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Research Article

Inequalities Involving A-Numerical Radius and Operator A-Norm for a Class of Operators Related to $(\alpha, \beta) - A$-Normal Operators

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In this article, we introduce and study a new class of operators, larger than $(\alpha, \beta) - A$-normal operators and different than $(\alpha, \beta) - A$-normal operators, named m-quasi-$(\alpha, \beta) - A$-normal operators. Considering the semi-inner product induced by a positive operator A, the m-quasi-$(\alpha, \beta) - A$-normal operators turn into a generalization (for this new structure) of classical m-quasi-(α, β)-normal operators. Several results concerning properties of this kind of operators are presented in the paper. Several inequalities for the A-numerical radius and A-operator norm for members of this class are established.

1. Introduction and Preliminaries

Let $\mathscr{B}_b[\mathscr{H}]$ denote the C^*-algebra of all bounded linear operators on an infinite dimension complex Hilbert space $\mathscr{H}$. $\mathscr{B}_b[\mathscr{H}]^+$ denotes the cone of positive operators of $\mathscr{H}$. It is well known that $A \in \mathscr{B}_b[\mathscr{H}]^+ \Leftrightarrow \langle Aw|w\rangle \geq 0\, \forall w \in \mathscr{H}$.

For $R \in \mathscr{B}_b(\mathscr{H})$, we denote by R^* the adjoint operator of R, range R by $\text{ran}(R)$, and its null space by $\ker(R)$. If $R \in \mathscr{B}_b[\mathscr{H}]$, then $\overline{\text{ran}(R)}$ stands for its closure in the norm topology of $\mathscr{H}$. We denote by $P_{\overline{\text{ran}(A)}}$ the orthogonal projection onto the closed linear subspace $\overline{\text{ran}(A)}$.

If A is a positive operator on $\mathscr{H}$, the bilinear functional $\langle \cdot|\cdot\rangle$: $\mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{C}$ defines a positive semidefinite sesquilinear form given by $(w, v) \mapsto \langle w|v\rangle_A = \langle Aw|v\rangle$.

Notice that

$$\langle w|w\rangle_A = 0 \Leftrightarrow \langle Aw|w\rangle = 0 \Leftrightarrow \|w\|_A^2 = 0 \Leftrightarrow w \in \ker(A). \quad (1)$$

An operator $R \in \mathscr{B}_b[\mathscr{H}]$ is said to be A-bounded if there exists $C > 0$ such that

$$\|Rw\|_A \leq C\|w\|_A, \quad \forall w \in \mathscr{H}. \quad (2)$$

The set of all A-bounded operators on $\mathscr{H}$ will be denoted by $\mathscr{B}_b[\mathscr{K}]_A$.

For any $R \in \mathscr{B}_b[\mathscr{H}]_A$, we take from [1],

$$\|R\|_A := \sup\left\{\frac{\|Rw\|_A}{\|w\|_A}, \quad w \in \overline{\text{ran}(A)}\right\} = \sup\{\|Rw\|_A, u \in \mathscr{H}: \|w\|_A = 1\}. \quad (3)$$

In [1], the authors have introduced the concept of A-self-adjoint for an operator $R \in \mathscr{B}_b[\mathscr{H}]$. An $S \in \mathscr{B}_b[\mathscr{H}]$ is called an A-adjoint operator of R if R and S satisfy

$$\langle Rw|v\rangle_A = \langle w|Sv\rangle_A, \quad \forall w, v \in \mathscr{H}. \tag{4}$$

We observe that

$$\langle Rw|v\rangle_A = \langle w|Sv\rangle_A, \quad \forall w, v \in \mathscr{H} \Leftrightarrow AR = S^* A. \tag{5}$$

R is an A-self-adjoint operator if R is an A-adjoint of itself, that is, $AR = R^* A$.

It was observed by Douglas Theorem [2] that an operator $R \in \mathscr{B}_b[\mathscr{H}]$ admits an A-adjoint if and only if $\operatorname{ran}(R^* A) \subseteq \operatorname{ran}(A)$. In the following, the set of all operators $R \in \mathscr{B}_b[\mathscr{H}]$ which admit an A-adjoint is denoted by $\mathscr{B}_b[\mathscr{H}]_A$. Throughout this paper, we consider that $A \in \mathscr{B}_b[\mathscr{H}]^+$ with $\operatorname{ran}(A) = \overline{\operatorname{ran}(A)}$. In [1], it was observed that if $R \in \mathscr{B}_b[\mathscr{H}]_A$, the solution of the equation $AX = R^* A$ is a distinguished A-adjoint operator of R, which is denoted by $R^\sharp$. However, $R^\sharp$ is given by $R^\sharp = A^\dagger R^* A$ where $A^\dagger$ is the Moore–Penrose inverse of A. The A-adjoint operator $R^\sharp$ satisfies the following axioms:

$$AR^\sharp = R^* A, \quad \operatorname{ran}\left(R^\sharp\right) \subseteq \operatorname{ran}(A) \text{ and } \ker\left(R^\sharp\right) = \ker(R^* A). \tag{6}$$

The authors in [1, 3, 4] have studied the following useful properties of $R^\sharp$ for $R \in \mathscr{B}[\mathscr{H}]_A$.

For $R \in \mathscr{B}_b[\mathscr{H}]_A$, then the following axioms hold:

(1) $R^\sharp \in \mathscr{B}_b[\mathscr{H}]_A$.

(2) $(R^\sharp)^\sharp = P_{\operatorname{ran}(A)} R P_{\operatorname{ran}(A)}$.

(3) If $S \in \mathscr{B}_b[\mathscr{H}]_A$, then $(RS)^\sharp = S^\sharp R^\sharp$.

In recent years, the theory of normal operators has known many extensions due to the work carried out by several authors ([5–8]). The study of operators in semi-Hilbertian spaces is of significant interest and is currently being done by a number of mathematicians around the world. Some developments toward this subject have been done in [1, 3, 4, 9–18].

Any operator $R \in \mathscr{B}_b[\mathscr{H}]$ is A-positive (symbolically $R \geq_A 0$) if $AR \in \mathscr{B}[\mathscr{H}]^+$, that is,

$$\langle ARw, w\rangle = \langle Rw, w\rangle_A \geq 0, \quad \forall w \in \mathscr{H}. \tag{7}$$

For $R, S \in \mathscr{B}_b[\mathscr{H}]$: $R \geq_A S \Leftrightarrow A(R - S) \geq 0 \Leftrightarrow \langle Rw, w\rangle_A \geq \langle Sw, w\rangle_A \, \forall w \in \mathscr{H}$.

The classes isometry, unitary, normal, and (n, m)-normal operators in Hilbert space have been generalized to the concepts of so called A-isometry, A-unary, and A-normal and $(n, m) - A$-normal operators by many authors. An operator $R \in \mathscr{B}[\mathscr{H}]_A$ is called

(1) A-isometry if $R^* AR = A$ ($\|Ru\|_A = \|u\|_A, \forall u \in \mathscr{H}$ [19]),

(2) A-unitary if $R^\sharp R = (R^\sharp)^\sharp R^\sharp = P_{\operatorname{ran}(A)}$ ([1]),

(3) A-normal if $R^\sharp R = RR^\sharp$ ([12, 17]),

(4) $(\alpha, \beta) - A$-normal, if $\beta^2 R^\sharp R \geq_A RR^\sharp \geq_A \alpha^2 R^\sharp R$, for $0 \leq \alpha \leq 1 \leq \beta$ ([11]),

(5) $(n, m) - A$-normal if $R^n (R^\sharp)^m = (R^\sharp)^m R^n$, for some positive integers n and m [9].

Of course, these extensions are not trivial since many difficulties arise.

Recently, the authors in [20] have introduced the concept of m-quasi totally-(α, β)-normal operators as a generalization of totally-(α, β)-normal operators in Hilbert space. An element $R \in \mathscr{B}[\mathscr{H}]$ is called m-quasi totally-(α, β)-normal if R satisfies

$$\alpha \left\|(R - \lambda) R^m w\right\| \leq \left\|(R - \lambda)^* R^m\right\| \leq \beta \left\|(R - \lambda) R^m w\right\|, \quad \forall w \in \mathscr{H}, \forall \lambda \in \mathbb{C}. \tag{8}$$

In this paper, our goal is to introduce a new class of operators, larger than $(\alpha, \beta) - A$-normal operators, named m-quasi-$(\alpha, \beta) - A$-normal operators in semi-Hilbertian spaces. First of all, we introduce notations and consider a few preliminary results which are useful to prove the main result of the paper. We give sufficient conditions on two m-quasi $(\alpha, \beta) - A$-normal operators defined on a semi-Hilbert space, which make their product and their tensor product m-quasi $(\alpha, \beta) - A$-normal. The inspiration for our investigation comes from [11, 20]. Moreover, we established various inequalities between the A-numerical radius $(\omega_A(\cdot))$ and A-operator norm $(\|\cdot\|_A)$ of m-quasi-$(\alpha, \beta) - A$-normal operators.

2. m-Quasi-$(\alpha, \beta) - A$-Normal Operators

In this section, we define a class of operators in semi-Hilbertian spaces, i.e., spaces generated by positive semidefinite sesquilinear forms. This kind of spaces appears in many problems concerning linear and bounded operators on Hilbert spaces and is intensively studied in the present. The concept of operator theory is semi-Hilbertian spaces have attracted attention. Recently, many extensions of some concepts of Hilbert space operators to semi-Hilbertian space operators have attracted much attention of various authors in several papers [1, 3, 4, 9–18]. In this framework, we are interested to introducing a new concept of quasinormality in semi-Hilbertian spaces known as m-quasi-$(\alpha, \beta) - A$-normal operators. We investigate various structural properties of this class of operators and study some relations about it.

Definition 1. An operator $R \in \mathscr{B}[\mathscr{H}]_A$ is called m-quasi-$(\alpha, \beta) - A$-normal for $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\beta^2 \left(R^\sharp\right)^{m+1} R^{m+1} \geq_A \left(R^\sharp\right)^m \left(RR^\sharp\right) R^m \geq_A \alpha^2 \left(R^\sharp\right)^{m+1} R^{m+1}, \tag{9}$$

for some positive integer m, or equivalently, if

$$\beta\left\|R^{m+1}u\right\|_A \geq \left\|R^{\sharp}R^m u\right\|_A \geq \alpha\left\|R^{m+1}u\right\|_A, \quad \text{for all } u \in \mathscr{H}. \tag{10}$$

Remark 1. The reader will easily see from Definition 1 that

(i) An $(\alpha,\beta)-A$-normal is an m-quasi-$(\alpha,\beta)-A$-normal operator.

(ii) Every m-quasi-$(\alpha,\beta)-A$-normal operator is $(m+1)$-quasi-$(\alpha,\beta)-A$-normal operator.

Remark 2. Notice that the converse of the statement (i) in Remark 1 is not true in general, as shown in the following example:

Example 1. Let us consider $\mathscr{H}=\mathbb{C}^2$ and $A=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ and $R=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$. Direct calculations show that $\text{ran}(R^*A)\subseteq\text{ran}(A)$.

$$\begin{aligned}&\sqrt{3}\left\|R^2 w\right\|_A \geq \left\|R^{\sharp}Rw\right\|_A \geq \left\|R^2 w\right\|_A, \quad \forall w \in \mathbb{C}^2,\\ &\left\|R^{\sharp}w_0\right\|_A \geq \sqrt{3}\left\|Rw_0\right\|_A, \quad \text{for some } w_0 \in \mathbb{C}^2.\end{aligned} \tag{11}$$

This shows that R is a quasi-$(1,\sqrt{3})-A$-normal, but it is not a $(1,\sqrt{3})-A$-normal.

Remark 3. From Example 1, it is obvious that the class of m-quasi-$(\alpha,\beta)-A$-normal operators contained the class of $(\alpha,\beta)-A$-normal operators as a proper subset.

Theorem 1. *Let $R\in\mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta)-A$-normal operator. If R^m has dense range, then R is $(\alpha,\beta)-A$-normal.*

Proof. Since R^m has a dense range, it follows that $\overline{\text{ran}(R^m)}=\mathscr{H}$. Let $u\in\mathscr{H}$, then there exists a sequence (u_n) in $\mathscr{H}$ such that $R^m u_n \longrightarrow u$ as $n\longrightarrow\infty$.

From the fact that R is m-quasi-$(\alpha,\beta)-A$-normal operator, we have

$$\alpha\left\|R^{m+1}w\right\|_A \leq \left\|R^{\sharp}R^m w\right\|_A \leq \beta\left\|R^{m+1}w\right\|_A, \tag{12}$$

for all $w\in\mathscr{H}$.

In particular, we have

$$\alpha\left\|R^{m+1}u_n\right\|_A \leq \left\|R^{\sharp}R^m u_n\right\|_A \leq \beta\left\|R^{m+1}u_n\right\|_A. \tag{13}$$

Consequently,

$$\alpha\|Ru\| \leq \left\|R^{\sharp}u\right\| \leq \beta\|Ru\|_A, \quad \forall u \in \mathscr{H}. \tag{14}$$

Therefore, R is $(\alpha,\beta)-A$-normal operator. □

Corollary 1. *Let $R\in\mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta)-A$-normal operator. If $R^m\neq 0$ and if R has no nontrivial R^m-invariant closed subspace, then R is $(\alpha,\beta)-A$-normal.*

Proof. From the fact that R^m has no nontrivial invariant closed subspace, it follows that R^m has no nontrivial hyperinvariant subspace. But, $\ker(R^m)$ and $\overline{\text{ran}(R^m)}$ are hyperinvariant subspaces, and $R^m\neq 0$, hence $\ker(R^m)=0$ and $\overline{\text{ran}(R^m)}=\mathscr{H}$. Therefore, R is $(\alpha,\beta)-A$-normal operator by [11], Definition 6. □

The following theorem gives a characterization of m-quasi-$(\alpha,\beta)-A$-normal operators.

Theorem 2. *Let $R\in\mathscr{B}_b[\mathscr{H}]_A$, then R is m-quasi-$(\alpha,\beta)-A$-normal operator if and only if*

$$\begin{cases}\lambda^2\left(R^{\sharp}\right)^m RR^{\sharp}R^m + 2\lambda\alpha^2\left(R^{\sharp}\right)^{m+1}R^{m+1} + \left(R^{\sharp}\right)^m RR^{\sharp}R^m \geq_A 0,\\ \lambda^2\left(R^{\sharp}\right)^{m+1}R^{m+1} + 2\lambda\left(R^{\sharp}\right)^m RR^{\sharp}R^m + \beta^4\left(R^{\sharp}\right)^{m+1}R^{m+1} \geq_A 0,\end{cases} \tag{15}$$

for all $\lambda\in\mathbb{R}$.

Proof. By using elementary properties of real quadratic forms, we infer

$$\begin{aligned}&\lambda^2\left(R^{\sharp}\right)^m RR^{\sharp}R^m + 2\lambda\alpha^2\left(R^{\sharp}\right)^{m+1}R^{m+1} + \left(R^{\sharp}\right)^m RR^{\sharp}R^m \geq_A 0\\ &\Leftrightarrow \lambda^2\left\|R^{\sharp}R^m u\right\|_A^2 + 2\lambda\alpha^2\left\|R^{m+1}u\right\|_A^2 + \left\|R^{\sharp}R^m u\right\|_A^2 \geq 0, \quad \forall u \in \mathscr{H} \text{ and } \forall \lambda \in \mathbb{R}\\ &\Leftrightarrow \alpha\left\|R^{m+1}u\right\|_A \leq \left\|R^{\sharp}R^m u\right\|_A, \quad \forall u \in \mathscr{H}.\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}&\lambda^2\left(R^{\sharp}\right)^{m+1}R^{m+1} + 2\lambda\left(R^{\sharp}\right)^m RR^{\sharp}R^m + \beta^4\left(R^{\sharp}\right)^{m+1}R^m \geq_A 0\\ &\Leftrightarrow \lambda^2\left\|R^{m+1}u\right\|_A^2 + 2\lambda\left\|R^{\sharp}R^m u\right\|_A^2 + \beta^4\left\|R^{m+1}u\right\|_A^2 \geq 0, \quad \forall u \in \mathscr{H} \text{ and } \forall \lambda \in \mathbb{R}\\ &\Leftrightarrow \beta\left\|R^m u\right\|_A \geq \left\|R^{\sharp}R^m u\right\|_A, \quad \forall u \in \mathscr{H}.\end{aligned} \tag{17}$$

Therefore, R is the m-quasi-$(\alpha,\beta)-A$-normal operator. □

In the following theorem, we show the relationship between m-quasi-$(\alpha,\beta)-A$-normal operator and it is A-adjoint.

Theorem 3. *Let $(\alpha,\beta)\in\mathbb{R}^2$ such that $0<\alpha\le 1\le\beta$ and let $R\in\mathscr{B}_b[\mathscr{H}]_A$. The following statements hold.*

(1) If R is an m-quasi-$(\alpha,\beta)-A$-normal such that $\operatorname{ran}(R^m)=\operatorname{ran}((R^\sharp)^m)$, then $R^\sharp$ is m-quasi-$((1/\beta),(1/\alpha))-A$-normal.

(2) If $R^\sharp$ is m-quasi-$((1/\beta),(1/\alpha))-A$-normal such that $\operatorname{ran}((R^m)=\operatorname{ran}((R^\sharp)^m)$ and $R(\ker(A)^\perp)\subseteq\ker(A)^\perp$, then R is an m-quasi-$(\alpha,\beta)-A$-normal.

Proof. (1) If R is m-quasi-$(\alpha,\beta)-A$-normal, it follows that

$$\alpha\|RR^m x\|_A\le\|R^\sharp R^m x\|_A\le\beta\|RR^m x\|_A,\quad\forall x\in\mathscr{H}. \tag{18}$$

Taking into account the assumption $\operatorname{ran}(R^m)=\operatorname{ran}((R^\sharp)^m))$, we infer

$$\alpha\|RR^{\sharp m}u\|_A\le\|R^\sharp R^{\sharp m}u\|_A\le\beta\|RR^{\sharp m}u\|_A,\quad\forall u\in\mathscr{H}. \tag{19}$$

Combining these inequalities,

$$\frac{1}{\alpha}\left\|\left(R^\sharp\right)^{m+1}u\right\|_A\ge\|RR^{\sharp m}u\|_A\ge\frac{1}{\beta}\left\|\left(R^\sharp\right)^{m+1}u\right\|_A. \tag{20}$$

So, $R^\sharp$ is m-quasi-$((1/\beta),(1/\alpha))-A$-normal as required.

(2) Since $R^\sharp$ is m-quasi-$(\alpha,\beta)-A$-normal operator, we have for all $w\in\mathscr{H}$,

$$\frac{1}{\beta}\left\|\left(R^\sharp\right)^{m+1}w\right\|_A\le\left\|\left(R^\sharp\right)^\sharp\left(R^\sharp\right)^m w\right\|_A\le\frac{1}{\alpha}\left\|\left(R^\sharp\right)^{m+1}w\right\|_A. \tag{21}$$

After rearranging it further, we get

$$\frac{1}{\beta}\left\|R^\sharp\left(R^\sharp\right)^m w\right\|_A\le\left\|P_{\operatorname{ran}(A)}RP_{\operatorname{ran}(A)}\left(R^\sharp\right)^m w\right\|_A\le\frac{1}{\alpha}\left\|R^\sharp\left(R^\sharp\right)^m w\right\|_A. \tag{22}$$

From the assumption that $R(\ker(A)^\perp)\subseteq\ker(A)^\perp$, we observe that $\ker(A)$ is a reducing subspace of R and it follows that $RP_{\operatorname{ran}(A)}=P_{\operatorname{ran}(A)}R$ and $AP_{\operatorname{ran}(A)}=P_{\operatorname{ran}(A)}A=A$. Thus, we have

$$\frac{1}{\beta}\left\|R^\sharp\left(R^\sharp\right)^m w\right\|_A\le\left\|R\left(R^\sharp\right)^m w\right\|_A\le\frac{1}{\alpha}\left\|R^\sharp\left(R^\sharp\right)^m w\right\|_A. \tag{23}$$

From the assumption that $\operatorname{ran}(R^m)=\operatorname{ran}((R^\sharp)^m)$, we infer

$$\frac{1}{\beta}\left\|R^\sharp R^m u\right\|_A\le\left\|R^{m+1}u\right\|_A\le\frac{1}{\alpha}\left\|R^\sharp R^m w\right\|_A,\quad\forall u\in\mathscr{H}. \tag{24}$$

This gives

$$\alpha\left\|R^{m+1}u\right\|_A\le\left\|R^\sharp R^m u\right\|_A\le\beta\left\|R^{m+1}u\right\|_A, \tag{25}$$

then T is always m-quasi-$(\alpha,\beta)-A$-normal. □

Corollary 2. *Under the same conditions of Theorem 3, if $\alpha\beta=1$, then R is m-quasi-$(\alpha,\beta)-A$-normal if and only if $R^\sharp$ is m-quasi-$(\alpha,\beta)-A$-normal.*

Theorem 4. *Let $R\in\mathscr{B}_b[\mathscr{H}]_A$ be m-quasi-$(\alpha,\beta)-A$-normal such that $\alpha\beta=1$, then*

$$\beta\|R^\sharp R^m u\|_A\ge\|R^{m+1}u\|_A\ge\alpha\|R^\sharp R^m u\|_A,\ \forall\, u\in\mathscr{H}. \tag{26}$$

Proof. By using the m-quasi-$(\alpha,\beta)-A$-normality of R and given condition, we have the following:

R is m-quasi-$(\alpha,\beta)-A$-normal if and only if

$$\beta^2\left(R^\sharp\right)^{m+1}R^{m+1}\ge_A\left(R^\sharp\right)^m RR^\sharp R^m\ge_A\alpha^2\left(R^\sharp\right)^{m+1}R^{m+1}. \tag{27}$$

Therefore,

$$\begin{cases}\alpha^2\beta^2\left(R^\sharp\right)^{m+1}R^{m+1}\ge_A\alpha^2\left(R^\sharp\right)^m RR^\sharp R^m\ge_A\alpha^4\left(R^\sharp\right)^{m+1}R^{m+1},\\ \beta^4\left(R^\sharp\right)^{m+1}R^{m+1}\ge_A\beta^2\left(R^\sharp\right)^m RR^\sharp R^m\ge_A\beta^2\alpha^2\left(R^\sharp\right)^{m+1}R^{m+1}.\end{cases} \tag{28}$$

Combining these inequalities, we get

$$\beta^2\left(R^\sharp\right)^m RR^\sharp R^m\ge_A\left(R^\sharp\right)^{m+1}R^{m+1}\ge_A\alpha^2\left(R^\sharp\right)^m RR^\sharp R^m, \tag{29}$$

and so,

$$\beta^2\langle\left(R^\sharp\right)^m RR^\sharp R^m u,u\rangle_A\ge\langle\left(R^\sharp\right)^{m+1}R^{m+1}u,u\rangle_A\ge\alpha^2\langle\left(R^\sharp\right)^m RR^\sharp R^m u,u\rangle_A. \tag{30}$$

This means that

$$\beta\left\|R^\sharp R^m u\right\|_A\ge\left\|R^{m+1}u\right\|_A\ge\alpha\left\|R^\sharp R^m u\right\|_A,\quad\forall u\in\mathscr{H}. \tag{31}$$

Hence, the result holds. □

Theorem 5. *Let $R,S\in\mathscr{B}_b[\mathscr{H}]_A$ be two m-quasi-$(\alpha,\beta)-A$-normal, then $T\oplus S$ is also m-quasi-$(\alpha,\beta)-A\oplus A$-normal, but tensor product $R\otimes S$ is m-quasi-$(\alpha^2,\beta^2)-(A\otimes A)$- normal.*

Proof. Since R and S are m-quasi-$(\alpha,\beta)-A$-normal, we have

$$\begin{aligned}\beta^2\left(\left((R\oplus S)^{\sharp}\right)^{m+1}\left[(R\oplus S)^{m+1}\right]\right) &= \beta^2\left(R^{\sharp}\right)^{m+1}R^{m+1}\oplus\beta^2\left(S^{\sharp}\right)^{m+1}S^{m+1}\\ &\geq_{A\oplus A} R^{\sharp}R^{m+1}\oplus S^{\sharp}S^{m+1}\\ &= (R\oplus S)^{\sharp}(R\oplus S)^{m+1}.\end{aligned} \tag{32}$$

Moreover,

$$\begin{aligned}(R\oplus S)^{\sharp}(R\oplus S)^{m+1} &= R^{\sharp}R^{m+1}\oplus S^{\sharp}S^{m+1}\\ &\geq_{A\oplus A}\alpha^2\left(R^{\sharp}\right)^{m+1}R^{m+1}\oplus\alpha^2\left(S^{\sharp}\right)^{m+1}S^{m+1}\\ &= \alpha^2\left[(R\oplus S)^{\sharp}\right]^{m+1}\left[(R\oplus S)^{m+1}\right].\end{aligned} \tag{33}$$

Therefore,

$$\beta^2\left(\left((R\oplus S)^{\sharp}\right)^{m+1}\left((R\oplus S)^{m+1}\right)\right)\geq_{A\oplus A}(R\oplus S)^{\sharp}(R\oplus S)^{m+1}\geq_{A\oplus A}\alpha^2\left(\left((R\oplus S)^{\sharp}\right)^{m+1}(R+S)^{m+1}\right). \tag{34}$$

Thus, $R\oplus S$ is m-quasi-$(\alpha,\beta)-A\oplus A$-normal operator.

To prove that $R\otimes S$ is m-quasi-$(\alpha^2,\beta^2)-A\otimes A$-normal operator, we observe that

$$\begin{aligned}\beta^4(A\otimes A)\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right) &= A\left(R^{\sharp}\right)^{m+1}R^{m+1}\otimes\beta^2A\left(S^{\sharp}\right)^{m+1}S^{m+1}\\ &\geq AR^{\sharp}R^{m+1}\otimes AS^{\sharp}S^{m+1}\\ &= (A\otimes A)(R\otimes S)^{\sharp}(R\otimes S)^{m+1}.\end{aligned} \tag{35}$$

We get

$$\beta^4\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right)\geq_{A\otimes A}(R\otimes S)^{\sharp}(R\otimes S)^{m+1}. \tag{36}$$

Similarly,

$$\begin{aligned}(A\otimes A)(R\otimes S)^{\sharp}(R\otimes S)^{m+1} &= AR^{\sharp}R^{m+1}\otimes AS^{\sharp}S^{m+1}\\ &\geq\alpha^2A\left(R^{\sharp}\right)^{m+1}R^{m+1}\otimes\alpha^2A\left(S^{\sharp}\right)^{m+1}S^{m+1}\\ &= \alpha^4(A\otimes A)\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right).\end{aligned} \tag{37}$$

Therefore,

$$(R\otimes S)^{\sharp}(R\otimes S)^{m+1}\geq_{A\otimes A}\alpha^4(A\otimes A)\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right). \tag{38}$$

Combining (36) and (38), we get

$$\beta^4\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right)\geq_{A\otimes A}(R\otimes S)^{\sharp}(R\otimes S)^{m+1}\geq_{A\otimes A}\alpha^4\left(\left((R\otimes S)^{\sharp}\right)^{m+1}(R\otimes S)^{m+1}\right). \tag{39}$$

□

Theorem 6. *The set*

$$\Gamma(\alpha,\beta)=\left\{R\in\mathcal{B}_b[\mathcal{H}]_A:\ \beta^2\left(R^{\sharp}\right)^{m+1}R^{m+1}\geq_A\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}\geq_A\alpha^2\left(R^{\sharp}\right)^{m+1}R^{m+1}\right\}, \tag{40}$$

and $(0 \le \alpha \le 1 \le \beta)\}$ *is Arcwise connected for* $m \in \mathbb{N}$.

Proof. Let $R \in \Gamma(\alpha,\beta)$ and $\lambda \in \mathbb{C}, \lambda \neq 0$. We need to prove that λR is m-quasi-$(\alpha,\beta) - A$-normal. Now, for $u \in \mathscr{H}$,

$$\begin{aligned}\beta^2\langle\left((\lambda R)^{\sharp}\right)^{m+1}(\lambda R)^{m+1}u,u\rangle_A &= |\lambda|^{2(m+1)}\beta^2\langle\left(R^{\sharp}\right)^{m+1}R^{m+1}u,u\rangle_A\\ &\ge |\lambda|^{2(m+1)}\langle\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}u,u\rangle_A\\ &= |\lambda|^{m+1}|\bar{\lambda}|^{m+1}\langle\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}u,u\rangle_A\\ &= \langle\left((\lambda R)^{\sharp}\right)^{m}(\lambda R)(\lambda R)^{\sharp}(\lambda R)^{m}u,u\rangle_A.\end{aligned} \tag{41}$$

Otherwise,

$$\begin{aligned}\langle\left((\lambda R)^{\sharp}\right)^{m}(\lambda R)(\lambda R)^{\sharp}(\lambda R)^{m}u,u\rangle_A &= |\lambda|^{m+1}|\bar{\lambda}|^{m+1}\langle\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}u,u\rangle_A\\ &\ge |\lambda|^{m+1}|\bar{\lambda}|^{m+1}\alpha^2\langle\left(R^{\sharp}\right)^{m+1}R^{m+1}u,u\rangle_A|\\ &= \alpha^2\langle\left((\lambda R)^{\sharp}\right)^{m+1}(\lambda R)^{m+1}u,u\rangle_A.\end{aligned} \tag{42}$$

Hence,

$$\beta\left\|(\lambda R)^{m+1}\right\|_A \ge \left\|(\lambda R)^{\sharp}(\lambda R)^{m}\right\|_A \ge \alpha\left\|(\lambda R)^{m+1}\right\|_A. \tag{43}$$

This means that the class of m-quasi-$(\alpha,\beta) - A$-normal operator is Arcwise connected. □

Proposition 1. *Let* $R, V \in \mathscr{B}_b[\mathscr{H}]_A$ *such that* $R(\ker(A)^{\perp}) \subseteq \ker(A)^{\perp}$ *and* $V(\ker(A)^{\perp}) \subseteq \ker(A)^{\perp}$. *If* R *is* m*-quasi-* $(\alpha,\beta) - A$*-normal for* $(0 \le \alpha \le 1 \le \beta)$ *and* V *is an* A*-isometry, then* $VRV^{\sharp}$ *is* m*-quasi-*$(\alpha,\beta) - A$*-normal operator.*

Proof. In view of Definition 1 and the fact that V is an A-isometry ([1], Proposition 3.6), we have

$$\beta^2\left(R^{\sharp}\right)^{m+1}R^{m+1} \ge_A \left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m} \ge_A \alpha^2\left(R^{\sharp}\right)^{m+1}R^{m+1} \text{ and } V^{\sharp}V = P_{\text{ran}(A)}. \tag{44}$$

Moreover, from the fact that $\ker(A)$ is a reducing subspace for both R and V, it follows that

$$\begin{aligned}P_{\text{ran}(A)}R &= RP_{\text{ran}(B)},\\ R^{\sharp}P_{\text{ran}(A)} &= P_{\text{ran}(A)}R^{\sharp},\\ VP_{\text{ran}(A)} &= P_{\text{ran}(A)}V,\\ V^{\sharp}P_{\text{ran}(A)} &= P_{\text{ran}(A)}V^{\sharp}.\end{aligned} \tag{45}$$

Direct calculations show that

$$\begin{aligned}\left(VRV^{\sharp}\right)^{k} &= \underbrace{\left(VRV^{\sharp}\right)\left(VRV^{\sharp}\right)\cdots\left(VRV^{\sharp}\right)}_{k\text{-times}}\\ &= \left(VRP_{\text{ran}(B)}RV^{\sharp}\right)\cdots\left(VRV^{\sharp}\right)\\ &= P_{\text{ran}(A)}VR^2V^{\sharp}\cdots\left(VRV^{\sharp}\right)\\ &= \vdots\\ &= P_{\text{ran}(A)}VR^{k}V^{\sharp}.\end{aligned} \tag{46}$$

This implies

$$\begin{aligned}\beta^2\left(\left(VRV^{\sharp}\right)^{\sharp}\right)^{m+1}\left(VRV^{\sharp}\right)^{m+1} &= \beta^2\left(\left(V^{\sharp}\right)^{\sharp}R^{\sharp}V^{\sharp}\right)^{m+1}\left(VRV^{\sharp}\right)^{m+1}\\ &= \left(P_{\text{ran}(A)}VP_{\text{ran}(A)}R^{\sharp}V^{\sharp}\right)^{m+1}\left(VRV^{\sharp}\right)^{m+1} = \beta^2\left(P_{\text{ran}(A)}V\left(R^{\sharp}\right)^{m+1}V^{\sharp}P_{\text{ran}(A)}VR^{m+1}V^{\sharp}\right)\\ &= \beta^2\left(VP_{\text{ran}(A)}\left(R^{\sharp}\right)^{m+1}R^{m+1}\left(VP_{\text{ran}(A)}\right)^{\sharp}\right) \ge_A VP_{\text{ran}(A)}\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}\left(VP_{\overline{\mathscr{R}(A)}}\right)^{\sharp} \text{ (by [[11], Lemma 1])}\\ &= \left(VRV^{\sharp}\right)^{\sharp m}\left(VRV^{\sharp}\right)\left(VRV^{\sharp}\right)^{\sharp}\left(VRV^{\sharp}\right)^{m}.\end{aligned} \tag{47}$$

Similarly, we have

$$\left(VRV^\sharp\right)^{\sharp m}\left(VRV^\sharp\right)\left(VRV^\sharp\right)^\sharp\left(VRV^\sharp\right)^m = VP_{\mathrm{ran}(A)}\left(R^\sharp\right)^m RR^\sharp R^m\left(VP_{\overline{\mathscr{R}(A)}}\right)^\sharp \geq \alpha^2 VP_{\mathrm{ran}(A)}\left(R^\sharp\right)^{m+1}R^{m+1}\left(VP_{\overline{\mathscr{R}(A)}}\right)^\sharp \geq \alpha^2\left(\left(VRV^\sharp\right)^\sharp\right)^{m+1}\left(VRV^\sharp\right)^{m+1}. \tag{48}$$

The conclusion holds. □

Proposition 2. *Let $R, S \in \mathscr{B}_b[\mathscr{H}]_A$ be commuting and such that R is m-quasi-$(\alpha,\beta)-A$-normal operator. The following statements hold.*

(1) If S is A-self-adjoint and $R^\sharp S = SR^\sharp$, then RS is m-quasi-$(\alpha,\beta)-A$-normal.

(2) If S is A-unitary, $R^\sharp S = SR^\sharp$ and $\ker(A)$ is invariant subspace for R, then RS is m-quasi-$(\alpha,\beta)-A$-normal.

Proof

(1) Since R is m-quasi-$(\alpha,\beta)-A$-normal, S is A-normal and $R^\sharp S = SR^\sharp$, it follows that for $u \in \mathscr{H}$,

$$\alpha\left\|(RS)^{m+1}u\right\|_A = \alpha\left\|R^{m+1}S^{m+1}u\right\|_A \leq \left\|R^\sharp R^m S^{m+1}u\right\|_A = \left\|SR^\sharp R^m S^m u\right\|_A = \left\|S^\sharp R^\sharp R^m S^m u\right\|_A = \left\|(RS)^\sharp (RS)^m u\right\|_A. \tag{49}$$

On the other hand,

$$\left\|(RS)^\sharp (RS)^m u\right\|_A = \left\|S^\sharp R^\sharp R^m S^m u\right\|_A = \left\|SR^\sharp R^m S^m u\right\|_A = \left\|R^\sharp R^m S^{m+1}u\right\|_A \leq \beta\left\|R^{m+1}S^{m+1}u\right\|_A = \beta\left\|(RS)^{m+1}u\right\|_A. \tag{50}$$

Consequently, RS is the m-quasi-$(\alpha,\beta)-A$-normal operator.

(2) Since $\ker(A)$ is invariant subspace for R, we observe that $RP_{\mathrm{ran}(A)} = P_{\mathrm{ran}(A)}R$ and $R^\sharp P_{\mathrm{ran}(A)} = P_{\mathrm{ran}(A)}R^\sharp$. In view of the fact that S is A-unitary, we have $S^\sharp S = SS^\sharp = P_{\mathrm{ran}(A)}$.

Now, direct calculations give

$$\beta^2\left(\left((RS)^\sharp\right)^{m+1}(RS)^{m+1}\right) = \beta^2\left(\left(R^\sharp\right)^{m+1}\left(S^\sharp\right)^{m+1}S^{m+1}R^{m+1}\right) = \beta^2\left(\left(R^\sharp\right)^{m+1}P_{\mathrm{ran}(A)}R^{m+1}\right) = \beta^2\left(P_{\mathrm{ran}(A)}\left(R^\sharp\right)^{m+1}R^{m+1}P_{\mathrm{ran}(A)}\right). \tag{51}$$

Since R is m-quasi-$(\alpha,\beta)-A$-normal, it follows immediately from [11], Lemma 1 that

$$\beta^2\left(\left((RS)^\sharp\right)^{m+1}(RS)^{m+1}\right) \geq_A \underbrace{\left(P_{\mathrm{ran}(A)}\left(R^\sharp\right)^m RR^\sharp R^m P_{\mathrm{ran}(A)}\right)}_{(1)} \geq_A \alpha^2\underbrace{\left(P_{\mathrm{ran}(A)}\left(R^\sharp\right)^{m+1}R^{m+1}P_{\mathrm{ran}(A)}\right)}_{(2)}. \tag{52}$$

From the quantity in (1), we get

$$\left(P_{\mathrm{ran}(A)}\left(R^\sharp\right)^m RR^\sharp R^{m+1}P_{\mathrm{ran}(A)} = R^m P_{\mathrm{ran}(A)}RR^\sharp R^m P_{\mathrm{ran}(A)} = \left((RS)^\sharp\right)^m (RS)(RS)^\sharp (RS)^m. \tag{53}$$

Similarly by (2), we have

$$\left(P_{\operatorname{ran}(A)}\left(R^{\sharp}\right)^{m+1}R^{m+1}P_{\operatorname{ran}(A)}=\left(R^{\sharp}\right)^{m+1}\left(S^{\sharp}\right)^{m+1}S^{m+1}R^{m+1}=\left((RS)^{\sharp}\right)^{m+1}(RS)^{m+1}. \tag{54}$$

This means that

$$\beta^2\left((RS)^{\sharp}\right)^{m+1}(RS)^{m+1}\geq_A\left((RS)^{\sharp}\right)^{m}(RS)(RS)^{\sharp}(RS)^{m}\alpha^2\left((RS)^{\sharp}\right)^{m+1}(RS)^{m+1}. \tag{55}$$

Therefore, RS is m-quasi-$(\alpha,\beta)-A$-normal operator. □

Theorem 7. *Let $R,S\in\mathscr{B}_b[\mathscr{H}]_A$ be commuting and $R^{\sharp}S=SR^{\sharp}$. If R is an m-quasi-$(\alpha,\beta)-A$-normal and S is an m-quasi-$(\alpha',\beta')-A$-normal for $0\leq\alpha,\alpha'\leq 1\leq\beta,\beta'$, then RS is an m-quasi-$(\alpha\alpha',\beta\beta')-A$-normal.*

Proof. By using the m-quasi-$(\alpha,\beta)-A$-normality of R and m-quasi-$(\alpha',\beta')-A$-normality of S and given condition, we have

$$\alpha\alpha'\left\|(RS)^{m+1}u\right\|_A=\alpha\alpha'\left\|R^{m+1}S^{m+1}u\right\|_A\leq\alpha'\left\|R^{\sharp}R^{m}S^{m+1}u\right\|_A\leq\left\|S^{\sharp}R^{\sharp}R^{m}S^{m}u\right\|_A,\quad\forall u\in\mathscr{H}. \tag{56}$$

Otherwise,

$$\begin{aligned}\left\|S^{\sharp}S^{m}R^{\sharp}R^{m}u\right\|_A&\leq\beta'\left\|SS^{m}R^{\sharp}R^{m}u\right\|_A=\beta'\left\|R^{\sharp}R^{m}SS^{m}u\right\|_A\leq\beta\beta'\left\|R^{m+1}S^{m+1}u\right\|_A,\\ \alpha\alpha'\left\|(RS)^{m+1}u\right\|_A&\leq\left\|(SR)^{\sharp}T^{m}S^{m}u\right\|_A\leq\beta\beta'\left\|{}_AT^{m+1}S^{m+1}u\right\|_A,\quad\forall u\in\mathscr{H}.\end{aligned} \tag{57}$$

□

Hence, the desired result is obtained.

3. Inequalities Involving A-Numerical Radius and Operator A-Norm

In this section, we are interested to study some inequalities concerning the A-numerical radius and A-norm of m-quasi-$(\alpha,\beta)-A$-normal operators. It is followed by several inequalities, refer to [21–29]. The authors in [20] have given various inequalities between the operator norm and the numerical radius of m-quasi-(α,β)-normal operators in Hilbert spaces. Motivated by this work, we will extend some of these inequalities to A-operator norm and A-numerical radius for m-quasi-$(\alpha,\beta)-A$-normal operators in semi-Hilbertian spaces by using some known results for vectors in inner product spaces.

Definition 2 (see [17]). The A-numerical radius of an operator $R\in\mathscr{B}_b[\mathscr{H}]$ is defined by

$$\omega_A(R)=\sup\left\{\left|\langle Ru,u\rangle_A\right|,\quad u\in\mathscr{H}\colon\|u\|_A=1\right\}. \tag{58}$$

Theorem 8. *If $R\in\mathscr{B}_b[\mathscr{H}]_A$ is m-quasi-$(\alpha,\beta)-A$ normal operator, then the following inequality holds:*

$$\left(1+\left(\frac{\alpha}{\beta}\right)^{2s}\right)\left\|R^{m+1}\right\|_A^2\leq\begin{cases}\dfrac{2}{\beta}\omega_A\left(\left(R^{\sharp}\right)^{m}R^{m+2}\right)+\dfrac{s^2}{\beta^2}\left\|\beta R^{m+1}-R^{\sharp}R^{m}\right\|_A^2, & \text{if } s\geq 1,\\[2ex] \dfrac{2}{\beta}\omega_A\left(\left(R^{\sharp}\right)^{m}R^{m+2}\right)+\dfrac{1}{\beta^2}\left\|\beta R^{m+1}-R^{\sharp}R^{m}\right\|_A^2, & \text{if } s<1.\end{cases} \tag{59}$$

Proof. From [28], we have inspired the following inequalities:

$$\|w\|^{2s}+\|v\|^{2s}-2\|w\|^{s-1}\|v\|^{s-1}\mathrm{Re}\langle w,v\rangle\le\begin{cases}s^2\|w\|^{2s-2}\|w-v\|^2, & \text{if } s\ge 1,\\ \|v\|^{2s-2}\|w-v\|^2, & \text{if } s<1,\end{cases}\tag{60}$$

for $s\in\mathbb{R}$ and $w,v\in\mathscr{H}$ with $\|w\|\ge\|v\|$.

By replacing w by $A^{1/2}w$ and v by $A^{1/2}v$ for $w,v\notin\ker(A^{1/2})$, we get

$$\left\|A^{1/2}w\right\|^{2s}+\left\|A^{1/2}v\right\|^{2s}-2\left\|A^{1/2}w\right\|^{s-1}\left\|A^{1/2}v\right\|^{s-1}\mathrm{Re}\langle A^{1/2}w,A^{1/2}v\rangle\le\begin{cases}s^2\left\|A^{1/2}w\right\|^{2s-2}\left\|A^{1/2}w-A^{1/2}v\right\|^2, & \text{if } s\ge 1,\\ \left\|A^{1/2}v\right\|^{2s-2}\left\|A^{1/2}w-A^{1/2}v\right\|^2, & \text{if } s<1,\end{cases}\tag{61}$$

for $s\in\mathbb{R}$ and $w,v\in\mathscr{H}$ with $w,v\notin\ker(A)$ and $\|A^{1/2}w\|\ge\|A^{1/2}v\|$.

This means

$$\|w\|_A^{2s}+\|v\|_A^{2s}-2\|w\|_A^{s-1}\|v\|_A^{s-1}\mathrm{Re}\langle w,v\rangle_A\le\begin{cases}s^2\|w\|_A^{2s-2}\|w-v\|_A^2, & \text{if } s\ge 1,\\ \|v\|_A^{2s-2}\|w-v\|_A^2, & \text{if } s<1,\end{cases}\tag{62}$$

for $s\in\mathbb{R}$ and $w,v\in\mathscr{H}$ with $w,v\notin\ker(A)$ and $\|w\|_A\ge\|v\|_A$.

If we take $w=\beta R^{m+1}u, v=R^{\sharp}R^m u$, we obtain the following inequality:

$$\begin{aligned}\left\|\beta R^{m+1}w\right\|_A^{2s}+\left\|R^{\sharp}R^m w\right\|_A^{2s}&\le s^2\left\|\beta R^{m+1}w\right\|_A^{2s-2}\left\|\beta R^{m+1}w-R^{\sharp}R^m w\right\|_A^2\\&\quad+2\left\|\beta R^{m+1}w\right\|_A^{s-1}\left\|R^{\sharp}R^m w\right\|_A^{s-1}\mathrm{Re}\langle\beta R^{m+1}w,R^{\sharp}R^m w\rangle_A,\end{aligned}\tag{63}$$

for any $w\in\mathscr{H}$ with $\|w\|_A=1$ and $s\ge 1$.

Hence,

$$\begin{aligned}\left(\alpha^{2s}+\beta^{2s}\right)\left\|R^{m+1}w\right\|_A^{2s}&=\alpha^{2s}\left\|R^{m+1}w\right\|_A^{2s}+\beta^{2s}\left\|R^{m+1}w\right\|_A^{2s}\\&\le\left\|R^{\sharp}R^m w\right\|_A^{2s}+\beta^{2s}\left\|R^{m+1}w\right\|_A^{2s}\\&\le s^2\beta^{2s-2}\left\|R^{m+1}w\right\|_A^{2s-2}\left\|\beta R^{m+1}w-R^{\sharp}R^m w\right\|_A^2\\&\quad+2\beta^{2s-1}\left\|R^{m+1}w\right\|_A^{2s-2}\left|\langle\left(R^{\sharp}\right)^m R^{m+2}w,w\rangle_A\right|.\end{aligned}\tag{64}$$

Applying sup over $w\in\mathscr{H}$, $\|w\|_A=1$ of both sides of the above inequality, we infer

$$\begin{aligned}\left(\alpha^{2s}+\beta^{2s}\right)\left\|R^{m+1}\right\|_A^{2s}&\le s^2\beta^{2s-2}\left\|R^{m+1}\right\|_A^{2s-2}\left\|\beta R^{m+1}-R^{\sharp}R^m\right\|_A^2\\&\quad+2\beta^{2s-1}\left\|R^{m+1}\right\|_A^{2s-2}\omega_A\left(\left(R^{\sharp}\right)^m R^{m+2}\right).\end{aligned}\tag{65}$$

This proves the first inequality.

Analogously, by considering similar techniques, we get the second inequality for $s<1$. □

Theorem 9. *If $R\in\mathscr{B}_b[\mathscr{H}]_A$ is m-quasi-$(\alpha,\beta)-A$- normal operator and if $\lambda\in\mathbb{C}$, then*

$$\alpha\left\|R^{m+1}\right\|_A^2\le\omega_A\left(\left(R^{\sharp}\right)^m R^{m+2}\right)+\frac{2\beta}{(1+|\lambda|\alpha)^2}\left\|R^{m+1}-\lambda R^{\sharp}R^m\right\|_A^2.\tag{66}$$

Proof. We give our proof depending on the following inequality inspired from [27],

$$\|w\|\|v\|\le|\langle w,v\rangle|+\frac{2\|w\|\|v\|\|w-v\|^2}{(\|w\|+\|v\|)^2},\tag{67}$$

for $w,v\in\mathscr{H}\backslash\{0\}$.

By choosing $w=A^{1/2}R^{m+1}u$ and $v=\lambda A^{1/2}R^{\sharp}R^m u$, we infer

$$\left\|R^{m+1}u\right\|_A\left\|\lambda R^{\sharp}R^m u\right\|_A\le\left|\langle R^{m+1}u,\lambda R^{\sharp}R^m u\rangle_A\right|+\frac{2\left\|R^{m+1}u\right\|_A\left\|\lambda R^{\sharp}R^m u\right\|_A\left\|R^{m+1}u-\lambda R^{\sharp}R^m u\right\|_A^2}{\left(\left\|R^{m+1}u\right\|_A+\left\|\lambda R^{\sharp}R^m u\right\|_A\right)^2}.\tag{68}$$

This gives

$$\alpha\|R^{m+1}u\|_A^2 \leq \left|\langle R^{m\sharp}R^2R^m u, u\rangle_A\right| + \frac{2\beta}{(1+|\lambda|\alpha)^2}\|R^{m+1}u - \lambda R^\sharp R^m u\|_A^2. \quad (69)$$

Taking the sup of both sides of the last inequality for $\|u\|_A = 1$, we infer the desired result. □

Theorem 10. *If $R \in \mathscr{B}_b[\mathscr{H}]_A$ is m-quasi-$(\alpha,\beta) - A$-normal operator and if $\lambda \in \mathbb{C}\backslash\{0\}$, then*

$$\left[\alpha^2 - \left(\frac{1}{|\lambda|} + \beta\right)^2\right]\|R^{m+1}\|_A^4 \leq \omega_A\left(R^{m\sharp}R^{m+2}\right)^2. \quad (70)$$

Proof. We give our proof depending on the following inequality inspired from [24]:

$$\|w\|^2\|v\|^2 \leq |\langle w, v\rangle|^2 + \frac{1}{|\lambda|^2}\|w\|^2\|w - \lambda v\|^2, \quad (71)$$

provided $w, v \in \mathscr{H}$ and $\lambda \in \mathbb{C}\backslash\{0\}$. By choosing $w = A^{1/2}R^{m+1}u$, $v = A^{1/2}R^\sharp R^m u$, we infer

$$\|R^{m+1}u\|_A^2\|R^\sharp R^m u\|_A^2 \leq \left|\langle R^{m+1}u, R^\sharp R^m u\rangle_A\right|^2 + \frac{1}{|\lambda|^2}\|R^{m+1}u\|_A^2\|R^{m+1}u - \lambda R^\sharp R^m u\|_A^2. \quad (72)$$

This gives

$$\alpha^2\|R^{m+1}u\|_A^4 - \frac{1}{|\lambda|^2}\|R^{m+1}u\|_A^4(1+|\lambda|\beta)^2 \leq \left|\langle\left(R^\sharp\right)^m R^2R^m u, u\rangle_A\right|^2. \quad (73)$$

Taking the sup of both sides of the last inequality for $\|u\|_A = 1$, the desired result will be obtained. □

Theorem 11. *If $R \in \mathscr{B}_b[\mathscr{H}]_A$ is m-quasi-$(\alpha,\beta) - A$-normal operator, then*

$$2\omega_A\left(R^{m+1}\right)\omega_A\left(R^\sharp R^m\right) \leq \beta\|R^{m+1}\|_A^2 + \omega_A\left(\left(R^\sharp\right)^m R^{m+2}\right). \quad (74)$$

Proof. In the following inequality [21],

$$2|\langle w, u\rangle\langle u, v\rangle| \leq \|w\|\|v\| + |\langle w, v\rangle|, \quad (75)$$

where $w, v, u \in \mathscr{H}$ and $\|u\| = 1$.

Choosing $u = A^{1/2}x$, $w = A^{1/2}R^{m+1}x$, and $v = A^{1/2}R^\sharp R^m x$ with $\|x\|_A = 1$, it follows

$$2\left|\langle A^{1/2}R^{m+1}x, x\rangle\langle x, A^{1/2}R^\sharp R^m x\rangle\right| \leq \|R^{m+1}x\|_A\|R^\sharp R^m x\|_A + \left|\langle R^{m+1}x, R^\sharp R^m x\rangle_A\right|. \quad (76)$$

Hence,

$$2\left|\langle R^{m+1}x, x\rangle_A\langle\left(R^\sharp\right)^m Rx, x\rangle_A\right| \leq \beta\|R^{m+1}\|_A^2 + \left|\langle\left(R^\sharp\right)^m R^{m+2}x, x\rangle_A\right|. \quad (77)$$

Taking the sup of both sides of the last inequality for $\|x\|_A = 1$, the required inequality will be obtained. □

Theorem 12. *If $R \in \mathscr{B}_b[\mathscr{H}]_A$ is m-quasi-$(\alpha,\beta) - A$-normal operator and if $p \geq 2$, then*

$$(1+\alpha^p)\|R^{m+1}\|_A^p \leq \frac{1}{2}\left(\|R^{m+1} + R^\sharp R^m\|_A^p + \|R^{m+1} - R^\sharp R^m\|_A^p\right). \quad (78)$$

Proof. We use the following inequality [26]:

$$\|w\|^p + \|v\|^p \le \frac{1}{2}\left(\|w+v\|^p + \|w-v\|^p\right), \tag{79}$$

for any $w, v \in \mathscr{H}$ and $p \ge 2$.

Now, if we choose $w = A^{1/2}R^{m+1}u$, $v = A^{1/2}R^{\sharp}R^{m}u$, we get

$$\left\|A^{1/2}R^{m+1}u\right\|^p + \left\|A^{1/2}R^{\sharp}R^{m}u\right\|^p \le \frac{1}{2}\left(\left\|A^{1/2}R^{m+1}u + A^{1/2}R^{\sharp}R^{m}u\right\|^p + \left\|A^{1/2}R^{m+1}u - A^{1/2}R^{\sharp}R^{m}u\right\|^p\right). \tag{80}$$

That means

$$\left\|R^{m+1}u\right\|_A^p + \left\|R^{\sharp}R^{m}u\right\|_A^p \le \frac{1}{2}\left(\left\|R^{m+1}u + R^{\sharp}R^{m}u\right\|_A^p + \left\|R^{m+1}u - R^{\sharp}R^{m}u\right\|_A^p\right). \tag{81}$$

Hence, since R is m-quasi-$(\alpha,\beta) - A$ normal operator, we have

$$\left(1+\alpha^p\right)\left\|R^{m+1}u\right\|_A^p \le \left\|R^{m+1}u\right\|_A^p + \left\|R^{\sharp}R^{m}u\right\|_A^p \le \frac{1}{2}\left(\left\|R^{m+1}u + R^{\sharp}R^{m}u\right\|_A^p + \left\|R^{m+1}u - R^{\sharp}R^{m}u\right\|_A^p\right). \tag{82}$$

Taking the sup over all $u \in \mathscr{H}$ with $\|u\|_A = 1$ in the above inequality, we get

$$\left(1+\alpha^p\right)\left\|R^{m+1}\right\|_A^p \le \frac{1}{2}\left(\left\|R^{m+1} + R^{\sharp}R^{m}\right\|_A^p + \left\|R^{m+1} - R^{\sharp}R^{m}\right\|_A^p\right). \tag{83}$$

□

Theorem 13. *Let $R \in \mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta) - A$-normal operator and $p \ge 2$. Then,*

$$\omega_A\left(\frac{\left(R^{\sharp}\right)^{m+1}R^{m+1} + \left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}}{2}\right)^{p/2} \le \frac{1+\beta^p}{4\left(1+\alpha^p\right)}\left(\left\|R^{m+1} + R^{\sharp}R^{m}\right\|_A^p + \left\|R^{m+1} - R^{\sharp}R^{m}\right\|_A^p\right). \tag{84}$$

Proof. Using the following elementary inequality,

$$2^{1-q}(a+b)^q \le a^q + b^q, \tag{85}$$

for $a, b \ge 0$ and $q \ge 1$.

For any $w \in \mathscr{H}$ with $\|w\|_A = 1$, take $a = \|A^{1/2}R^{m+1}w\|^2$, $b = \|A^{1/2}R^{\sharp}R^{m}w\|^2$, and $q = (p/2)$ in the above inequality to get

$$2^{1-(p/2)}\left(\left\|A^{1/2}R^{m+1}w\right\|^2 + \left\|A^{1/2}R^{\sharp}R^{m}w\right\|^2\right)^{p/2} \le \left(\left\|A^{1/2}R^{m+1}w\right\|^2\right)^{p/2} + \left(\left\|A^{1/2}R^{\sharp}R^{m}w\right\|^2\right)^{p/2}. \tag{86}$$

Then,

$$2^{1-(p/2)}\left(\left\|R^{m+1}w\right\|_A^2 + \left\|R^{\sharp}R^{m}w\right\|_A^2\right)^{p/2} \le \left(\left\|R^{m+1}w\right\|_A^2\right)^{p/2} + \left(\left\|R^{\sharp}R^{m}w\right\|_A^2\right)^{p/2}, \tag{87}$$

whence

$$\left(\frac{\left\|R^{m+1}w\right\|^{2}+\left\|R^{\sharp}R^{m}w\right\|_{A}^{2}}{2}\right)^{p/2}\leq\frac{1}{2}\left\|R^{m+1}w\right\|_{A}^{p}+\left\|R^{\sharp}R^{m}w\right\|_{A}^{p}. \tag{88}$$

Since R is m-quasi-$(\alpha,\beta)-A$-normal, we have

$$\left\|R^{m+1}w\right\|_{A}^{p}+\left\|R^{\sharp}R^{m}w\right\|_{A}^{p}\leq\left(1+\beta^{p}\right)\left\|R^{m+1}w\right\|_{A}^{p}. \tag{89}$$

Hence, by Theorem 12, we obtain

$$\left(\frac{\left\|R^{m+1}w\right\|^{2}+\left\|R^{\sharp}R^{m}w\right\|_{A}^{2}}{2}\right)^{p/2}\leq\frac{1+\beta^{p}}{4\left(1+\alpha^{p}\right)}\left(\left\|R^{m+1}+R^{\sharp}R^{m}\right\|_{A}^{p}+\left\|R^{m+1}-R^{\sharp}R^{m}\right\|_{A}^{p}\right)\|w\|_{A}. \tag{90}$$

But,

$$\left|\left\langle\left(\frac{\left(R^{\sharp}\right)^{m+1}R^{m+1}+\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}}{2}\right)w,w\right\rangle_{A}\right|^{p/2}=\left(\frac{\left\|R^{m+1}w\right\|^{2}+\left\|R^{\sharp}R^{m}w\right\|_{A}^{2}}{2}\right)^{p/2}. \tag{91}$$

Therefore,

$$\left|\left\langle\left(\frac{\left(R^{\sharp}\right)^{m+1}R^{m+1}+\left(R^{\sharp}\right)^{m}RR^{\sharp}R^{m}}{2}\right)w,w\right\rangle_{A}\right|^{p/2}\leq\frac{\left(1+\beta^{p}\right)\|w\|_{A}}{4\left(1+\alpha^{p}\right)}\left(\left\|R^{m+1}+R^{\sharp}R^{m}\right\|_{A}^{p}+\left\|R^{m+1}-R^{\sharp}R^{m}\right\|_{A}^{p}\right). \tag{92}$$

Taking the sup of both sides of the last inequality for $\|w\|_A=1$, we infer the desired result. □

Theorem 14. *Let $R\in\mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta)-A$-normal operator. If $p\in\mathbb{R}$ with $1<p<2$ and $\max\{|\lambda|-|\mu|\beta,\alpha|\mu|-|\lambda|\}\geq 0$, then*

$$\begin{aligned}&\left(\left(|\lambda|+\alpha|\mu|\right)^{p}\left\|R^{m+1}\right\|_{A}^{p}+\left(\max\{|\lambda|-|\mu|\beta,\alpha|\mu|-|\lambda|\}\right)^{p}\left\|R^{m+1}\right\|_{A}^{p}\\&\quad\leq\left\|\lambda R^{m+1}+\mu R^{\sharp}R^{m}\right\|_{A}^{p}+\left\|\lambda R^{m+1}-\mu R^{\sharp}R^{m}\right\|_{A}^{p}.\end{aligned} \tag{93}$$

Proof. We use the following inequality inspired from [26],

$$\left(\|w\|+\|v\|\right)^{p}+\left|\|w\|-\|v\|\right|^{p}\leq\|w+v\|^{p}+\|w-v\|^{p}, \tag{94}$$

for any $w,v\in\mathscr{H}$ and $p\in\mathbb{R}$: $1<p<2$.

Put $w=\lambda A^{1/2}R^{m+1}u$ and $v=\mu A^{1/2}R^{\sharp}R^{m}u$ for $u\in\mathscr{H}$ to get

$$\begin{aligned}&\left(\left\|\lambda R^{m+1}u\right\|_{A}+\left\|\mu R^{\sharp}R^{m}u\right\|_{A}\right)^{p}+\left|\left\|\lambda R^{m+1}u\right\|_{A}-\left\|\mu R^{m+1}u\right\|_{A}\right|^{p}\\&\quad\leq\left\|\lambda R^{m+1}u+\mu R^{\sharp}R^{m}u\right\|_{A}^{p}+\left\|\lambda R^{m+1}u-\mu R^{\sharp}R^{m}u\right\|_{A}^{p}.\end{aligned} \tag{95}$$

Since R is m-quasi-$(\alpha,\beta)-A$-normal operator, it follows that

$$\begin{aligned}&\left(|\lambda|+\alpha|\mu|\right)^{p}\left\|R^{m+1}u\right\|_{A}^{p}\leq\left(\left\|\lambda R^{m+1}u\right\|_{A}+\left\|\mu R^{\sharp}R^{m}u\right\|_{A}\right)^{p},\\&\left(|\lambda|-|\mu|\beta\right)\left\|R^{m+1}u\right\|_{A}\leq\left\|\lambda R^{m+1}u\right\|_{A}-\left\|\mu R^{\sharp}R^{m}u\right\|_{A}\leq\left(|\lambda|-\alpha|\mu|\right)\left\|R^{m+1}u\right\|_{A}.\end{aligned} \tag{96}$$

By observing that if $\max\{|\lambda|-|\mu|\beta,\alpha|\mu|-|\lambda|\}=|\lambda|-|\mu|\beta\geq 0$, then

$$(|\lambda|-|\mu|\beta)\|R^{m+1}u\|_A \le \|\lambda R^{m+1}u\|_A - \|\mu R^{\sharp}R^m u\|_A = \left|\|\lambda R^{m+1}u\|_A - \|\mu R^{\sharp}R^m u\|_A\right|, \tag{97}$$

and therefore

$$(|\lambda|-|\mu|\beta)^p\|R^{m+1}u\|_A^p \le \left|\|\lambda R^{m+1}u\|_A - \|\mu R^{\sharp}R^m u\|_A\right|^p. \tag{98}$$

However, if $\max\{|\lambda|-|\mu|\beta, \alpha|\mu|-|\lambda|\} = |\mu|\alpha-|\lambda| \ge 0$, then

$$\|\mu R^{\sharp}R^m u\|_A - \|\lambda R^{m+1}u\|_A \ge (\alpha|\mu|-|\lambda|)\|R^{m+1}u\|_A \ge 0, \tag{99}$$

and therefore

$$\left|\|\lambda R^{m+1}u\|_A - \|\mu R^{\sharp}R^m u\|_A\right|^p = \left(\|\mu R^{\sharp}R^m u\|_A - \|\lambda R^{m+1}u\|_A\right)^p \ge (|\mu|\alpha-|\lambda|)^p\|R^{m+1}u\|_A^p. \tag{100}$$

From either case, we get

$$\left|\|\lambda R^{m+1}u\|_A - \|\mu R^{\sharp}R^m u\|_A\right|^p \ge (\max\{|\lambda|-|\mu|\beta, \alpha|\mu|-|\lambda|\})^p\|R^{m+1}\|_A^p. \tag{101}$$

Hence,

$$\begin{aligned}&\left((|\lambda|+\alpha|\mu|)^p\|R^{m+1}u\|_A^p + \max\{|\lambda|-|\mu|\beta, \alpha|\mu|-|\lambda|\}\right)\|R^{m+1}u\|_A^p\\ &\le \left(\|\lambda R^{m+1}u\|_A + \|\mu R^{\sharp}R^m u\|_A\right)^p + \left|\|\lambda R^{m+1}u\|_A - \|\mu R^{m+1}u\|_A\right|^p\\ &\le \|\mu R^{m+1}u + \mu R^{\sharp}R^m u\|_A^p + \|\lambda R^{m+1}u - \mu R^{\sharp}R^m u\|_A^p.\end{aligned} \tag{102}$$

Taking the sup of both sides over $u \in \mathscr{H}$ with $\|u\|_A = 1$, we will deduce the desired result. □

Theorem 15. *Let $R \in \mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta)-A$-normal operator and $s \ge 0$. If*

$$\|\lambda R^{\sharp}R^m - R^{m+1}\|_A \le s \le \inf\left\{|\lambda|\|R^{\sharp}R^m u\|_A, \quad u \in \mathscr{H}, \|u\|_A = 1\right\}, \tag{103}$$

then the following inequality holds:

$$\alpha^2\|R^{m+1}\|_A^4 \le \omega_A\left(R^{m\sharp}R^{m+2}\right)^2 + \frac{s^2}{|\lambda|^2}\|R^{m+1}\|_A^2. \tag{104}$$

Proof. We use the following inequality [22]:

$$\|w\|^2\|v\|^2 \le [\mathrm{Re}\langle w, v\rangle]^2 + s^2\|v\|^2, \tag{105}$$

provided $\|w-v\| \le s \le \|v\|$.

Setting $v = \lambda A^{1/2}R^{\sharp}R^m u$ and $w = A^{1/2}R^{m+1}u$ for $u \in \mathscr{H}$, we get

$$\|R^{m+1}u\|_A^2\|\lambda R^{\sharp}R^m u\|_A^2 \le \left[\mathrm{Re}\langle R^{m+1}u, \lambda R^{\sharp}R^m u\rangle_A\right]^2 + s^2\|R^{m+1}u\|_A^2. \tag{106}$$

Hence,

$$|\lambda|^2\alpha^2\|R^{m+1}u\|_A^4 \le |\lambda|^2\left[\mathrm{Re}\langle R^{m\sharp}R^{m+2}u, u\rangle_A\right]^2 + s^2\|R^{m+1}\|_A^2. \tag{107}$$

Taking the sup on both sides of the last inequality on $u \in \mathscr{H}$, $\|u\|_A = 1$, we get the desired result. □

Theorem 16. *Let $R \in \mathscr{B}_b[\mathscr{H}]_A$ be an m-quasi-$(\alpha,\beta)-A$-normal operator, $s \ge 0$.*

If $\|\lambda R^{\sharp}R^m - R^{m+1}\|_A \le s$ for $\lambda \in \mathbb{C}, \lambda \ne 0$, then the following inequality holds:

$$\alpha\|R^{m+1}\|_A^2 \le \omega_A\left[\left(R^{\sharp}\right)^m R^{m+2}\right] + \frac{s^2}{2|\lambda|}. \tag{108}$$

Proof. Using the following reverse of the Schwarz inequality inspired from [23],

$$\|w\|\|v\| \leq [\mathrm{Re}\langle w, v\rangle] + \frac{s^2}{2}, \quad (109)$$

provided $\|w - v\| \leq s$.Setting $v = \lambda A^{1/2} R^{\sharp} R^m u$ and $w = A^{1/2} R^{m+1} u$ for $u \in \mathscr{H}$, we get

$$\left\|R^{m+1}u\right\|_A \left\|\lambda R^{\sharp} R^m u\right\|_A \leq \left[\mathrm{Re}\langle R^{m+1}u, \lambda R^{\sharp} R^m u\rangle_A\right] + \frac{s^2}{2}. \quad (110)$$

From this, we obtain

$$|\lambda|\alpha\left\|R^{m+1}u\right\|_A^2 \leq |\lambda|\left|\mathrm{Re}\langle R^{m\sharp} R^{m+2}u, u\rangle_A\right| + \frac{s^2}{2}. \quad (111)$$

Taking the sup of both sides of the last inequality for $\|u\|_A = 1$, we infer the required result. □

4. Conclusion

The study of classes of operators and related topics is one of the hottest areas in operator theory in Hilbert and semi-Hilbert spaces. In the work, we introduce a new class of operators known as m-quasi-$(\alpha, \beta) - A$-normal operators, which is a generalization of $(\alpha, \beta) - A$-normal and m-quasi-(α, β)-normal operators. Several properties are proved by exploiting the special kind of structure associated with such operators. In the course of our investigation, we find some properties of (α, β)-normal and m-quasi-(α, β)-normal operators which are retained by m-quasi-$(\alpha, \beta) - A$-normal operators. Some inequalities between the A-operator norm and A-numerical radius of m-quasi-$(\alpha, \beta) - A$-normal operators are proved.

Data Availability

No data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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