $q = 2rs + 1$ is a prime for some primes r, s (possibly equal); $\mathrm{PSL}_2(p^m)$ where $p \in \{3, 5\}$, m is an odd prime and $p^m - 1 = 2rs^a$, $a \in \{0, 1\}$; $\mathrm{PSL}_2(2^4)$, $\mathrm{PSL}_2(2^r)$ with r a prime such that $2^r - 1 = r_1 r_2^{a_2}$ for primes r_1, r_2 and $a_2 \in \{0, 1\}$

Let $n = 2$.

In Lemma 8, we have considered the groups $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$ and $\mathrm{PSL}_2(9) \cong A_6$. So $q = 7$ or $q = 8$ or $q \geq 11$. Two cases are considered now.

(a) If q is odd, then $k = \gcd(2, q-1) = 2$, so by Table 1, $E_q : C_{(q-1)/2} \in \max\mathrm{PSL}_2(q)$ and $(q-1)/2 \in \mathrm{cd}(E_q : C_{(q-1)/2})$. Hypothesis shows that $(q-1)/2 = r^2$ for a prime r, or $(q-1)/2 = rs$ for different primes r, s

(i) $(q-1)/2 = r^2$

If $r = 2$, then $q = 9$, so G is isomorphic to $\mathrm{PSL}_2(9) \cong A_6$ which is considered in Lemma 8. So $r \geq 3$.

If q is a prime, then $q = 2r^2 + 1$ with $\gcd(2, r) = 1$. We see from Table 1 that $\mathrm{PSL}_2(q)$ possibly contains $E_q : C_{(q-1)/2}, D_{q+1}, D_{q+1}, S_4, A_5, A_4$ as its maximal subgroups, so every $H \in \max\mathrm{PSL}_2(q)$ is a T-group, so $\mathrm{PSL}_2(q)$ is a TS-group.

If q is a prime power, then by Lemma 5, $q = 3^5$, $r = 11$, or $q = p^2$ for some prime $p \geq 5$. If $q = p^2$ for some prime $p \geq 5$, we see that $(q-1)/2 = (p^2-1)/2 = (p-1)(p+1)/2$ is divisible by $4 = 2^2$. It follows from hypothesis and Lemma 4 that $q = 9$ and so$p = 3 \geq 5$, a contradiction. It is easy to check that $\mathrm{PSL}_2(3^5)$ is a TS-group by Table 1.

(ii) $(q-1)/2 = rs$ for different primes r, s

Without loss of generality, we can assume that $s > r$. If $r = 2$, then $q = 4s + 1 \geq 13$.

If q is a prime power, say $q = p^m$, then $m \geq 3$ is odd (if m is even, $8 | (q-1)$ forces $4 = rs \geq 6$, a contradiction).

If $m = r_1^{a_1} r_2^{a_2}$, then $\mathrm{PSL}_2(q)$ has a subgroup of the form $\mathrm{PSL}_2(q_0).\gcd(2, r_1)$ with $q = q_0^{r_1}$. Note that $\mathrm{PSL}_2(q_0).2$ or $\mathrm{PSL}_2(q_0)$ is a subgroup of $\mathrm{PSL}_2(q)$ and that

$$\mathrm{cd}(\mathrm{PSL}_2(q_0)) = \left\{1, \frac{q_0 + (-1)^{(q-1)/2}}{2}, q_0, q_0 - 1, q_0 + 1\right\},$$
$$\mathrm{cd}(\mathrm{PSL}_2(q_0).2) = \{1, q_0, q_0 - 1, q_0 + 1\}, \quad (6)$$

by [16]. As $q_0 > 9$ is odd, one of the numbers $q_0 - 1$ and $q_0 + 1$ has the form $4s$ for some $s > 1$, so $\mathrm{PSL}_2(q_0).2$ and $\mathrm{PSL}_2(q_0)$ are not T-groups. It follows that $m \geq 3$ is a prime. If $p \geq 7$, then $\mathrm{PSL}_2(p)$ is a subgroup of $\mathrm{PSL}_2(q)$. Since $\mathrm{PSL}_2(p)$ is not a T-group for $p \geq 7$, one has that $\mathrm{PSL}_2(q)$ is not a TS-group. Thus, $p = 3, 5$. Now, Table 1 shows that $\max\mathrm{PSL}_2(p^m)$ possibly contains $E_q : C_{(q-1)/2}, D_{q+1}, D_{q+1}, S_4, A_5, A_4$ as its members. Thus, $\mathrm{PSL}_2(p^m)$ where m is an odd prime and $p \in \{3, 5\}$ with $p^m - 1 = 2rs^a$, $a \in \{0, 1\}$, is a TS-group.

If q is a prime, then $q = 2rs + 1$ is a prime. By Table 1, we get that $\mathrm{PSL}_2(q)$ is a TS-group.

Table 1: $\mathrm{PSL}_2(q)$, $q \geq 5$ (Chap II Theorem 8.27 of [15]).

	max H	Condition
$\mathscr{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathscr{C}_2$	$D_{2(q-1)/k}$	$q \in \{5, 7, 9, 11\}$
C_3	$D_{2(q+1)/k}$	$q \in \{7, 9\}$
$\mathscr{C}_5$	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathscr{C}_6$	S_4	$q = p \equiv \pm 1 (\mathrm{mod}\ 8)$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 (\mathrm{mod}\ 40)$
$\mathscr{S}$	A_5	$q \equiv \pm 1 (\mathrm{mod}\ 10)$, $F_q = F_p\left[\sqrt{5}\right]$

(b) If q is even, then $q = 8$ or $q \geq 2^4$, say $q = 2^s$, $s \geq 4$

Let $q = 8$, then by [9] (pp. 6), $\max\mathrm{PSL}_2(8) = \{E_{2^3} : C_7, D_{18}, D_{14}\}$, so $\mathrm{PSL}_2(8)$ is a TS-group.

Let $s = r_1^{a_1} r_2^{a_2} \geq 4$ with $a_1, a_2 \geq 1$ integers and r_1, r_2 primes. Without loss of generality, we can assume that $r_1 > r_2$, and then, $\mathrm{PSL}_2(2^s)$ has a subgroup of the form $\mathrm{PSL}_2(2^{s/r_2^{a_2}})$. We see that $2^{s/r_2^{a_2}} \in \mathrm{cd}(\mathrm{PSL}_2(2^{s/r_2^{a_2}}))$, so $\omega(2^{s/r_2^{a_2}}) > 3$ because $r_1 > r_2 \geq 2$ shows $s/r_2^{a_2} \geq r_1 > r_2 \geq 2$. Thus, $s := r_1^{a_1}$ say.

If $r_1 = 2$ and $a_1 \geq 2$, then let $q_0 = 2^{2^{a_1 - 1}}$; we get that $\mathrm{PSL}_2(2^s)$ has a subgroup $\mathrm{PSL}_2(q_0)$. If $a_1 = 2$, then $\mathrm{PSL}_2(2^4)$ is a TS-group as $\mathrm{PSL}_2(q_0)$ is a T-subgroup. If $a_1 > 2$, then let $\chi \in \mathrm{Irr}(\mathrm{PSL}_2(q_0))$ with $\chi(1) = q_0$, $\omega(\chi(1)) = 2^{a_1 - 1} > 2^{2-1} = 2$, so $\mathrm{PSL}_2(q_0)$ is not a T-group.

If $r := r_1 \geq 5$ is odd, then $a_1 \geq 2$ or $a_1 = 1$. Note that

$$\max\mathrm{PSL}_2(2^s) = \left\{E_{2^s} : C_{2^s - 1}, D_{2(2^s - 1)}, D_{2(2^s + 1)}\right\}, \quad (7)$$

and that $\mathrm{PSL}_2(2^{r^{a_1}})$ has a subgroup $\mathrm{PSL}_2(2^{r^{a_1 - 1}})$. Note that $2^{r^{a_1 - 1}} \in \mathrm{PSL}_2(2^{r^{a_1 - 1}})$, and for $a_1 \geq 2$, $\omega(2^{r^{a_1 - 1}}) = r^{a_1 - 1} \geq r^{2-1} = r \geq 3$, so $\mathrm{PSL}_2(2^s)$ is not a TS-group. If $a_1 = 1$, then $\mathrm{PSL}_2(2^s)$ with s an odd prime and $2^s - 1 = rs^a$ for $a \in \{0, 1\}$ is a TS-group.

Let $n = 3$. If $q = 2$, then $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ is considered as above. So $q \geq 3$. If $q = q_0^b$ with b a prime, then $\mathrm{PSL}_3(q)$ has $\mathrm{PSL}_3(q_0).\gcd(b, \gcd(3, q - 1))$ as its subgroup. So $\mathrm{PSL}_3(q_0) \in \sum\mathrm{PSL}_3(q)$. By [17],

$$(q_0 - 1)\left(q_0^2 + q_0 + 1\right), (q_0 + 1)\left(q_0^2 + q_0 + 1\right) \in \mathrm{cd}(\mathrm{PSL}_3(q_0)). \quad (8)$$

It is easy to see that for at least one of the numbers $(q_0 - 1)(q_0^2 + q_0 + 1), (q_0 + 1)(q_0^2 + q_0 + 1)$, say d, we have $\omega(d) > 2$ if q is a prime power. Thus, $q \geq 3$ is a prime. By pp. 191 of [18], $\mathrm{SO}_3(q) \in \max\mathrm{PSL}_3(q)$. We know that $\mathrm{SO}_3(q) \cong \mathrm{SL}_2(q)$ and that

$$\mathrm{cd}(\mathrm{SL}_2(q)) = \left\{1, \frac{q-1}{2}, \frac{q+1}{2}, q - 1, q, q + 1\right\}, \quad (9)$$

by [16], so $q = 3, 5$ because one of the numbers $q - 1$ and $q + 1$ with $q \geq 7$ has the form $4s$ for some $s \geq 2$. If $q = 3$, then $3^2.2.S_4 \in \max\mathrm{PSL}_3(3)$, so by [19], $2^4 \in \mathrm{cd}(3^2.2.S_4)$; if $q = 5$, then $5^2 : 4S_5 \in \max\mathrm{PSL}_3(5)$, so by [19], $2^5 \cdot 3 \in \mathrm{cd}(5^2 : 4S_5)$. Thus, $\mathrm{PSL}_3(q)$ with $q = 3, 5$ is not a TS-group. Now, we have shown that $\mathrm{PSL}_3(q)$ with $q \geq 3$ is not a TS-group and so is $\mathrm{SL}_3(q)$ as

$$\mathrm{PSL}_3(q) = \frac{\mathrm{SL}_3(q)}{Z(\mathrm{SL}_3(q))}, \quad \mathrm{cd}(\mathrm{PSL}_3(q)) \subseteq \mathrm{cd}(\mathrm{SL}_3(q)), \quad (10)$$

by [20]. In particular, neither $\mathrm{SL}_3(q)$ nor $\mathrm{PSL}_3(q)$ is a T-group.

Let $n \geq 4$. If $n = 4$ and $q = 2$, then $\mathrm{PSL}_4(2) \cong A_8$ is considered in Lemma 8. Now, $n \geq 5$, so by Lemma 6, we obtain that

$$\text{either } \mathrm{PSL}_3(q) < \mathrm{PSL}_n(q) \text{ or } \mathrm{SL}_3(q) < \mathrm{PSL}_n(q). \quad (11)$$

This implies that $\mathrm{PSL}_n(q)$ with $n \geq 4$ is not a TS-group since $\mathrm{PSL}_3(q)$ and $\mathrm{SL}_3(q)$ are not T-groups.

Case 2: $\mathrm{PSU}_n(q)$ with $n \geq 3$

Let $n = 3$. If $q = 2$, then $\mathrm{PSU}_3(2)$ is solvable, so $q \geq 3$. By [9], we have that

$$\begin{cases} \mathrm{ES}\left(3^{1+2}\right): C_8 \in \max\mathrm{PSU}_3(3), \\ \mathrm{ES}\left(4^{1+2}\right): C_{15} \in \max\mathrm{PSU}_3(4), \\ \mathrm{ES}\left(5^{1+2}\right): C_8 \in \max\mathrm{PSU}_3(5), \\ \mathrm{ES}\left(7^{1+2}\right): C_{48} \in \max\mathrm{PSU}_3(7), \\ \mathrm{ES}\left(8^{1+2}\right): C_{21} \in \max\mathrm{PSU}_3(8), \end{cases} \quad (12)$$

so by [19], we get that $\mathrm{PSU}_3(q)$ for $q \in \{3, 4, 5, 7, 8\}$ is not a TS-group. Thus, we can assume that $q \geq 9$; then, by pp. 200 of [18], $\mathrm{ES}(q^{1+2})$: $C_{(q^2-1)/k} \in \max\mathrm{PSU}_3(q)$ where $k = \gcd(3, q + 1)$, and $(q^2 - 1)/k \in \mathrm{cd}(\mathrm{ES}(q^{1+2})$: $C_{(q^2-1)/k})$. Observe that $\mathrm{ES}(q^{1+2})$: $C_{(q^2-1)/k}$ is a Frobenius group and that there in G does exist an irreducible character $\chi \in \mathrm{Irr}(\mathrm{ES}(q^{1+2}))$ with $\chi(1) = q$ (note that $|Z(\mathrm{ES}(q^{1+2}))| = q$, and $\mathrm{ES}(q^{1+2})/Z(\mathrm{ES}(q^{1+2}))$ is abelian, so by Theorem 2.31 of [21], for $\chi \in \mathrm{Irr}(G)$, $\chi(1)^2 = |\mathrm{ES}(q^{1+2})$: $Z(\mathrm{ES}(q^{1+2}))| = q^2$; hence, $\chi(1) = q$).

If q is even, then $\omega(q) > 4$ as $q > 2^3$. If q is odd, then $q \geq 9$ and

$$\frac{q^2 - 1}{k} \in \mathrm{cd}\left(\mathrm{ES}\left(q^{1+2}\right): C_{(q^2-1)/k}\right). \quad (13)$$

Observe that $(q^2 - 1)/k$ is divisible by eight and that $k \in \{1, 3\}$, so $\omega((q^2 - 1)/k)) \geq 3$.

It follows that $\mathrm{PSU}_3(q)$ with $q \geq 3$ is a non-TS-group.

Let $n \geq 4$. If $n = 4$ and $q = 2$, then $\mathrm{PSU}_4(2)$ has $2^4 : A_5$ as a subgroup. As $2^2 \cdot 5 \in \mathrm{cd}(2^4 : A_5)$, $2^4 : A_5$ is not a T-group. Now, assume that $n \geq 5$; then, by Lemma 6, we have that

$$\text{either } \mathrm{PSU}_3(q) < \mathrm{PSU}_n(q) \text{ or } \mathrm{SU}_3(q) < \mathrm{PSU}_n(q). \quad (14)$$

For $q \geq 3$, by [20], we have that $\mathrm{SU}_3(q)$ and $\mathrm{PSU}_3(q)$ are not T-groups. It follows that $\mathrm{PSU}_n(q)$ with $n \geq 4$ is not a TS -group.

Case 3: $\Omega_{2n+1}(q)$ with $n \geq 3$, q odd

If $n = 3$, then $[q^7]$: $(1/2)(\mathrm{GL}_2(q) \times \mathrm{SO}_3(q))$ by [18] (pp. 213). We know that $(q+1)^2 \in \mathrm{cd}(\mathrm{GL}_2(q) \times \mathrm{SO}_3(q))$, so $\omega((q+1)^2) > 2$ for an odd q. Thus, $\Omega_7(q)$ is not a TS-group. In particular, $\Omega_7(q)$ is not a T-group.

Let $n \geq 4$. Then, by Lemma 6, a subgroup series is obtained:

$$\Omega_7(q) < \Omega_9(q) < \cdots < \Omega_{2n+1}(q), \quad (15)$$

so $\Omega_{2n+1}(q)$ is not a TS-group since $\Omega_7(q)$ is not a T-group.

Case 4: $\mathrm{PSp}_{2n}(q)$ with $n \geq 2$

Let $n = 2$. If $q = 2$, then $\mathrm{PSp}_4(2) \cong S_6$ is not simple, so $q \geq 3$, and by pp. 209 of [18] $\mathrm{PSp}_4(q)$ has a subgroup $\mathrm{PSp}_2(q^2).2 \cong \mathrm{PSL}_2(q^2).2$. We know that $\mathrm{PSL}_2(q^2)$ is a normal subgroup of $\mathrm{PSL}_2(q^2).2$ and that $q^4 - 1 \in \mathrm{cd}(\mathrm{PSL}_2(q^2))$, so $\mathrm{PSp}_4(q)$ is not a TS-group; in particular, $\mathrm{PSp}_4(q)$ is not a T-group.

Let $n \geq 3$. Then from Lemma 6, $\mathrm{PSp}_{2n}(q)$ contains a subgroup $\mathrm{PSp}_4(q)$, so $\mathrm{PSp}_{2n}(q)$ is not a TS-group.

Case 5: $\mathrm{P}\Omega^{\varepsilon}_{2n}(q)$ with $n \geq 4$ and $\varepsilon = \pm$

If $n = 4$, then $\mathrm{P}\Omega^{\varepsilon}_8(q)$ has a subgroup $\Omega_7(q)$, so by Case 3, $\Omega_7(q)$ is neither a T-group nor a TS-group.

If $n \geq 5$, then by Lemma 6, $\mathrm{P}\Omega^{\varepsilon}_{2n}(q)$ is not a TS-group as $\mathrm{P}\Omega^{\varepsilon}_{2n}(q)$ contains a subgroup isomorphic to either $\Omega_7(q)$ with q odd or $\mathrm{PSp}_4(q)$ with q even. Note that $\Omega_7(q)$ with q odd and $\mathrm{PSp}_4(q)$ with q are non-T-groups, so we rule out this case. □

Lemma 10. *Let G be a simple group of exceptional Lie type. Assuming that G is a TS-group, then G is isomorphic to* $^2B_2(8)$.

Proof. We see that G is isomorphic to $^2B_2(q)$ with $q = 2^{2m+1} \geq 8$, $^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$, $G_2(q)$, $^3D_4(q)$, $F_4(q)$, $^2F_4(q^2)$, $E_6^{\varepsilon}(q)$, $E_7(q)$, or $E_8(q)$. We deal with these case by case.

The following subgroup series are obtained from Table 2:

$$\begin{aligned} &G_2(q) > \mathrm{SU}_3(q^2).2 > \mathrm{SL}_3(q^2), \\ &^2F_4(q^2) > \mathrm{SU}_3(q^2).2 > \mathrm{SU}_3(q^2). \end{aligned} \quad (16)$$

We know that $\mathrm{PSL}_n(q) \cong (\mathrm{SL}_n(q))/(Z(\mathrm{SL}_n(q)))$ and $\mathrm{PSU}_n(q) \cong (\mathrm{SU}_n(q))/(Z(\mathrm{SU}_n(q)))$, so $\mathrm{SU}_3(q^2)$ and $\mathrm{SL}_3(q^2)$ are non-TS-groups since Cases 1 and 2 in Lemma 9 show that $\mathrm{PSU}_3(q^2)$ and $\mathrm{PSL}_3(q^2)$ are non-T-groups. So $G_2(q)$ and $^2F_2(q^2)$ are not TS-groups. Note that $^2F_4(2)'$ is not a TS -group by [9]. From Table 2, $^3D_4(q)$, $F_4(q)$, and $E_6^{\varepsilon}(q)$ are non-TS-groups. Now, if $\mathrm{P}\Omega^+_{12}(q)$ is not a TS-group, then $E_7(q)$ and $E_8(q)$ are not TS-groups. Now, we will show that $\mathrm{P}\Omega^+_{12}(q)$ is not a TS-group. In fact, $\mathrm{P}\Omega^+_{12}(q)$ contains a subgroup $\Omega_7(q)$ or $\mathrm{PSp}_4(q)$ which are not a T-group, so $\mathrm{P}\Omega^+_{12}(q)$ is not a TS-group.

Table 2: Exceptional groups of Lie type.

G	$H \in \max G$	Reference
$G_2(q)$	$\mathrm{SL}_3(q).2$, $\mathrm{SU}_3(q^2).2$	Table 1 of [22]
$^3D_4(q)$	$G_2(q)$	Table 1 of [22]
$F_4(q)$	$^3D_4(q^3)$	Table 1 of [22]
$E_6^{\varepsilon}(q)$	$F_4(q)$	Table 1 of [22]
$E_7(q)$	$\mathrm{P}\Omega^+_{12}(q)$	Table 1 of [22]
$E_8(q)$	$E_7(q))$	Table 1 of [22]
$^2F_4(q^2)$	$\mathrm{SU}_3(q^2).2$	Main theorem of [23]

Now, we can conclude that G is possibly isomorphic to $^2B_2(q)$ with $q = 2^{2m+1} \geq 8$ or $^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$. Thus, two cases are considered.

Case 1: $^2B_2(q)$ with $q = 2^{2m+1} \geq 8$

We know from pp. 385 of [10] that $^2B_2(2^{2m+1})$ has a maximal subgroup of the form $E_2^{(2m+1)+(2m+1)} : C_{2^{2m+1}-1}$ which is a Frobenius group. So there is an irreducible character $\chi \in \mathrm{Irr}(E_2^{(2m+1)+(2m+1)})$ with a maximal degree with respect to divisibility in $\mathrm{cd}(E_2^{(2m+1)+(2m+1)})$. Observe that if $m \geq 2$, then by Theorems 13.3 and 13.8 of [24],

$$\chi(1)\left(2^{2m+1} - 1\right) \in \mathrm{cd}\left(E_2^{(2m+1)+(2m+1)}\right), \quad (17)$$

so by hypothesis, $2^{2m+1} - 1$ is a prime. On the other hand, $Z(E_2^{(2m+1)+(2m+1)})$ is of order 2^{2m+1}, so $E_2^{(2m+1)+(2m+1)}/Z(E_2^{(2m+1)+(2m+1)})$ is a 2-group, so there is a nonlinear character $\chi \in \mathrm{Irr}(E_2^{(2m+1)+(2m+1)})$ such that $\chi(1)^2$ divides 2^{2m+1}, i.e., $\chi(1) | 2^m$. Notice that $\chi(1) = \max\mathrm{cd}(E_2^{(2m+1)+(2m+1)})$. If $\chi(1) = 2$ and let $P = E_2^{(2m+1)+(2m+1)}$, then $|P/P'| = 2$, and $P' = \Phi(P)$, so $Z(P) = P'$, P is an extraspecial 2-group, so $m = 1$. (In fact, we know from (17) that for $m \geq 2$, $E_2^{(2m+1)+(2m+1)} : C_{2^{2m+1}-1}$ is not a T-group.) So by [9] (pp. 28), $\max {}^2B_2(8) = \{2^{3+3} : 7, 13 : 4, 5 : 4, D_{14}\}$. By [19], we may get that the groups $2^{3+3} : 7$, $13 : 4$, $5 : 4$, and D_{14} are T-groups, so $^2B_2(8)$ is a TS-group.

Case 2: $^2G_2(q)$ with $q = 3^{2m+1} \geq 3^3$

From [10] (pp. 398), $^2G_2(q)$ has a maximal subgroup $E_q^{1+1+1} : C_{q-1}$ which is a Frobenius group. Let $\chi(1) = k \neq 1$ for $\chi \in \mathrm{Irr}(E_q^{1+1+1} : C_{q-1})$. Then, by Theorems 13.3 and 13.8 of [24], $k(q-1) \in \mathrm{cd}(E_q^{1+1+1} : C_{q-1})$. It follows that $^2G_2(q)$ is not a TS-group. □

Lemma 11. *A simple group of sporadic group is not a TS -group.*

Proof. By [9] (pp. 18), $PSL_2(11) \in \max M_{11}$ and $12 \in cd(PSL_2(11))$, so $PSL_2(11)$ is not a T-group. Now, [9] (pp. 238) shows that M_{11} is a subgroup of these groups: M_{12}, M_{23}, HS, M_{24}, McL, Suz, ON, Co_3, Co_2, Fi_{22}, HN, Ly, Fi_{23}, Co_1, J_4, Fi_{24}, B, and M.

By checking [9] J_1, M_{22}, J_2, J_3, He, Ru, and Th are not TS-groups as $PSL_2(11)$ is a subgroup of J_1, M_{22}, $PSL_2(9)$ is a subgroup of J_2, J_3, Th, and $PSL_3(2) \cong PSL_2(7)$ is a subgroup of He, Ru. □

Theorem 12. *Let G be a nonabelian simple TS-group. Then, G is isomorphic to the following:*

(1) $PSL_2(q)$ where $q = 2rs^a + 1$ with $a \in \{0, 1\}$ is a prime for some primes r, s (possibly equal)

(2) $PSL_2(p^m)$ where $p \in \{2, 3, 5\}$, m is a prime and $(p^m - 1)/(\gcd(2, p^m - 1)) = rs^a$ for some primes r, s, and $a \in \{0, 1\}$

(3) $PSL_2(2^4)$

(4) ${}^2B_2(8)$

Proof. We conclude the result from Lemmas 8, 10, and 11. □

Theorem 13. *Let G be an almost simple TS-group with socle S, $S \le G \le Aut(S)$. Then, G is isomorphic to one of the groups:*

(1) $PSL_2(q)$ where $q = 2rs + 1$ is a prime for some primes r, s (possibly equal)

(2) $PSL_2(p^m)$ where $p \in \{2, 3, 5\}$, and m is a prime such that $(p^m - 1)/(\gcd(2, q - 1)) = rs^a$ for primes r, s, and $a \in \{0, 1\}$

(3) ${}^2B_2(8)$, $PSL_2(2^4)$

(4) S_5

Proof. If G is simple, then by Theorem 12, we have (1)-(3). So three cases are considered, and also, we assume that G is nonsimple.

Case 1: $PSL_2(q)$ where $q = 2rs^a + 1$ is a prime for some primes r, s (possibly equal) and $a \in \{0, 1\}$

In this case, G is possibly isomorphic to $PGL_2(q)$. Note that $PGL_2(q)$ has a normal subgroup $PSL_2(q)$ with index $\gcd(2, q - 1)$ and that

$$cd(PSL_2(q)) = \left\{1, \frac{q + (-1)^{(q-1)/2}}{2}, q - 1, q, q + 1\right\}, \quad (18)$$

by [16], so hypothesis shows that $q + 1 = 4$ or $q - 1 = 4$ (in fact, if q is odd, then one of the numbers $q - 1$ or $q - 1$ can be written by $4r$ for some $r \ge 1$). It follows that G is isomorphic to S_5.

Case 2: $PSL_2(p^m)$ where $p \in \{2, 3, 5\}$, m is a prime and $(p^m - 1)/(\gcd(2, q - 1)) = rs^a$ for primes r, s, and $a \in \{0, 1\}$

(i) Let $p = 2$

If $m = 2$, then $PSL_2(4) \cong PSL_2(5)$ is done in Case 1. If $m \ge 3$, then G is possibly isomorphic to $PSL_2(2^m).m$. We see that $PSL_2(2^m).m$ has $PSL_2(2^m)$ as its subgroup and that $2^m \in cd(PSL_2(2^m))$, so hypothesis forces $m \le 2$, a contradiction. It means that $PSL_2(2^m).m$ is not a TS-group.

(ii) Let $p = 3$

If $m = 2$, then the order of the outer-automorphism group of $PSL_2(9)$ is 4, so by [9], G is possibly isomorphic to $PSL_2(9).2_1$, $PSL_2(9).2_2$, $PSL_2(9).2_3$, or $PSL_2(9).2^2$. Notice that $PSL_2(9)$ is a normal subgroup of these groups ($PSL_2(9).2_1$, $PSL_2(9).2_2$, $PSL_2(9).2_3$, and $PSL_2(9).2^2$) and that $cd(PSL_2(9)) = \{1, 5, 8, 9, 10\}$, so $PSL_2(9)$ is not a T-group. It follows that the groups $PSL_2(9).2_1$, $PSL_2(9).2_2$, $PSL_2(9).2_3$, and $PSL_2(9).2^2$ are non-TS-groups.

Let $m \ge 3$. Then, the order of the outer-automorphism group of $PSL_2(3^m)$ is $2m$, and so by Corollary 6.5 of [25], G possibly has one of the structures: $PGL_2(3^m)$, $PSL_2(3^m).m$, $PGL_2(3^m).m$, and $PSL_2(3^m).(2m)$. Note that $PGL_2(3^m)$, $PSL_2(3^m).m$, $PGL_2(3^m).m$, and $PSL_2(3^m).(2m)$ have a subgroup of the form $PSL_2(3^m)$. We know that $3^m \in cd(PSL_2(3^m))$, so hypothesis shows that $m \le 2$. If $m = 1$, then $PSL_2(3)$ is solvable; if $m = 2$, then $8 \in cd(PSL_2(3^2))$. Thus, $PGL_2(3^m)$, $PSL_2(3^m).m$, $PGL_2(3^m).m$, and $PSL_2(3^m).(2m)$ are not TS-groups.

(iii) Let $p = 5$

Then, we can get from Corollary 6.5 of [25] that the possible groups G are isomorphic to $PSL_2(3^m).m$, $PGL_2(3^m)$, $PSL_2(3^m).(2m)$, and $PGL_2(3^m).m$. Note that these groups always have a subgroup $PSL_2(3^m)$ which is not a T-group. Thus, $PSL_2(3^m).m$, $PGL_2(3^m)$, $PSL_2(3^m).(2m)$, and $PGL_2(3^m).m$ are not TS-groups

Case 3: ${}^2B_2(8)$ and $PSL_2(2^4)$

In this case, the outer-automorphism group of ${}^2B_2(8)$ is of order 3, so G is possibly isomorphic to ${}^2B_2(8).3$. Now, by [9] (pp. 28), $13 : 12 \in \max {}^2B_2(8).3$, so $2^2 \cdot 3 \in cd(13 : 12)$. It follows that $13 : 12$ is not a T-group and so ${}^2B_2(8).3$ is not a TS-group.

We can get from [9] (pp. 12) that $PSL_2(2^4).2$ and $PSL_2(2^4).4$ are not TS-groups as $PSL_2(2^4)$ is a subgroup of both $PSL_2(2^4).2$ and $PSL_2(2^4).4$. □

Now, we will prove Theorem 3.

Proof. Let N be a minimal nonabelian normal subgroup of G. Then, we get that

$$N = S_1 \times S_2 \times \cdots \times S_n, \quad (19)$$

where S_i is isomorphic to a nonabelian simple group S for each $1 \le i \le n$. If $n \ge 2$, we obtain by Theorem 4.21 of [21] that for $\tau_1 \in Irr(S_1) \setminus Lin(S_1)$, $\tau_2 \in Irr(S_2) \setminus Lin(S_2)$ and $\lambda \in Lin(S_i)$, for $i \ge 3$,

$$\tau_1 \times \tau_2 \times \lambda \cdots \times \lambda \in Irr(N). \quad (20)$$

Thus, N is a non-T-group, a contradiction. Now, we get that $N = S$ is a nonabelian simple group which is isomorphic to one of the groups satisfying Theorem 12.

We know that N is isomorphic to a subgroup of $G/C_G(N)$ and that $G/C_G(N)$ is isomorphic to a subgroup of Aut(N), the automorphism group of N; then, $G/C_G(N)$ is an almost simple group satisfying Theorem 13.

Let $\mathfrak{S}$ be the set of the groups:

(i) $PSL_2(q)$ where $q = 2rs + 1$ is a prime for some primes r, s (possibly equal)

(ii) $PSL_2(p^m)$ where $p \in \{2, 3, 5\}$, and m is a prime such that $(p^m - 1)/(\gcd(2, q-1)) = rs^a$ for primes r, s, and $a \in \{0, 1\}$

(iii) ${}^2B_2(8)$, $PSL_2(2^4)$

(iv) S_5

If $M \in \mathfrak{S} \setminus \{S_5, A_5\}$, we have that $G/C_G(N)$ is isomorphic to S and that $NC_G(N) \lhd G$. From $N \cap C_G(N) = 1$, we conclude that G is isomorphic to $N \times C_G(N)$ (if $G > NC_G(N)$, $NC_G(N)$ is a T-group, a contradiction). If $C_G(N) \neq 1$, one has that N is a T-group, a contradiction. It follows that G is isomorphic to S.

If $M = S_5$, we may get that G is isomorphic to $S_5 \times C_G(N)$ as $S_5 \cap C_G(N) = 1$. Notice that in this case, N is isomorphic to $PSL_2(5)$. If $C_G(N)$ is nonabelian, we have from the fact, $PSL_2(5) \cong A_5 < S_5$, that $PSL_2(5) \times C_G(N)$ is a T-group, a contradiction. So $C_G(N)$ is abelian.

If $M = A_5 \cong PSL_2(5)$, one has that $N \cong PSL_2(5)$ and that $G' \cap C_G(PSL_2(5)) \leq C_2$ as the order of the Schur multiplier of $PSL_2(5)$ is two. If $G' \cap C_G(PSL_2(5)) = 1$, we have

$$G' \cong \frac{G'}{G' \cap C_G(PSL_2(5))} \cong G' C_G(PSL_2(5))/C_G(PSL_2(5)) \cong PSL_2(5), \quad (21)$$

and so G is isomorphic to $PSL_2(5) \times C_G(N)$. By hypothesis, $C_G(N)$ is a T-group. Let $A = C_G(N)$. Then, we easily get that G is a TS-group if $cd(A) \subseteq \{1, p\}$ for some prime p and a non-TS-group if for different primes $p, r, pr \in cd(A)$, or $p, r \in cd(A)$.

If $G' \cap C_G(PSL_2(5)) \cong C_2$, we have $G/K \cong SL_2(5)$ for some normal subgroup K of G. If K is nonabelian, we assume that $cd(K) = \{1, p\}$ for some prime p by above arguments. Let $\theta \in Irr(K)$ with $\theta(1) = p$ and $I = I_G(\theta)$ be the inertia subgroup of θ in G. Then, I is isomorphic to K, S_4K, $Q_{12}K$, $(E_5 : C_4)K$, or G by [10] (pp. 377). If $I = K$, we have that $I_{S_4K}(\theta) = I \cap (S_4K) = I$, $\theta^{S_4K} \in Irr(S_4K)$ by Theorem 6.16 of [21], and so $\theta^{S_4K}(1) = |S_4K : I|\theta(1) = 2^3 \cdot 3 \cdot p$. Now, by Corollary 11.29 of [21], for $\chi \in (Irr S_4K|\theta)$, $2^3 \cdot 3 = \chi(1)/\theta(1) | (|S_4K|/|K|)$. It follows that there is an irreducible character $\delta \in Irr(S_4K)$ with $12 | \delta(1)$. Now, S_4K is a non-T-group, and so we also rule out the case $I = S_4K$. If $I = Q_{12}K$, we have that I has a subgroup isomorphic to C_6K, so hypothesis shows that C_6K is a T-group. We see that $C_6 \cong C_6K/K$ is cyclic and so $6 \in cd(C_6K)$. Note that $I \cap C_6K = I_{C_6K}(\theta)$ where $\theta \in Irr(K)$ with $\theta(1) = p$. It follows from Corollary 6.17 of [21] that $6p \in cd(S_4K)$, a contradiction. Similarly, we also rule out when $I = (E_5 : C_4)K$. If $I = G$, we can get a contradiction by Corollary 6.17 of [21] too. It follows that K is abelian, the desired result. □

4. Conclusion

In this paper, we change the condition from "the degrees of a group are direct products of two prime numbers" to "the degrees of all proper subgroups of a group are direct products of two prime numbers" and get that if a nonsolvable group that all proper subgroups have degrees which are the direct products of at most two prime numbers, then it has a section isomorphic to ${}^2B_2(8)$ or $PSL_2(q)$ for certain q. Note that A_7 is a T-group but not a TS-group.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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