Hindawi
Journal of Mathematics
Volume 2022, Article ID 1568632, 11 pages
https://doi.org/10.1155/2022/1568632

*Research Article*

# Discrete Analogues of the Erdélyi Type Integrals for Hypergeometric Functions

**Yashoverdhan Vyas**,[1] **Anand V. Bhatnagar**,[1] **Kalpana Fatawat**,[2] **D. L. Suthar**,[3] and **S. D. Purohit**[4]

[1] *Department of Mathematics, School of Engineering, Sir Padampat Singhania University, Bhatewar, Udaipur 313601, Rajasthan, India*
[2] *Techno India NJR Institute of Technology, Plot SPL-T, Bhamashah (RIICO) Industrial Area, Kaladwas, Udaipur 313003, Rajasthan, India*
[3] *Department of Mathematics, Wollo University, P. O. Box 1145, Dessie, Ethiopia*
[4] *Department of HEAS (Mathematics), Rajasthan Technical University, Kota 324010, India*

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

Received 5 July 2022; Accepted 22 August 2022; Published 7 October 2022

Academic Editor: Serkan Araci



Gasper followed the fractional calculus proof of an Erdélyi integral to derive its discrete analogue in the form of a hypergeometric expansion. To give an alternative proof, we derive it by following a procedure analogous to a triple series manipulation-based proof of the Erdélyi integral, due to "Joshi and Vyas". Motivated from this alternative way of proof, we establish the discrete analogues corresponding to many of the *Erdélyi type integrals* due to "Joshi and Vyas" and "Luo and Raina" in the form of new hypergeometric expansion formulas. Moreover, the applications of investigated discrete analogues in deriving some expansion formulas involving orthogonal polynomials of the Askey-scheme and a new generalization of Whipple's transformation for a balanced ${}_4F_3$ in the form of an ${}_{m+4}F_{m+3}$ transformation, are also discussed.

## 1. Introduction, Motivation, and Preliminaries

The Gauss hypergeometric function ([1], p. 243, Equation (III.3)), see also ([2], p. 18, Equation (17)):

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\, z\right] := \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}\frac{z^n}{n!} \tag{1}$$

is an instance of the following generalized hypergeometric ${}_{r+1}F_r$ function ([2], p. 19, Equation (23)):

$$ \begin{aligned} {}_{r+1}F_r\left[\begin{matrix}\alpha_1,\ldots,\alpha_{r+1};\\ \beta_1,\ldots,\beta_r;\end{matrix}\, z\right] &:= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_{r+1})_n}{(\beta_1)_n\ldots(\beta_r)_n}\frac{z^n}{n!}\\ &= {}_{r+1}F_r\left[\begin{matrix}(\alpha);\\ (\beta);\end{matrix}\, z\right], \end{aligned} \tag{2}$$

where $r$ can be either 0 or any natural number. The parameter vector is

$$ (\boldsymbol{\alpha};\boldsymbol{\beta}) := (\alpha_1,\ldots,\alpha_{r+1};\beta_1,\ldots,\beta_r)\in\mathbb{C}^{r+1}\times\mathbb{C}^r, \tag{3}$$

in which all the parameters can take any real or complex value under the restriction that $\beta_j$ can neither be zero nor a negative integer. The series in (2) converges or diverges in accordance with the value of $|z|$ less than or greater than 1, respectively. Further, if we take

$$ \lambda := \sum_{j=1}^{r}\beta_j - \sum_{j=1}^{r+1}\alpha_j, \tag{4}$$

then for $|z| = 1$, the series in (2) converges or diverges in accordance with the value of $\Re(\lambda)$ is less than or greater than 0, respectively. For a detailed discussion on convergence of (2), we refer to ([2], p. 20) and ([3], p. 43).The symbol $(\rho)_n$ in (2) is the well-known symbol of Pochhammer defined by

$$(\rho)_n := \rho(\rho+1)\ldots(\rho+n-1),\\ (\rho)_0 := 1. \tag{5}$$

Also, we have

$$(\rho)_n := \frac{\Gamma(\rho+n)}{\Gamma(\rho)}, \tag{6}$$

$$(\rho)_{-n} := \frac{\Gamma(\rho-n)}{\Gamma(\rho)} = \frac{(-1)^n}{(1-\rho)_n}. \tag{7}$$

For further such relations, we refer to ([1], pp. 239–240, Appendix I).The following result

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\; z\right] = \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\cdot\int_0^1 u^{\alpha_2-1}(1-u)^{\beta_1-\alpha_2-1}(1-zu)^{-\alpha_1}\mathrm{d}u, \tag{8}$$

where $\Re(\beta_1) > \Re(\alpha_2) > 0$, is known as Euler's integral ([4], p. 47, Theorem 16). An extension of (8) was developed by Bateman [5] which is as follows:

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\; z\right] = \frac{\Gamma(\beta_1)}{\Gamma(\mu)\Gamma(\beta_1-\mu)}\int_0^1 u^{\mu-1}(1-u)^{\beta_1-\mu-1}\cdot {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \mu;\end{matrix}\; zu\right]\mathrm{d}u, \tag{9}$$

where $\Re(\beta_1) > \Re(\mu) > 0$.About eighty years ago, Erdélyi [6] gave the extensions of Euler's integral (8) and Bateman's integral (9), in the form of following equations (10)–(12), as given in the following equation:

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\; z\right] = \frac{\Gamma(\beta_1)}{\Gamma(\mu)\Gamma(\beta_1-\mu)}\int_0^1 u^{\mu-1}(1-u)^{\beta_1-\mu-1}(1-zu)^{\lambda-\alpha_1-\alpha_2}\cdot {}_2F_1\left[\begin{matrix}\lambda-\alpha_1,\lambda-\alpha_2;\\ \mu;\end{matrix}\; zu\right]{}_2F_1\left[\begin{matrix}\alpha_1+\alpha_2-\lambda,\lambda-\mu;\\ \beta_1-\mu;\end{matrix}\; \frac{(1-u)z}{1-zu}\right]\mathrm{d}u, \tag{10}$$

where $\Re(\beta_1) > \Re(\mu) > 0$,

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\; z\right] = \frac{\Gamma(\beta_1)}{\Gamma(\mu)\Gamma(\beta_1-\mu)}\int_0^1 u^{\mu-1}(1-u)^{\beta_1-\mu-1}(1-zu)^{-\alpha'_1}\cdot {}_2F_1\left[\begin{matrix}\alpha'_1-\alpha_1,\alpha_2;\\ \mu;\end{matrix}\; zu\right]{}_2F_1\left[\begin{matrix}\alpha'_1,\alpha_2-\mu;\\ \alpha_1-\mu;\end{matrix}\; \frac{(1-u)z}{1-zu}\right]\mathrm{d}u, \tag{11}$$

where $\Re(\beta_1) > \Re(\mu) > 0$, and

$$ {}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2;\\ \beta_1;\end{matrix}\; z\right] = \frac{\Gamma(\beta_1)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\mu+\beta_1-\lambda-\nu)}\cdot\int_0^1 u^{\nu-1}(1-u)^{\mu+\beta_1-\lambda-\nu-1}\cdot {}_2F_1\left[\begin{matrix}\mu-\lambda,\beta_1-\lambda;\\ \mu+\beta_1-\lambda-\nu;\end{matrix}\; 1-u\right]{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\mu;\\ \alpha_1,\nu;\end{matrix}\; zu\right]\mathrm{d}u, \tag{12}$$

where $\Re(\lambda,\nu,\mu+\beta_1-\lambda-\nu) > 0$. Further, equations (8)–(12) satisfy the constraints $|z| \neq 1$ and $|\arg(1-z)| < \pi$. The derivations of above equations in [6] were based on the fractional calculus. For some important applications of (10), as shown in [7].

Thirty five years hence Erdélyi [6], Gasper [7] derived the discrete analogue of Erdélyi's integral (10) as given in the following equation ([7], Equation (27)):

$$ {}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,-n;\\ \beta_1,\delta;\end{matrix}\; 1\right] = \sum_{k=0}^{n}\frac{(\mu)_k(\delta+\lambda-\alpha_2-\alpha_1)_k(\beta_1-\mu)_{n-k}(-n)_k(-1)^k}{(\beta_1)_n(\delta)_k k!}\cdot {}_3F_2\left[\begin{matrix}\lambda-\alpha_1,\lambda-\alpha_2,-k;\\ \mu,\delta+\lambda-\alpha_2-\alpha_1;\end{matrix}\; 1\right]{}_3F_2\left[\begin{matrix}\lambda-\mu,\alpha_1+\alpha_2-\lambda,k-n;\\ \beta_1-\mu,\delta+k;\end{matrix}\; 1\right]. \tag{13}$$

Gasper [7] proved (13) by following the steps analogous to Erdélyi's fractional calculus proof of (10). He used three transformation formulas ([7], Equations (22)–(24)) to prove (13), where one of the transformation formulas ([7], Equations (23)) is as follows:

$$ {}_3F_2\left[\begin{matrix}-n,\alpha_1,\beta_1-\alpha_2;\\ \beta_1,1+\alpha_1-\delta-n;\end{matrix}\; 1\right] = \frac{(\delta)_n}{(\delta-\alpha_1)_n}{}_3F_2\left[\begin{matrix}-n,\alpha_1,\alpha_2;\\ \beta_1,\delta;\end{matrix}\; 1\right]. \tag{14}$$

As explained in Gasper ([7], p. 208) and ([8], p. 2), replacing $n,k,\delta$ by $Nz, Nzt, -N$, respectively, in (13) and choosing $Nz \longrightarrow \infty$ through integer values of $Nz$ with fixed $z$ such that $z$ lies between 0 and 1, and finally using analytical continuation with respect to $z$, we get (10). For this conversion, we utilize the fundamental definition of definite integral and a limit relation involving Pochhammer symbols ([4], p. 11). Moving a few steps ahead, Gasper [8] studied ${}_4\Phi_3$ expansions to provide generalizations of (13) and other results, along with the applicability to orthogonal polynomials [7, p. 6]. Lievens [9] discussed the association of such expansions with the invariant group of the Lie algebra. The ${}_4\Phi_3$ expansions given by [8] were specialized to give expansion formulas for the Racah polynomials, the Askey–Wilson polynomials and their $q$-analogues, where the Racah polynomials $R_n$ ([10], p. 25, Equation (1.2.1)) and the Askey–Wilson polynomials $W_n$ ([10], p. 23, Equation (1.1.1)) are defined as follows:

$$
\begin{aligned}
&R_n(\rho(x);\alpha_1,\alpha_2,\beta_1,\beta_2)\\
&\quad = {}_4F_3\left[\begin{matrix} -n,-x,n+\alpha_1+\alpha_2+1,x+\beta_1+\beta_2+1; \\ \alpha_1+1,\alpha_2+\beta_2+1,\beta_1+1; \end{matrix}\, 1\right],\\
&n\in\mathbb{N}\cup\{0\},\quad \rho(x)=x\left(x+\beta_2+\beta_1+1\right), \qquad (15)\\
&\alpha_2+\beta_2+1=-N \text{ or}\\
&\alpha_1+1=-N \text{ or}\\
&\beta_1+1=-N,
\end{aligned}
$$

with $N$ a positive integer.

$$
\frac{W_n\left(x^2;\alpha_1,\alpha_2,\beta_1,\beta_2\right)}{(\alpha_1+\alpha_2)_n(\alpha_1+\beta_1)_n(\alpha_1+\beta_2)_n} = {}_4F_3\left[\begin{matrix} n+\alpha_1+\alpha_2+\beta_1+\beta_2-1,\alpha_1+ix,\alpha_1-ix,-n; \\ \alpha_1+\alpha_2,\alpha_1+\beta_1,\alpha_1+\beta_2; \end{matrix}\, 1\right]. \qquad (16)
$$

As a matter of fact, the further generalizations of these ${}_4\Phi_3$ expansions were developed by Joshi and Vyas [11], in the form of ${}_{12}\Phi_{11}(q)$ and ${}_r\Phi_s(z)$ expansions, along with the applicability to biorthogonal rational functions ([11], p. 221, Section 3).

Many researchers, for example, [7, 8, 11–19] have studied and investigated several kinds of integrals that involve and represent the hypergeometric functions, on account of the numerous applications of these integrals, for example, as shown in [13].

Following, Erdélyi [6] and Gasper [7, 8]; Joshi and Vyas [15] gave the alternative proofs of three integrals of Erdélyi by using the series rearrangement method ([4], see also [3]) and the three classical summation theorems, namely, the Gauss, Vandermonde, and Saalschütz summation theorems. Motivated from the above way of proof, they investigated many new integrals for certain ${}_{r+1}F_r$, which were similar to the Erdélyi integrals due to the inclusion of the powers and hypergeometric functions and hence named, *Erdélyi type integrals*. Furthermore, they obtained many other varieties of Erdélyi's integrals, for example ([15], p. 132, Equations (4.1), (4.4), (4.5), (4.7) to (4.9) and (5.2)). In the sequel, they [16] also discussed $q$-extensions of the Erdélyi type integrals, by using series rearrangement. Certain classical series rearrangements were also used by [20, 21] to derive extensions of the Bailey transform, which in turn, were applied to establish remarkable ordinary and $q$-hypergeometric identities.

Recently, Luo and Raina [22] have obtained two new Erdélyi type integrals ([22], Theorems 3.1 and 3.2), which are extensions of (10). They further investigated the generalization of Erdélyi type integral ([15], p. 132, Equation (4.1)) using the extended version of Saalschütz summation theorem [23]:

$$
\begin{aligned}
&{}_{r+3}F_{r+2}\left[\begin{matrix} \alpha_1,\alpha_2,-n, & (\theta_r+m_r); \\ 1+m-n+\alpha_1+\alpha_2-\beta_1,\beta_1, & (\theta_r); \end{matrix}\, 1\right] \qquad (17)\\
&\quad = \frac{(\beta_1-\alpha_1-m)_n(\beta_1-\alpha_2-m)_n}{(c)_n(\beta_1-\alpha_1-\alpha_2-m)_n}\,\frac{(\eta_m+1)_n}{(\eta_m)_n},
\end{aligned}
$$

where $(\eta_m)$ are the non-zero roots of the associated parametric equation $Q_m(\omega)=0$ of degree $m$, where

$$
Q_m(\omega)=\sum_{p=0}^{m} B_p(\alpha_1)_p(\alpha_2)_p(\omega)_p G_{m,p}(\omega), \qquad (18)
$$

with

$$
\begin{aligned}
B_p &= (-1)^p A_p(\beta_1-\alpha_1-m+p)_{m-p}\\
&\quad (\beta_1-\alpha_2-m+p)_{m-p},
\end{aligned} \qquad (19)
$$

$$
G_{m,p}(\omega) = {}_3F_2\left[\begin{matrix} -(m-p),\beta_1-\alpha_1-\alpha_2-m,\omega+p; \\ \beta_1-\alpha_1-m+p,\beta_1-\alpha_2-m+p; \end{matrix}\, 1\right]. \qquad (20)
$$

The hypergeometric series $G_{m,p}(\omega)$ is a polynomial in $\omega$ of degree $m-p$ where $0\le p\le m$. The coefficients $A_p$ are defined by

$$
\begin{aligned}
A_p &= \sum_{j=p}^{m} S_j^{(p)}\sigma_{m-j},\\
A_0 &= (\theta_1)_{m_1}\cdots(\theta_r)_{m_r},\\
A_m &= 1,
\end{aligned} \qquad (21)
$$

where $S_j^{(p)}$ stands for the second kind Stirling number ([24], Chapter 6). The coefficients $\sigma_j$ can be obtained from the following generating relation:

$$
(\theta_1+x)_{m_1}\cdots(\theta_r+x)_{m_r} = \sum_{j=0}^{m}\sigma_{m-j}x^j,\quad 0\le j\le m. \qquad (22)
$$

The application of one of the extended Erdélyi type integrals due to ([22], Theorem 3.2) in deriving a generalization of Thomae-type transformation for ${}_{p+1}F_p(z)$ with integral parameter differences ([22], p. 11, Equation (4.6)) is also discussed.

However, Gasper [7, 8] developed the discrete analogues for Erdélyi's integrals but, the discrete analogues corresponding to the Erdélyi type integrals due to ([22], p. 9, Section 4) and ([15], Equations (4.1) and (3.1) to (3.7)) have not been investigated till now.

It will be our endeavor in this paper to conjecture and prove the discrete analogues of the Erdélyi type integrals due to [15, 22] in the form of new hypergeometric expansion formulas by following the method analogous to [15], that is, the utilization of series manipulation techniques in conjunction with certain classical summation theorems, namely, the Vandermonde theorem ([1], p. 243, Equation (III.4)), the Saalschütz summation theorem ([1], p. 243, Equation (III.2)), the extended Saalschütz summation theorem [23], the Dougall theorems for ${}_5F_4(1)$ ([1], p. 244, Equation (III.13)) and the ${}_7F_6(1)$ ([1], p. 244, Equation (III.14)), and the transformation formula (14). To avoid repetition, the citation details for the well-known summation theorems mentioned above will be omitted in the remaining part of the paper.

Our primary aim here is to develop the discrete analogues of the Erdélyi type integrals due to Joshi and Vyas [15] and Luo and Raina [22]. The organization of the current research work is as follows.

In Section 2, a step-by-step demonstration of the alternative proof of the hypergeometric expansion or discrete analogue (13) of the Erdélyi integral, along the lines of a triple series manipulation-based derivation of the Erdélyi's integral (10) by Joshi and Vyas [15], is detailed. In Section 3, the statements and derivations of the discrete analogues of the Erdélyi type integrals due to [15, 22] in the form of new hypergeometric expansion theorems, are presented. In Section 4, some remarkable applications of the investigated results are discussed, where the application of Theorem 2 provides the expansion formulas involving the orthogonal polynomials (Racah and Askey–Wilson). Further, the application of Theorem 1 enable us to develop a generalization of the well-known Whipple's transformation of ${}_4F_3$ into ${}_4F_3$ ([25], p. 140, Theorem 3.3.3), in the form of a transformation of ${}_{m+4}F_{m+3}$ into ${}_{r+4}F_{r+3}$. This transformation of ${}_{m+4}F_{m+3}$ into ${}_{r+4}F_{r+3}$ leads to an extended Thomae-type transformation due to Luo and Raina ([22], Equation (4.6)) (see also [26], p. 114, Theorem 2) as a special case. Thus, this paper shows the efficiency of the series rearrangement technique in deriving the discrete analogues of the integrals due to ([22], p. 9) and ([15], Equations (4.1) and (3.1) to (3.7)) with the applications.

## 2. Alternative Proof for the Discrete Analogue of Erdélyi's Integral

In the present section, we are providing the alternative proof of (13) by using the series manipulation techniques, the transformation formula (14), the Vandermonde sum, and the Saalschütz sum.

*Proof*

Step 1: Denoting the right-side of (13) by $\Omega$ and writing the ${}_3F_2$'s in their series form, we get

$$\Omega = \sum_{k,j=0}^{\infty}\sum_{i=0}^{k}\frac{(-n)_{k+j}(\mu)_k(\delta+\lambda-\alpha_2-\alpha_1)_k(\alpha_1+\alpha_2-\lambda)_j}{(\delta)_{k+j}(1+\mu-\beta_1-n)_k(\lambda+\delta-\alpha_1-\alpha_2)_i}\cdot\frac{(\lambda-\mu)_j(\lambda-\alpha_1)_i(\lambda-\alpha_2)_i(\beta_1-\mu)_n(-1)^i}{(\beta_1-\mu)_j(\beta_1)_n(\mu)_i i!\,j!k-i!}. \tag{23}$$

Step 2: Next, by applying the following double series manipulation lemma [3, 4], we get

$$\sum_{k=0}^{\infty}\sum_{i=0}^{k}Q(i,k)=\sum_{k,i=0}^{\infty}Q(i,k+i), \tag{24}$$

on equation (23) and then taking an inner series in $k$, we can write

$$\Omega = \sum_{i,j=0}^{\infty}\frac{(-n)_{i+j}(\mu)_i(\alpha_1+\alpha_2-\lambda)_j(\lambda-\mu)_j(\lambda-\alpha_1)_i(\lambda-\alpha_2)_i}{(\delta)_{i+j}(1+\mu-\beta_1-n)_i(\beta_1-\mu)_j}\cdot\frac{(\beta_1-\mu)_n(-1)^i}{(\beta_1)_n(\mu)_i i!\,j!}\,{}_3F_2\left[\begin{matrix}-n+i+j,\mu+i,\lambda+\delta-\alpha_1-\alpha_2+i;\\ \delta+i+j,1+\mu-\beta_1-n+i;\end{matrix}\;1\right]. \tag{25}$$

Step 3: Now, replacing

$$\begin{aligned}\alpha_1&\longrightarrow\mu+i,\\ \alpha_2&\longrightarrow\alpha_2+\alpha_1-\lambda+j,\\ \beta_1&\longrightarrow\delta+i+j,\\ \delta&\longrightarrow\beta_1+i+j,\end{aligned} \tag{26}$$

and $n\longrightarrow n-i-j$ in (14) and applying the resulting transformation on ${}_3F_2$ of the right side of (27), we can obtain

$$\Omega = \sum_{k=0}^{\infty}\sum_{i,j=0}^{\infty}\frac{(-n)_{i+j+k}(\mu)_{i+k}(\alpha_2+\alpha_1-\lambda)_{j+k}(\lambda-\mu)_j(\lambda-\alpha_1)_i(\lambda-\alpha_2)_i}{(\delta)_{i+j+k}(\beta_1)_{i+j+k}(\mu)_i i!\,j!k!}. \tag{27}$$

Step 4: Next, on applying the following triple series rearrangement:

$$\sum_{k,i,j=0}^{\infty}Q(k,i,j)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{k-i}Q(k-i-j,i,j), \tag{28}$$

to the triple series (27), and then taking an inner series in $j$, we obtain

$$\Omega = \sum_{k,i=0}^{\infty,k}\frac{(-n)_k(\mu)_k(\alpha_2+\alpha_1-\lambda)_{k-i}(\lambda-\alpha_1)_i(\lambda-\alpha_2)_i(-k)_i(-1)^i}{(\delta)_k(\mu)_i(\beta_1)_k i!k!}\cdot{}_2F_1\left[\begin{matrix}\lambda-\mu,i-k;\\ 1-\mu-k;\end{matrix}\;1\right]. \tag{29}$$

Step 5: Now, applying Vandermonde sum to simplify the ${}_2F_1$ present in (29) and then taking an inner series in $i$, we can write

$$\Omega = \sum_{k=0}^{\infty}\frac{(\lambda)_k(\alpha_2+\alpha_1-\lambda)_k(-n)_k}{(\delta)_k(\beta_1)_k k!}\,{}_3F_2\left[\begin{matrix}-k,\lambda-\alpha_2,\lambda-\alpha_1;\\ \lambda,1-\alpha_1-\alpha_2+\lambda-k;\end{matrix}\;1\right]. \tag{30}$$

Finally, the application of the Saalschütz sum and further simplifications lead to the left side of (13). □

*Remark 1.* In the next Section 3, we shall prove Theorems 1 and 5 to 10 by following the steps similar to those we have elaborated in the present Section. It may be noted that Theorems 2–4 follow as special cases of Theorems 1 and 3.

## 3. Discrete Analogues of the Erdélyi Type Integrals

Here, we state and prove a number of new hypergeometric expansions as discrete analogues for the Erdélyi type integrals due to ([22], pp. 9–10, Equation (4.1)) and ([15], Equations (4.1) and (3.1) to (3.7)) in the form of Theorems 1, 2, and 4 to 10, respectively. We also obtain an additional expansion as Theorem 3, which is the generalization of Theorem 4.

**Theorem 1.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_{m+4}F_{m+3}\left[\begin{matrix}\lambda,\xi,\nu,-N, & (\eta_m+1); \\ \gamma,\delta,\epsilon, & (\eta_m);\end{matrix}\; 1\right] = \frac{(\gamma-\mu)_N}{(\gamma)_N} \\
&\cdot\sum_{k=0}^{N}\frac{(\epsilon-\xi-\nu+\delta-m)_k(\mu)_k(-N)_k}{(1+\mu-\gamma-N)_k(\epsilon)_k k!} \\
&\cdot{}_{r+4}F_{r+3}\left[\begin{matrix}\delta-\nu-m,\lambda,\delta-\xi-m,-k, & (f_r+m_r); \\ \mu,\delta,\epsilon-\xi-\nu+\delta-m, & (f_r);\end{matrix}\; 1\right] \\
&\cdot{}_3F_2\left[\begin{matrix}\xi+\nu-\delta+m,\lambda-\mu,-N+k; \\ \gamma-\mu,\epsilon+k;\end{matrix}\; 1\right],
\end{aligned}
\tag{31}
$$

*where the lower parameters $(\eta_m)$ have a similar interpretation as given by equations (18)–(22) provided the following replacements are applied:*

$$
\begin{aligned}
\alpha_1 &\longrightarrow \delta-\nu-m, \\
\alpha_2 &\longrightarrow \delta-\xi-m, \\
\beta_2 &\longrightarrow \delta.
\end{aligned}
\tag{32}
$$

*Proof.* The proof of the above expansion follows the same steps as discussed in Section 2, except using the Vandermonde and the extended Saalschütz sum (Equations (17)–(22)), one by one, to simplify the appearing inner series. □

*Remark 2.* To convert Theorem 1 into the Erdélyi type integral ([22], pp. 9–10, Equation (4.1)), first, we replace $N, k, \varepsilon$ by $Nz, Nzt, -N$, respectively and then, we choose $Nz \longrightarrow \infty$ through integer values of $Nz$ with fixed $z$ such that $z$ lies between 0 and 1. Finally, we use equations (5)–(7), a limit relation involving Pochhammer symbols ([4], p. 11), and analytical continuation with respect to $z$ to get the desired result. The same procedure may be applied to convert Theorems 2, 4–10) to their corresponding Erdélyi type integrals ([15], Equations (4.1), and (3.1) to (3.7)), *respectively*.

**Theorem 2.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\lambda,\nu,\xi,-N; \\ \gamma,\delta,\epsilon;\end{matrix}\; 1\right] = \frac{(\gamma-\mu)_N}{(\gamma)_N}\sum_{k=0}^{N}\frac{(\mu)_k(\epsilon-\xi-\nu+\delta)_k(-N)_k}{(1+\mu-\gamma-N)_k(\epsilon)_k k!} \\
&\cdot{}_4F_3\left[\begin{matrix}\delta-\xi,\delta-\nu,\lambda,-k; \\ \mu,\delta,\epsilon-\xi-\nu+\delta;\end{matrix}\; 1\right]{}_3F_2\left[\begin{matrix}\xi+\nu-\delta,\lambda-\mu,-N+k; \\ \gamma-\mu,\epsilon+k;\end{matrix}\; 1\right].
\end{aligned}
\tag{33}
$$

*Proof.* It can be easily observed that the assertion in Theorem 2 follows from Theorem 1 when $m = 0$. □

**Theorem 3.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_{m+4}F_{m+3}\left[\begin{matrix}\alpha,\beta,\alpha'+\nu,-N, & (\eta_m+1); \\ \gamma,\alpha+\nu,\epsilon, & (\eta_m);\end{matrix}\; 1\right] \\
&= \frac{(\gamma-\lambda)_N}{(\gamma)_N}\sum_{k=0}^{N}\frac{(\epsilon-\alpha'-m)_k(\lambda)_k(-N)_k}{(1+\lambda-\gamma-N)_k(\epsilon)_k k!} \\
&\cdot{}_{r+4}F_{r+3}\left[\begin{matrix}\alpha-\alpha'-m,\beta,\nu-m,-k, & (f_r+m_r); \\ \nu+\alpha,\lambda,\epsilon-\alpha'-m, & (f_r);\end{matrix}\; 1\right] \\
&\cdot{}_3F_2\left[\begin{matrix}\alpha'+m,\beta-\lambda,-N+k; \\ \gamma-\lambda,\epsilon+k;\end{matrix}\; 1\right].
\end{aligned}
\tag{34}
$$

*Proof.* The following replacements are

$$
\begin{aligned}
\lambda &\longrightarrow \beta, \\
\xi &\longrightarrow \alpha'+\nu, \\
\nu &\longrightarrow \alpha, \\
\delta &\longrightarrow \alpha+\nu, \\
\mu &\longrightarrow \lambda,
\end{aligned}
\tag{35}
$$

in Theorem 1 lead us to Theorem 3. □

**Theorem 4.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\alpha,\beta,\alpha'+\nu,-N; \\ \gamma,\alpha+\nu,\epsilon;\end{matrix}\; 1\right] = \frac{(\gamma-\lambda)_N}{(\gamma)_N}\sum_{k=0}^{N}\frac{(\epsilon-\alpha')_k(\lambda)_k(-N)_k}{(1+\lambda-\gamma-N)_k(\epsilon)_k k!} \\
&\cdot{}_4F_3\left[\begin{matrix}\alpha-\alpha',\beta,\nu,-k; \\ \nu+\alpha,\lambda,\epsilon-\alpha';\end{matrix}\; 1\right]{}_3F_2\left[\begin{matrix}\alpha',\beta-\lambda,-N+k; \\ \gamma-\lambda,\epsilon+k;\end{matrix}\; 1\right].
\end{aligned}
\tag{36}
$$

*Proof.* The case $m = 0$ of Theorem 3 leads us to the expansion formula given in Theorem 4. □

**Theorem 5.** *The following assertion holds true:*

$$
{}_4F_3\left[\begin{matrix}\alpha, \dfrac{\gamma-\beta}{2}, \dfrac{\gamma-\beta+1}{2}, -N;\\ \\ \dfrac{\gamma}{2}, \dfrac{\gamma+1}{2}, \epsilon;\end{matrix}\; 1\right] = \frac{(\alpha+\beta)_N}{(\gamma)_N}
\cdot \sum_{k=0}^{N} \frac{(\gamma-\alpha-\beta)_k (\epsilon-\beta)_k (-N)_k}{(1-\alpha-\beta-N)_k (\epsilon)_k k!} {}_3F_2\left[\begin{matrix}\alpha-\beta, \gamma-\beta, -k;\\ \gamma-\alpha-\beta, \epsilon-\beta;\end{matrix}\; 1\right]
\cdot {}_6F_5\left[\begin{matrix}\dfrac{\beta}{2}, \dfrac{\beta+1}{2}, \alpha, \alpha+\beta+N-k, 1-\epsilon-\beta-m, -N+k;\\ \\ \dfrac{\alpha+\beta}{2}, \dfrac{\alpha+\beta+1}{2}, \gamma+N, 1-\epsilon+\beta-k, \epsilon+k;\end{matrix}\; 1\right]. \tag{37}
$$

*Proof.* The proof of above theorem follows the steps explained in Section 2 and utilizes the Saalschütz sum twice. □

**Theorem 6.** *The following assertion holds true:*

$$
{}_4F_3\left[\begin{matrix}\alpha, \beta, 1+\gamma-\mu-\lambda, -N;\\ 1+\gamma-\mu, 1+\gamma-\lambda, \epsilon;\end{matrix}\; 1\right] = \frac{(1+\gamma-\beta)_N}{(1+\gamma)_N}
\cdot \sum_{k=0}^{N} \frac{(\beta)_k (\epsilon-\mu)_k (-N)_k}{(\beta-\gamma-N)_k (\epsilon)_k k!} {}_4F_3\left[\begin{matrix}\alpha-\mu, \beta+k, \epsilon-\mu+k, -N+k;\\ \epsilon+k, \epsilon-\mu, \beta-\gamma-N+k;\end{matrix}\; -1\right]
\cdot {}_7F_6\left[\begin{matrix}\gamma, 1+\dfrac{\gamma}{2}, \alpha, \lambda, \mu, \quad 1-\beta+\gamma+N-n-k, -k;\\ \\ \dfrac{\gamma}{2}, 1+\gamma-\lambda, 1+\gamma-\mu, 1+\gamma-\beta, \quad 1+\gamma+N, 1-\epsilon+\mu-n-k;\end{matrix}\right]. \tag{38}
$$

*Proof.* Following the steps discussed in Section 2 and applying the Vandermonde and the Dougall ${}_5F_4$ sum, one by one, we can easily get the assertion stated in Theorem 6. □

**Theorem 7.** *The following assertion holds true:*

$$
{}_5F_4\left[\begin{matrix}\alpha, \dfrac{\gamma-\mu}{2}, \dfrac{\gamma-\mu+1}{2}, \beta, -N;\\ \\ \gamma-\mu, \dfrac{\gamma}{2}, \dfrac{\gamma+1}{2}, \varepsilon;\end{matrix}\; 1\right] = \frac{(\gamma-\beta)_N}{(\gamma)_N}
\cdot \sum_{k=0}^{N} \frac{(\varepsilon-\mu)_k (\beta)_k (-N)_k}{(1+\beta-\gamma-N)_k (\varepsilon)_k k!} {}_2F_1\left[\begin{matrix}\alpha-\mu, -k;\\ \varepsilon-\mu;\end{matrix}\; 1\right]
\cdot {}_4F_3\left[\begin{matrix}\alpha, \mu, \beta+k, -N+k;\\ \gamma-\beta, \gamma+N, \varepsilon+k;\end{matrix}\; -1\right]. \tag{39}
$$

*Proof.* To prove the above assertion, we follow the method discussed in Section 2, and apply the Vandermonde and the Saalschütz sum. □

**Theorem 8.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}\alpha,\beta,1+\alpha-\beta-\gamma+\mu,\gamma,-N;\\ 1+\alpha-\beta,\beta+\gamma-\mu,1+\alpha-\gamma,\epsilon;\end{matrix}\ 1\right]=\frac{(1+\alpha-\mu-\lambda)_N}{(1+\alpha-\mu)_N}\\
&\cdot\sum_{k=0}^{N}\frac{(\lambda)_k(\epsilon-\mu)_k(-N)_k}{(\lambda-\alpha+\mu-N)_k(\alpha+1)_k k!}{}_3F_2\left[\begin{matrix}\mu,-\lambda+\alpha,-N+k;\\ \alpha+1-\lambda-\mu,\epsilon+k;\end{matrix}\ 1\right]\\
&\cdot{}_9F_8\left[\begin{matrix}\alpha-\mu,1+\dfrac{\alpha-\mu}{2},\dfrac{\alpha}{2},\dfrac{\alpha+1}{2},\beta-\mu,\gamma-\mu,\quad\lambda+k,1+\alpha-\beta-\gamma,-k;\\ \dfrac{\alpha-\mu}{2},\dfrac{\lambda}{2},\dfrac{\lambda+1}{2},1+\alpha-\beta,1+\alpha-\gamma,\quad\beta+\gamma-\mu,1+\alpha-\mu+N,\epsilon-\mu;\end{matrix}\ 1\right].
\end{aligned}
\tag{40}
$$

*Proof.* Following the steps mentioned in Section 2 and using the Vandermonde and the Dougall ${}_7F_6$ sum, we get the desired expansion. □

**Theorem 9.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix}\dfrac{\alpha+1}{2},1+\dfrac{\alpha}{2},1+\alpha-\mu-\lambda,\beta,\gamma,-N;\\ \dfrac{\beta+\gamma}{2},\dfrac{\beta+\gamma+1}{2},1+\alpha-\mu,1+\alpha-\lambda,\epsilon;\end{matrix}\ 1\right]\\
&=\frac{(2\gamma+\beta-\alpha-1)_N}{(\gamma+\beta)_N}\sum_{k=0}^{N}\frac{(1+\alpha-\gamma+m)_k(\epsilon-\gamma-n)_k(-N)_k}{(2+\alpha-2\gamma-\beta-N-n)_k(\epsilon)_k k!}\\
&\cdot{}_5F_4\left[\begin{matrix}\dfrac{\gamma}{2},\dfrac{\gamma+1}{2},\gamma+\beta-\alpha-1,2\gamma+\beta-1-\alpha+N,-N+k;\\ \dfrac{2\gamma+\beta-1-\alpha}{2},\dfrac{2\gamma+\beta-\alpha}{2},\gamma+\beta+N+m,\epsilon+k;\end{matrix}\ 1\right]\\
&\cdot{}_8F_7\left[\begin{matrix}\alpha,1+\dfrac{\alpha}{2},\beta,\mu,\lambda,\gamma+2n,\quad 2\gamma+\beta-1-\alpha-k+n+N,-k;\\ \dfrac{\alpha}{2},\gamma-\alpha-m-k,1+\alpha-\mu,1+\alpha-\lambda,\gamma+\beta+N,\quad 1-\epsilon+\gamma+n-k,2\gamma+\beta-1-\alpha+2n;\end{matrix}\ 1\right].
\end{aligned}
\tag{41}
$$

*Proof.* To prove the above assertion we follow the similar steps as given in Section 2 and will use the Saalschütz and the Dougall ${}_5F_4$ sum. □

**Theorem 10.** *The following assertion holds true:*

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}\alpha,\beta,\gamma,1+2\alpha-\beta-\gamma-\mu,\dfrac{\mu+1}{2},1+\dfrac{\mu}{2},-N;\\ \\ 1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma+\mu-\alpha,\ \dfrac{\lambda}{2},\dfrac{\lambda+1}{2},\epsilon;\end{matrix}\ 1\right]\\
&=\frac{(\lambda-\alpha)_N}{(\lambda)_N}\sum_{k=0}^{N}\frac{(\alpha)_k(\epsilon+\mu-\alpha)_k(-N)_k}{(1+\alpha-\lambda-N)_k(\epsilon)_k k!}\\
&\cdot{}_4F_3\left[\begin{matrix}\alpha-\mu,\lambda-\mu-1,\alpha+k+m,-N+k;\\ \lambda-\alpha,\lambda+N+m,\epsilon+k;\end{matrix}\ -1\right]\\
&\cdot{}_7F_6\left[\begin{matrix}\mu,1+\dfrac{\mu}{2},\beta+\mu-\alpha,\gamma+\mu-\alpha,1+\alpha-\beta-\gamma,\alpha+k,-k;\\ \\ \dfrac{\mu}{2},1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma+\mu-\alpha,\lambda+N,\epsilon+\mu-\alpha;\end{matrix}\ 1\right].
\end{aligned}
\tag{42}
$$

*Proof.* The proof of Theorem 10 follows the steps explained in Section 2 and will make use of the Vandermonde and the Dougall ${}_7F_6$ sum. □

## 4. Applications

In this Section, we shall apply Theorem 2 to investigate the expansion formulas involving the Racah and the Askey–Wilson polynomials and product of hypergeometric functions. Further, Theorem 1 will be utilized to derive the generalized Whipple transformation of ${}_{m+4}F_{m+3}$ into ${}_{r+4}F_{r+3}$ by using the Saalschütz sum and then, applying the double series manipulation (24) and the Vandermonde sum.

(1) On applying the following replacements

$$
\begin{aligned}
&\lambda\longrightarrow -x,\\
&\varepsilon\longrightarrow\alpha+1,\\
&\gamma\longrightarrow\beta+\delta+1,\\
&\delta\longrightarrow\gamma+1.
\end{aligned}
\tag{43}
$$

and then, $\nu\longrightarrow x+\alpha+\beta+1,\xi\longrightarrow N+\gamma+\delta+1$ in Theorem 2 lead to an expansion for the Racah polynomials as

$$
\begin{aligned}
&R_x(\rho(N);\alpha,\beta,\gamma,\delta)=\frac{(\beta-\mu+\delta+1)_N}{(\beta+\delta+1)_N}\sum_{k=0}^{N}\frac{(-N-\beta-\delta-x)_k}{(\mu-\beta-\delta-N)_k}\\
&\cdot\frac{(\mu)_k(-N)_k}{(\alpha+1)_k k!}{}_4F_3\left[\begin{matrix}-N-\delta,\gamma-\beta-\alpha-x,-x,-k;\\ \mu,\gamma+1,-N-\beta-\delta-x;\end{matrix}\ 1\right]\\
&\cdot{}_3F_2\left[\begin{matrix}1+\delta+\beta+\alpha+x+N,-x-\mu,-N+k;\\ \beta+\delta+1-\mu,1+\alpha+k;\end{matrix}\ 1\right].
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
&x\in\mathbb{N}\cup\{0\},\\
&\rho(N)=N(N+\gamma+\delta+1),
\end{aligned}
\tag{45}
$$

and one of the denominator parameters out of the three that is, $\beta+\delta+1$, $\alpha+1$, and $\gamma+1$ in $R_x$ should be equal to $-n$.

(2) The following replacements

$$\begin{aligned}
\lambda &\longrightarrow -x,\\
\varepsilon &\longrightarrow \alpha+1,\\
\gamma &\longrightarrow \gamma+1,\\
\delta &\longrightarrow \beta+\delta+1,
\end{aligned} \tag{46}$$

and then $\nu \longrightarrow x+\delta+\gamma+1$, $\xi \longrightarrow N+\beta+\alpha+1$, in Theorem 1 lead to an expansion for the Racah polynomials as

$$\begin{aligned}
&R_N(\rho(x);\alpha,\beta,\gamma,\delta)\\
&= \frac{(\gamma+1-\mu)_N}{(\gamma+1)_N}\sum_{k=0}^{N}\frac{(\mu)_k(-N-\gamma-x)_k(-N)_k}{(\alpha+1)_k(-N+\mu-\gamma)_k k!}\\
&\cdot {}_4F_3\left[\begin{matrix}\delta-N-\alpha,\beta-x-\gamma,-x,-k;\\ \beta+\delta+1,\mu,-N-x-\gamma;\end{matrix}\ 1\right]\\
&\cdot {}_3F_2\left[\begin{matrix}N+\alpha+x+\gamma+1,-\mu-x,-N+k;\\ \gamma+1+\mu,1+\alpha+k;\end{matrix}\ 1\right],
\end{aligned} \tag{47}$$

subject to the restrictions similar to those mentioned with equation (44).

(3) Next, letting

$$\begin{aligned}
\lambda &= a+b+c+d+N-1,\\
\nu &= a+ix,\\
\xi &= a-ix,\\
\varepsilon &= a+b,\\
\gamma &= a+c,
\end{aligned} \tag{48}$$

and $\delta = a+d$ in Theorem 2, we get the following expansion for the Askey-Wilson polynomials

$$\begin{aligned}
&\frac{W_N\left(x^2;a,b,c,d\right)}{(a+b)_N(a+d)_N} = (a+c-\mu)_N\\
&\sum_{k=0}^{N}\frac{(\mu)_k(d+b)_k(-N)_k}{(a+b)_k(1+\mu-c-a-N)_k k!}\\
&\cdot {}_4F_3\left[\begin{matrix}d-ix,d+ix,a+b+c+d+N-1,-k;\\ \mu,a+d,b+d;\end{matrix}\ 1\right]\\
&\cdot {}_3F_2\left[\begin{matrix}a-d,a+b+c+d+N-1-\mu,-N+k;\\ a+c-\mu,a+b+k;\end{matrix}\ 1\right].
\end{aligned} \tag{49}$$

(4) In Theorem 1, we can apply the Saalschütz sum to simplify the ${}_3F_2$ series under the following condition:

$$1+\lambda+\xi+\nu-N+m = \delta+\gamma+\varepsilon, \tag{50}$$

which converts Theorem 1 into the following result:

$$\begin{aligned}
&{}_{m+4}F_{m+3}\left[\begin{matrix}\lambda,\xi,\nu,-N, & (\eta_m+1);\\ \gamma,\delta,\varepsilon, & (\eta_m);\end{matrix}\ 1\right]\\
&= \frac{(\varepsilon+\mu-\lambda)_N(\gamma-\lambda)_N}{(\varepsilon)_N(\gamma)_N}\sum_{k=0}^{N}\frac{(\mu)_k(-N)_k}{(\varepsilon+\mu-\lambda)_k k!}\\
&\cdot {}_{r+4}F_{r+3}\left[\begin{matrix}\delta-\nu-m,\lambda,\delta-\xi-m,-k, & (f_r+m_r);\\ \mu,\delta,\varepsilon-\xi-\nu+\delta-m, & (f_r);\end{matrix}\ 1\right].
\end{aligned} \tag{51}$$

Next, expressing the ${}_{m+4}F_{m+3}$ in its series form, applying the double series manipulation lemma (24), taking an inner ${}_2F_1$ series in $k$ and then, summing this ${}_2F_1$ series by the Vandermonde theorem, we get

$$\begin{aligned}
&{}_{m+4}F_{m+3}\left[\begin{matrix}\lambda,\xi,\nu,-N, & (\eta_m+1);\\ \gamma,\delta,\varepsilon, & (\eta_m);\end{matrix}\ 1\right] = \frac{(\gamma-\lambda)_N(\varepsilon-\lambda)_N}{(\varepsilon)_N(\gamma)_N}\\
&\cdot {}_{r+4}F_{r+3}\left[\begin{matrix}\delta-\nu-m,\lambda,\delta-\xi-m,-N, & (f_r+m_r);\\ \delta,1-\gamma+\lambda-N,1-\varepsilon+\lambda-N, & (f_r);\end{matrix}\ 1\right].
\end{aligned} \tag{52}$$

The equation (52) is a generalization of the Whipple transformation of ${}_4F_3$ into ${}_4F_3$ mentioned in ([25], p. 140, Theorem 3.3.3), which follows from (52) when $m = 0$.

(5) Now, replacing $\varepsilon$ by $1 + \lambda + \xi + \nu - \delta - \gamma - N + m$ in (52) and then taking a limit as $N \longrightarrow \infty$, it leads us to an extended Thomae-type transformation due to Luo and Raina ([22], Equation (4.6)) see also Kim et al. ([23], p. 114, Theorem 2). At this point, it may be observed that ([22], Equation (4.6)) is equivalent to ([23], p. 114, Theorem 2).

## 5. Conclusion

This research paper clearly shows that how the series manipulation technique analogous to Joshi and Vyas [15] can effectively be applied to derive many new hypergeometric expansion formulas as discrete analogues for the Erdélyi type integrals. Further, the derived discrete analogues of this paper are also applied to provide the expansion formulas involving the Racah and the Askey–Wilson polynomials and a very important generalization of the Whipple transformation of ${}_4F_3$ into ${}_4F_3$ in the form of a transformation of ${}_{m+4}F_{m+3}$ into ${}_{r+4}F_{r+3}$. Moreover, it will not be out of place to observe here that Gasper [8] investigated the $q$-analogues and generalizations of the discrete analogues of the three integrals of Erdélyi to provide new expansion formulas for ${}_4F_3$ and ${}_4\Phi_3$ along with the applicability to orthogonal polynomials and some new double Kampé de Fériet summations, which in turn were further extended by Joshi and Vyas [11] in the form of ${}_{12}\Phi_{11}(q)$ and ${}_r\Phi_s(z)$ expansions, along with the applicability to biorthogonal rational functions. Some future research directions on this work can be the investigation of further generalizations of the expansion formulas of this paper along the lines of [8, 11]. A work along these directions is going on and in the foreseeable future, we may hope to submit these new results for publication.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

## References

[1] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, UK, 1966.

[2] H. M. Srivastava and P. W. Karlsson, "Multiple Gaussian hypergeometric series," *Ellis Horwood Series on Mathematics and Its Applications*, Halsted Press Ellis Horwood Limited, Chichester, UK, 1985.

[3] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester, UK, 1984.

[4] E. D. Rainville, *Special Functions*, Macmillan Company, New York, NY, USA, 1960.

[5] H. Bateman, "The solution of linear differential equations by means of definite integrals," *Trans. Cambridge Philos. Soc*, vol. 21, pp. 171–196, 1909.

[6] A. Erdélyi, "Transformation of hypergeometric integrals by means of fractional integration by parts," *The Quarterly Journal of Mathematics*, vol. 10, no. 1, pp. 176–189, 1939.

[7] G. Gasper, "Formulas of the dirichlet-mehler type," in *Fractional Calculus and its Applications, Proceedings of the International Conference*University of New Haven, West Haven, CT, USA, 1974.

[8] G. Gasper, "$q$-extensions of Erdélyi's fractional integral representations for hypergeometric functions," in *q-Series from a Contemporary Perspective: AMS-IMS-SIAM Joint Summer Research Conference on q-Series, Combinatorics, and Computer Algebra*American Mathematical Society, Providence, RI, USA, 1998.

[9] S. Lievens, "Expansion formulas for terminating balanced ${}_4F_3$ (1) series from the Biedenharn-Elliot identity for su (1;1)," *Journal of Computational and Applied Mathematics*, vol. 169, no. 2, pp. 419–430, 2003.

[10] R. Koekoek, A. Lesky Peter, and F. Swarttouw René, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Science and Business Media, Berlin, Germany, 2010.

[11] C. M. Joshi and Y. Vyas, "Extensions of two $q$-series expansions with application to biorthogonal rational functions," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 219–222, 2007.

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, NY, USA, 1953.

[13] H. Exton, *Handbook of Hypergeometric Integrals: Theory, Applications, Tables*, Wiley, Hoboken, NJ, USA, 1978.

[14] I. S. Gradshteyn and I. M. Ryzhik, "Table of integrals, series, and products," *Corrected and Enlarged*, Academic Press, California, MA, USA, 2014.

[15] C. M. Joshi and Y. Vyas, "Extensions of certain classical integrals of Erdélyi for Gauss hypergeometric functions," *Journal of Computational and Applied Mathematics*, vol. 160, no. 1-2, pp. 125–138, 2003.

[16] C. M. Joshi and Y. Vyas, "$q$-Extensions of certain Erdélyi type integrals," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 642–648, 2006.

[17] P. Orlik and H. Terao, *Arrangements and Hypergeometric Integrals*, Mathematical Society of Japan, Tokyo, Japan, 2001.

[18] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Gordon and Breach Publishers, Philadelphia, PA, USA, 1990.

[19] S. B. Yakubovich and Y. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, Springer, Berlin, Germany, 2012.

[20] C. M. Joshi and Y. Vyas, "Extensions of Bailey's transform and applications," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, Article ID 240386, 15 pages, 2005.

[21] C. M. Joshi and Y. Vyas, "Bailey type transforms and applications," *Jñānābha*, vol. 45, pp. 53–80, 2015.

[22] M. J. Luo and R. K. Raina, “Erdélyi-type integrals for generalized hypergeometric functions with integral parameter differences,” *Integral Transforms and Special Functions*, vol. 28, no. 6, pp. 476–487, 2017.
[23] Y. S. Kim, A. K. Rathie, and R. B. Paris, “An extension of Saalschütz’s summation theorems for the series ${}_{r+3}F_{r+2}$,” *Integral Transforms and Special Functions*, vol. 24, no. 11, pp. 916–921, 2013.
[24] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1994.
[25] G. E. Andrews, R. Askey, and R. Roy, *Special Functions. Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1999.
[26] Y. S. Kim, A. K. Rathie, and R. B. Paris, “On two Thomae-type transformations for hypergeometric series with integral parameter differences,” *Mathematical Communications*, vol. 19, no. 1, pp. 111–118, 2014.