Hindawi
Journal of Mathematics
Volume 2023, Article ID 9841465, 1 page
https://doi.org/10.1155/2023/9841465

Retraction

Retracted: New Types of μ-Proximity Spaces and Their Applications

Journal of Mathematics

Received 17 October 2023; Accepted 17 October 2023; Published 18 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. A. Hosny, T. M. Al-shami, and A. Mhemdi, "New Types of μ-Proximity Spaces and Their Applications," *Journal of Mathematics*, vol. 2022, Article ID 1657993, 10 pages, 2022.

Hindawi
Journal of Mathematics
Volume 2022, Article ID 1657993, 10 pages
https://doi.org/10.1155/2022/1657993

Research Article

New Types of μ-Proximity Spaces and Their Applications

Rodyna A. Hosny,[1] **Tareq M. Al-shami**,[2] **and Abdelwaheb Mhemdi**[3]

[1]*Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt*
[2]*Department of Mathematics, Sana'a University, Sana'a, Yemen*
[3]*Department of Mathematics, College of Sciences and Humanities in Aflaj Prince Sattam bin Abdulaziz University, Riyadh, Saudi Arabia*

Correspondence should be addressed to Tareq M. Al-shami; tareqalshami83@gmail.com

Received 21 October 2021; Accepted 20 December 2021; Published 17 January 2022

Academic Editor: Lazim Abdullah

Near set theory supplies a major basis for the perception, differentiation, and classification of elements in classes that depend on their closeness, either spatially or descriptively. This study aims to introduce a lot of concepts; one of them is μ-clusters as the useful notion in the study of μ-proximity (or μ-nearness) spaces which recognize some of its features. Also, other types of μ-proximity, termed $R\mu$-proximity and $O\mu$-proximity, on $\mathcal{X}$ are defined. In a μ-proximity space $(\mathcal{X}, \delta_\mu)$, for any subset K of $\mathcal{X}$, one can find out nonempty collections $\delta_\mu[K] = \{G \subseteq \mathcal{X} \mid K\overline{\delta}_\mu G\}$, which are hereditary classes on $\mathcal{X}$. Currently, descriptive near sets were presented as a tool of solving classification and pattern recognition problems emerging from disjoint sets; hence, a new approach to basic μ-proximity structures, which depend on the realization of the structures in the theory of hereditary classes, is introduced. Also, regarding to specific options of hereditary class operators, various kinds of μ-proximities can be distinguished.

1. Introduction

A proximity (or nearness) space is a sort of structured set, that consists of a nonempty set $\mathcal{X}$ and a binary relation between the subsets of $\mathcal{X}$. In constructive mathematics, any one of these relations may be possessed as major, and the others defined utilizing it; thence, we can differentiate, constructively, among a set-set nearness space "δ," a set-set apartness space "$\overline{\delta}$"(negation of δ or nonnear), and a set-set neighborhood space "$\ll$."

Initiatively, descriptive near sets have established to be valuable in an assortment of applications as topology [1, 2], solving a lot of problems that rely on human perception [3, 4] that arises in fields of image analysis [5], image processing [6], face recognition [7], rough set [8, 9], environmental space [10], and information systems [11–13], as well as science problems [14]. Also, Peters and Wasilewski [15] put in an approach to the foundations of information science which are formulated in the context of near sets.

The concept of proximity spaces was introduced by Naimpally and Warrack in [16]. A spatial nearness relation [1] δ is defined by $\delta = \{(A, B) \in P(\mathcal{X}) \times P(\mathcal{X}) \mid clA \cap clB \neq \varnothing\}$. It has ever after proven to be a valuable model in the rating of topological spaces. By introducing f-proximities that depends on certain functions f, Thron [17] created a framework for kinds of proximities. Generalized proximity structures have been widely investigated by several articles including [18, 19]. In [20, 21], several generalized proximities have been established utilizing Ef-proximity and ideals. In [22], Kandil et al. introduced an approach to proximity structures that depend on the recognition of many of the entities important in the theory of ideals. Also, they proposed the concept of g-proximities and showed that, for different choice of "g," one can obtain many of the known types of generalized proximities. Kandil et al. [23] presented approach of proximity and generalized proximity based on the soft sets. Also, they generalized the notions of compact, proximity relation and proximal neighborhood in the multiset context [24].

Many researchers have worked with weaker axioms than those of the fundamental concept ofEfremovic proximity space [25] enabling them to introduce an arbitrary topology on the underlying set with nice properties, and the theory possesses deep results, rich machinery, and tools. In 2019, Mukherjee et al. [26] constructed a generalized proximity structure, named μ-proximity on set $\mathcal{X}$, which induces a generalized topology(GT) on $\mathcal{X}$. Also, Yildirim [27] constructed a generalized μ–proximity structure by using hereditary class on a set. Császár studies attracted many researchers concentration, inducing their considerable studies which involve an extension of generalized topologies utilizing some specific sort of classes of sets called hereditary classes [28].

In this study, the concept of μ-clusters in the study of μ-proximity (or μ-nearness) spaces is presented and some of its features are investigated. As a generalization of [22], the theory of basic μ-proximities in terms of hereditary classes is developed. A new approach to basic μ-proximity structures, which depend on the realization of the structures in the theory of hereditary classes, is introduced. Also, regarding to specific options of hereditary class operators, various kinds of μ-proximities are distinguished.

2. Preliminaries

To outline this paper as self-sufficient as possible, we recall the next definitions and results which are due to different references.

Definition 1 (see [28]). A nonempty family $\mathcal{H}$ of subsets of $\mathcal{X}$ is called hereditary class if it is closed under subsets. The set of all hereditary classes on $\mathcal{X}$ is denoted by $\mathbb{H}$.

Definition 2 (see [26]). A binary relation δ_μ on the power set $P(\mathcal{X})$ of a set $\mathcal{X}$ is called a μ-proximity (μ-nearness) on $\mathcal{X}$ and $(\mathcal{X}, \delta_\mu)$ is a μ-proximity (μ-nearness) space if, for all $G, K \subseteq \mathcal{X}$, δ_μ satisfies the following axioms:

(1) $G\delta_\mu K \Rightarrow K\delta_\mu G$

(2) $G\delta_\mu K$, $G\subseteq U$, and $K\subseteq V \Rightarrow U\delta_\mu V$

(3) $\{x\}\delta_\mu\{x\}$, $\forall x \in \mathcal{X}$

(4) $G\overline{\delta}_\mu K \Rightarrow \exists E\subseteq\mathcal{X}$ s.t. $G\overline{\delta}_\mu E$, $(\mathcal{X}\backslash E)\overline{\delta}_\mu K$

A relation δ_μ on $\mathcal{X}$ is called a basic μ-proximity if it satisfies only conditions (1), (2), and (3). We denote by $\wp(\mathcal{X})$ the family of all basic μ-proximities on $\mathcal{X}$. Henceforth, we write $x\delta_\mu G$ for $\{x\}\delta_\mu G$.

Several properties of the relation δ_μ on $\mathcal{X}$ have been mentioned with details in [26].

Remark 1 (see [26]). A generalized topology μ is compatible with the μ-proximity relation of sets δ_μ, denoted $\mu_- \sim \delta_\mu$, $\tau_{\delta_\mu} = \mu_-$.

Definition 3 (see [16]). If δ_{μ_1} and δ_{μ_2} are two μ-proximities on a set $\mathcal{X}$, then δ_{μ_2} is called finer than δ_{μ_1} (in symbols $\delta_{\mu_1} \prec \delta_{\mu_2}$) if $G\delta_{\mu_2}K$ implies $G\delta_{\mu_1}K$.

Definition 4 (see [26]). A subset G of a μ-proximity space $(\mathcal{X}, \delta_\mu)$ is a δ_μ nbhood. of a set K if $K\overline{\delta}_\mu(\mathcal{X}\backslash G)$. The set of all δ_μ nbhood. of K with respect to δ_μ is denoted by $\mathcal{N}(\delta_\mu, K)$ or simply, $\mathcal{N}_\mu(K)$, i.e., $\mathcal{N}_\mu(K) = \{G \mid K\overline{\delta}_\mu(\mathcal{X}\backslash G)\}$.

Lemma 1 (see [26]). *For all subsets G, K of a basic μ-proximity space $(\mathcal{X}, \delta_\mu)$, then the following statements hold:*

(1) If $G\subseteq K$, then $\mathcal{N}_\mu(K)\subseteq\mathcal{N}_\mu(G)$

(2) $K \in \mathcal{N}_\mu(G)$ iff $(\mathcal{X}\backslash G) \in \mathcal{N}_\mu(\mathcal{X}\backslash K)$

(3) $\mathcal{N}_\mu(\varnothing) = P(\mathcal{X})$

Proposition 1 (see [26]). *Let $(\mathcal{X}, \delta_\mu)$ be a μ-proximity space and $\tau_{\delta_\mu} = \mu_-$. Then, the μ-closure $c_\mu(G)$ of a set G in $(\mathcal{X}, \mu)$ is given by $c_\mu(G) = \{x\colon x\delta_\mu G\}$.*

In the following proposition, one can deduce some useful properties of δ_μ.

Proposition 2. *Let $\mathcal{X}$ be a nonempty set. For each $x \in \mathcal{X}$ and $G, K\subseteq\mathcal{X}$,*

(1) $x\delta_\mu G$, $x\delta_\mu K \Rightarrow x\delta_\mu G\cup K$

(2) $x \in G \Rightarrow x\delta_\mu G$

(3) $x\delta_\mu G$, $y\delta_\mu K \forall y \in G \Rightarrow x\delta_\mu K$

(4) If there is a point $p \in \mathcal{X}$ s.t. $G\delta_\mu p$ and $p\delta_\mu K$, then $G\delta_\mu K$

3. On μ-Clusters

Let us consider the concept of μ-cluster from μ-proximity spaces and explore some of its properties.

Definition 5. Let $(\mathcal{X}, \delta_\mu)$ be a μ-proximity space and $G, K\subseteq\mathcal{X}$. A μ-cluster is a nonempty collection σ_μ of subsets of $\mathcal{X}$ s.t.

(1) If $G \in \sigma_\mu$, $K \in \sigma_\mu$, then $G\delta_\mu K$

(2) If $G\delta_\mu K$, for every $K \in \sigma_\mu$, then $G \in \sigma_\mu$

The family of all μ-clusters of $\mathcal{X}$ is denoted by $\aleph_\mu(\mathcal{X})$.

Theorem 1. *If $\sigma_{1\mu}, \sigma_{2\mu}$ are μ-clusters in a μ-proximity space $(\mathcal{X}, \delta_\mu)$ and if $\sigma_{2\mu}\subseteq\sigma_{1\mu}$, then $\sigma_{1\mu} = \sigma_{2\mu}$.*

Proof. Let $G \notin \sigma_{2\mu}$. Then, there exists $B \in \sigma_{2\mu}$ s.t. $B\overline{\delta}_\mu G$. Since $\sigma_{2\mu}\subseteq\sigma_{1\mu}$, then there exists $B \in \sigma_{1\mu}$ s.t. $B\overline{\delta}_\mu G$. By (1) of Definition 5, $G \notin \sigma_{1\mu}$. Hence, $\sigma_{1\mu}\subseteq\sigma_{2\mu}$, so $\sigma_{1\mu} = \sigma_{2\mu}$. □

Remark 2. If $\aleph_\mu(\mathcal{X})$ is a family of finite nested μ-clusters of $\mathcal{X}$, *i.e.*, $\sigma_{n\mu}\subseteq\cdots\subseteq\sigma_{2\mu}\subseteq\sigma_{1\mu}$, then $\aleph_\mu(\mathcal{X}) = \{\sigma_{1\mu}\}$.

In view of Definition 2 and Proposition 2, the following examples are given.

Lemma 2. *Let $(\mathcal{X}, \delta_\mu)$ be a μ-proximity space. A collection σ_μ in $\mathcal{X}$ is a μ-cluster iff all sets $G \in \sigma_\mu$ are near point x. In other words, the collection $\sigma_\mu(x) = \{G\subseteq\mathcal{X} \mid x\delta_\mu G\}$ is a μ-cluster. $\sigma_\mu(x)$ is called a principal μ-cluster or point μ-cluster.*

Remark 3. $G \in \sigma_\mu(x)$ iff $x \in c_\mu(G)$.

Example 1. If δ_μ is defined on $\mathcal{X}$, the family of all natural numbers, as $G\delta_\mu K$ iff both G and K are nonempty sets. Then, the collection $\sigma_\mu = \{G\subseteq\mathcal{X} \mid G \neq \varnothing\}$ is a μ-cluster. Also, $\sigma_\mu = \sigma_\mu(x)$, for every $x\in\mathcal{X}$.

Theorem 2. *If σ_μ is a μ-cluster in a μ-proximity space $(\mathcal{X},\delta_\mu)$, then*

(1) $\sigma_\mu(x)\neq\varnothing$

(2) $G\in\sigma_\mu$ *and* $G\subseteq K \Rightarrow K\in\sigma_\mu$

(3) $G\in\sigma_\mu$ *iff* $c_\mu(G)\in\sigma_\mu$

(4) *If there is a point* $x\in\mathcal{X}$ *s.t.* $\{x\}\in\sigma_\mu$, *then* $\sigma_\mu=\sigma_\mu(x)$

(5) $A\in\sigma_\mu$ *iff for all* $G,K\subseteq\mathcal{X}$, $G\in\sigma_\mu$ *and* $(\mathcal{X}\backslash A)\cup K=\mathcal{X}\Rightarrow G\delta_\mu K$

Proof

(1) Obvious.

(2) Let $G\in\sigma_\mu$ and $G\subseteq K$. Suppose $K\notin\sigma_\mu$; then, there exists $B\in\sigma_\mu$ s.t. $B\overline{\delta}_\mu K$. Since $G,B\in\sigma_\mu$, then from (2) of Definition 2 and (1) of Definition 5, $B\delta_\mu G$ and so $B\delta_\mu K$. This is a contradiction. Hence, $K\in\sigma_\mu$.

(3) If $G\in\sigma_\mu$, then, from (1), $c_\mu(G)\in\sigma_\mu$. In the other side, suppose $G\notin\sigma_\mu$. Then, $B\overline{\delta}_\mu G$, for some $B\in\sigma_\mu$. In view of Lemma 2.8 of [26], $c_\mu(B)\overline{\delta}_\mu c_\mu(G)$. So, $B\overline{\delta}_\mu c_\mu(G)$. Then, from (1) of Definition 5, $c_\mu(G)\notin\sigma_\mu$. Hence, the proof has been completed.

(4) Let $G\in\sigma_\mu$. Since $\{x\}\in\sigma_\mu$, then, by (1) of Definition 5, $x\delta_\mu G$. Hence, $G\in\sigma_\mu(x)$, i.e., $\sigma_\mu\subseteq\sigma_\mu(x)$. On the contrary, let $G\in\sigma_\mu(x)$; then, $x\delta_\mu G$. Suppose $G\notin\sigma_\mu$; then, $B\overline{\delta}_\mu G$ for some $B\in\sigma_\mu$. Since $\{x\},B\in\sigma_\mu$, then $x\delta_\mu B$. According to (4) of Proposition 2, $B\delta_\mu G$. This is a contradiction. Hence, $G\in\sigma_\mu$. Then, $\sigma_\mu(x)\subseteq\sigma_\mu$, so $\sigma_\mu=\sigma_\mu(x)$.

(5) ($\Rightarrow$) Let $A\in\sigma_\mu$ and $G\in\sigma_\mu$; then, $G\delta_\mu A$. Suppose $(\mathcal{X}\backslash A)\cup K=\mathcal{X}$; then, every element of A belongs to K. Hence, from (2), $K\in\sigma_\mu$, so $G\delta_\mu K$. ($\Leftarrow$) Assume that, for all $G,K\subseteq\mathcal{X}$, $G\in\sigma_\mu$ and $(\mathcal{X}\backslash A)\cup K=\mathcal{X}$ imply $G\delta_\mu K$. Choose $K=A$; then, $G\delta_\mu A$, for all $G\in\sigma_\mu$. So, $A\in\sigma_\mu$.

Now, we define a function α from $P(\mathcal{X})$ into a family $\aleph_\mu(\mathcal{X})$ of all μ-clusters of $\mathcal{X}$ by

$$\alpha(G)=\left\{\sigma_\mu\in\aleph_\mu(\mathcal{X}) \mid G\in\sigma_\mu\right\}. \tag{1}$$

□

Theorem 3.

(1) $\alpha(\varnothing)=\varnothing$ *and* $\alpha(\mathcal{X})=\aleph_\mu(\mathcal{X})$

(2) $G\subseteq K\Rightarrow\alpha(G)\subseteq\alpha(K)$

(3) $\alpha(G)\cup\alpha(K)\subseteq\alpha(G\cup K)$ *and* $\alpha(G\cap K)\subseteq\alpha(G)\cap\alpha(K)$

(4) $\alpha(G)\cap\alpha(K)\neq\varnothing$ *iff* $G\delta_\mu K$

Proof

(1) In view of (2) of Theorem 2, $\alpha(\varnothing)=\left\{\sigma_\mu\in\aleph_\mu(\mathcal{X}) \mid \varnothing\in\sigma_\mu\right\}=\varnothing$ and $\alpha(\mathcal{X})=\left\{\sigma_\mu\in\aleph_\mu(\mathcal{X}) \mid \mathcal{X}\in\sigma_\mu\right\}=\aleph_\mu(\mathcal{X})$.

(2) Let $\sigma_\mu\in\alpha(G)$; then, $G\in\sigma_\mu$. Since $G\subseteq K$, hence, in view of (2) of Theorem 2, $K\in\sigma_\mu$, so $\sigma_\mu\in\alpha(K)$, i.e., $\alpha(G)\subseteq\alpha(K)$.

(3) It is obvious from (2).

(4) ($\Rightarrow$) Suppose $\alpha(G)\cap\alpha(K)\neq\varnothing$; then, there exists $\sigma_\mu\in\alpha(G)\cap\alpha(K)$. Hence, $\sigma_\mu\in\alpha(G)$ and $\sigma_\mu\in\alpha(K)$ which imply that G and $K\in\sigma_\mu$. From Definition 5, $G\delta_\mu K$. ($\Leftarrow$) Assume that $G\delta_\mu K$ and $\alpha(G)\cap\alpha(K)=\varnothing$; then, $\sigma_\mu\notin\alpha(G)$ for every $\sigma_\mu\in\alpha(K)$. Hence, $G\delta_\mu K$, for every $K\in\sigma_\mu$ which imply that $G\in\sigma_\mu$. It is a contradiction.

Next, we shall introduce an appropriate proximity Φ on $\aleph_\mu(\mathcal{X})$. $\Lambda,\Gamma\subseteq\aleph_\mu(\mathcal{X})\Lambda\Phi\Gamma\Leftrightarrow\Lambda\subseteq\alpha(G)$ and $\Gamma\subseteq\alpha(K)$ imply $G\delta_\mu K$, for all $G,K\subseteq\mathcal{X}$. □

Theorem 4. *The structure* $(\aleph_\mu(\mathcal{X}),\Phi)$ *is a* μ*-proximity space.*

Proof

(1) $\Lambda\Phi\Gamma\Rightarrow\Gamma\Phi\Lambda$.

(2) Suppose $\Omega\overline{\Phi}\Upsilon$, $\Lambda\subseteq\Omega$, and $\Gamma\subseteq\Upsilon$; then, $\Omega\subseteq\alpha(G)$, $\Upsilon\subseteq\alpha(K)$, and $G\overline{\delta}_\mu K$, for some $G,K\subseteq\mathcal{X}$. From hypothesis $\Lambda\subseteq\Omega$ and $\Gamma\subseteq\Upsilon$, hence, $\Lambda\subseteq\alpha(G)$, $\Gamma\subseteq\alpha(K)$, and $G\overline{\delta}_\mu K$, for some $G,K\subseteq\mathcal{X}$. It follows that $\Lambda\overline{\Phi}\Gamma$. Consequently, $\Lambda\Phi\Gamma$, $\Lambda\subseteq\Omega$, and $\Gamma\subseteq\Upsilon$ imply $\Omega\Phi\Upsilon$

(3) Let σ be a μ-cluster of $\mathcal{X}$. If $\{\sigma\}\subseteq\alpha(G)$ and $\{\sigma\}\subseteq\alpha(K)$, then from (4) of Theorem 3, $G\delta_\mu K$, for all $G,K\subseteq\mathcal{X}$. Hence, $\{\sigma\}\Phi\{\sigma\}$.

(4) Suppose $\Lambda\overline{\Phi}\Gamma$; then, $\Lambda\subseteq\alpha(G)$, $\Gamma\subseteq\alpha(K)$, and $G\overline{\delta}_\mu K$, for some $G,K\subseteq\mathcal{X}$. □

Corollary 1. $\Lambda\cap\Gamma\neq\varnothing\Rightarrow\Lambda\Phi\Gamma$.

Proof. Let $\Lambda\cap\Gamma\neq\varnothing$; then, there exists σ s.t. $\sigma\in\Lambda$ and $\sigma\in\Gamma$. Suppose $\Lambda\overline{\Phi}\Gamma$; then, $\Lambda\subseteq\alpha(G)$, $\Gamma\subseteq\alpha(K)$, and $G\overline{\delta}_\mu K$, for some $G,K\subseteq\mathcal{X}$. Hence, $\sigma\subseteq\alpha(G)$ and $\sigma\subseteq\alpha(K)$. According to (4) of Theorem 3, $G\delta_\mu K$. It is a contradiction. So, $\Lambda\Phi\Gamma$. □

4. On μ-Proximity with Hereditary Class

In accordance with principal μ-clusters notion, we will turn to the concept of hereditary classes.

Definition 6. Let $(\mathcal{X},\delta_\mu)$ be a μ-proximity space and $x\in\mathcal{X}$; then, $\left\{G\subseteq\mathcal{X}\colon G\notin\sigma_\mu(x)\right\}$ is a hereditary class on $\mathcal{X}$.

Remark 4. Let $(\mathcal{X},\delta_\mu)$ be a μ -proximity space, $G\subseteq\mathcal{X}$ and $x\in\mathcal{X}$. Then,

(1) $\delta_\mu[x]=\left\{G\subseteq\mathcal{X}\colon x\overline{\delta}_\mu G\right\}$

(2) $G\in\delta_\mu[x]$ *iff* $x\in\delta_\mu[G]$

(3) $c_\mu(G)=\left\{x\in\mathcal{X} \mid x\notin\delta_\mu[G]\right\}$

In the next section, we introduce the notion $\delta_\mu[K]$ for any subset K of $\mathcal{X}$ as a generalization of $\delta_\mu[x]$ for any $x\in\mathcal{X}$.

Definition 7. Let $(\mathscr{X}, \delta_\mu)$ be a μ-proximity space and $k \subseteq \mathscr{X}$; then, we define

$$\delta_\mu[K] = \left\{G \subseteq \mathscr{X} \mid K \overline{\delta}_\mu G\right\}. \quad (2)$$

Next example shows that $\delta_\mu[K]$ is not an ideal on $\mathscr{X}$, for any set $K \subseteq \mathscr{X}$.

Example 2. Let $\mathscr{X} = \{a, b, c\}$ and let δ_μ be a μ-proximity on $\mathscr{X}$ defined as $G \delta_\mu K \Leftrightarrow G = K$.

If $G_1 = \{a\}, G_2 = \{b\}$, and $K = \{a, b\}$, then $G_1, G_2 \in \delta_\mu[K]$ but $G_1 \cup G_2 \notin \delta_\mu[K]$.

Example 3. Evidently, $\delta_\mu[\varnothing] = P(\mathscr{X})$ and $\delta_\mu[\mathscr{X}] = \{\varnothing\}$.

Next, we reformulate Definition 2 in terms of $\delta_\mu[.]$ as follows:

Definition 8. A binary relation δ_μ on $P(\mathscr{X})$ is called a μ-proximity on $\mathscr{X}$ if, for all $G, K \subseteq \mathscr{X}$, δ_μ satisfies the following axioms:

(1) $G \in \delta_\mu[K] \Rightarrow K \in \delta_\mu[G]$
(2) $U \in \delta_\mu[V]$, $G \subseteq U$, and $K \subseteq V \Rightarrow G \in \delta_\mu[K]$
(3) $\{x\} \notin \delta_\mu[\{x\}]$, $\forall x \in \mathscr{X}$
(4) $G \in \delta_\mu[K] \Rightarrow$ there exists $E \subseteq \mathscr{X}$ s.t. $G \in \delta_\mu[\mathscr{X} \backslash E]$ and $E \in \delta_\mu[K]$

A relation δ_μ is called a basic μ-proximity if it satisfies only conditions (1), (2), and (3). We write $x \in \delta_\mu[K]$ for $\{x\} \in \delta_\mu[K]$ and $\delta_\mu[x]$ for $\delta_\mu[\{x\}]$.

It is clear that (2) in the axiomatical definition of μ-proximity relation δ_μ can be equivalently replaced by $(2)^* \delta_\mu[G \cup K] \subseteq \delta_\mu[G] \cap \delta_\mu[K]$, for every $G, K \subseteq \mathscr{X}$.

In the following, we will display considerable of the properties of $\delta_\mu[.]$.

From Definition 7, the next lemmas follow directly.

Lemma 3. *Let $(\mathscr{X}, \delta_\mu)$ be a μ -proximity space. Then,*

(1) $G_1 \cup G_2 \in \delta_\mu[K] \Rightarrow G_1 \in \delta_\mu[K]$ and $G_2 \in \delta_\mu[K] \Rightarrow G_1 \cap G_2 \in \delta_\mu[K]$
(2) $G_r \in \delta_\mu[K]$, $r \in \Lambda \Rightarrow \cap_{r \in \Lambda} G_r \in \delta_\mu[K]$
(3) $G \in \delta_\mu[K] \Rightarrow x \in \delta_\mu[K] \forall x \in G$
(4) $\mathscr{X} \notin \delta_\mu[x]$ and $x \notin \delta_\mu[\mathscr{X}]$

Lemma 4. *For all subsets G, K of a μ-proximity space $(\mathscr{X}, \delta_\mu)$, the following statements hold:*

(1) $G \subseteq K \Rightarrow \delta_\mu[K] \subseteq \delta_\mu[G]$
(2) $\delta_\mu[G] \cap \delta_\mu[K] \subseteq \delta_\mu[G \cap K]$
(3) $A \in \delta_\mu[G]$, $B \in \delta_\mu[K] \Rightarrow A \cap B \in \delta_\mu[G \cap K]$

Regarding to hereditary classes on $\mathscr{X}$, one can introduce μ-proximity relations on $\mathscr{X}$ as we show in the following examples.

Example 4. Let $\mathscr{X}$ be a set with any hereditary class $\mathscr{H}$ and $\{x\} \notin \mathscr{H}$. For any subsets G and K of $\mathscr{X}$, we define

$$G \in \delta_\mu[K] \Leftrightarrow G \cap K \in \mathscr{H}. \quad (3)$$

Then, the relation δ_μ is a μ-proximity on $\mathscr{X}$.

Example 5. Let $\mathscr{X}$ be a set with any hereditary class $\mathscr{H}$ and $\{x\} \notin \mathscr{H}$. For any subsets G and K of $\mathscr{X}$, define $G \in \delta_\mu[K] \Leftrightarrow G$ or $K \in \mathscr{H}$.

Then, the relation δ_μ is a μ-proximity on $\mathscr{X}$.

Theorem 5. *Let $(\mathscr{X}, \delta_\mu)$ be a μ-proximity space and $G, K \subseteq \mathscr{X}$. If μ-closures and μ-interiors are taken with respect to $\mu - = \tau_{\delta_\mu}$, then the following properties are true:*

(1) $G \in \delta_\mu[K]$ implies $G \cap K = \varnothing$
(2) $G \in \delta_\mu[K]$ iff $(\mathscr{X} \backslash G) \in \mathscr{N}_\mu(K)$
(3) $K is \delta_\mu$ closed WRT $\mu - = \tau_{\delta_\mu}$ if $x \in \delta_\mu[K]$, $\forall x \notin K$
(4) $K \in \tau_{\delta_\mu}$ WRT $\mu - = \tau_{\delta_\mu}$ if $x \in \delta_\mu[\mathscr{X} \backslash K]$, $\forall x \in K$
(5) $G \in \delta_\mu[K]$ implies $c_\mu(G) \in \delta_\mu[K]$ and $G \in \delta_\mu[c_\mu(K)]$
(6) $G \in \delta_\mu[K]$ iff $c_\mu(G) \in \delta_\mu[c_\mu(K)]$
(7) $x \in \delta_\mu[K]$ iff $c_\mu(\{x\}) \in \delta_\mu[c_\mu(K)]$

Proof. Direct to prove. □

Lemma 5. *Let $\delta_{\mu_1}, \delta_{\mu_2}$ be two μ-proximities on a set $\mathscr{X}$. Then, $\delta_{\mu_1} \prec \delta_{\mu_2}$ iff $\delta_{\mu 1}[G] \subseteq \delta_{\mu 2}[G]$, $\forall G \subseteq \mathscr{X}$.*

Proof. Accessible consequence of Definition 3. □

Theorem 6. *Let μ_1 and μ_2 be two μ-completely regular generalized topologies on $\mathscr{X}$ and δ_{μ_1} and δ_{μ_2} be the μ-proximities on $\mathscr{X}$ defined as $G \in \delta_{\mu_j}[K] \Leftrightarrow G$ and K are functionally distinguishable WRT μ_j, respectively, $j = 1, 2$.*

Then, $\mu_1 \subseteq \mu_2$ implies $\delta_{\mu_1} \prec \delta_{\mu_2}$

Proof. If $G \in \delta_{\mu_1}[K]$, then there exists a $\mu_1$continuous function $f\colon (\mathscr{X}, \mu_1) \longrightarrow [0, 1]$, where $[0, 1]$ is endowed with the subspace generalized topology induced by κ on R (where κ is the generalized topology on the set R of reals generated by the base $\beta = \{(-\infty, p)\colon p \in R\} \cup \{(p, \infty)\colon p \in R\})$ s.t. $f(G) = \{0\}$ and $f(K) = \{1\}$. Since $\mu_1 \subseteq \mu_2$, then f is a μ_2 continuous function from $(\mathscr{X}, \mu_2)$ to $[0, 1]$ s.t. $f(G) = \{0\}$ and $f(K) = \{1\}$. So, $G \in \delta_{\mu_2}[K]$. According to Lemma 5, $\delta_{\mu_1} \prec \delta_{\mu_2}$. □

Theorem 7. *Let $\delta_{\mu_1}, \delta_{\mu_2}$ be two μ-proximities on a set $\mathscr{X}$. Then, the following statements are equivalent:*

(1) $\delta_{\mu 1}[x] = \delta_{\mu 2}[x]$, $\forall x \in \mathscr{X}$
(2) $c_{\delta_{\mu 1}}(G) = c_{\delta_{\mu 2}}(G)$, $\forall G \subseteq \mathscr{X}$
(3) $\mathscr{N}_{\delta_{\mu 1}}(\{x\}) = \mathscr{N}_{\delta_{\mu 2}}(\{x\})$, $\forall x \in \mathscr{X}$

Proof. Easy to prove. □

Definition 9. A μ-proximity space $(\mathscr{X}, \delta_\mu)$ is T_0 iff for any two distinct points x, y of $\mathscr{X}$, $x \overline{\delta}_\mu y$.

Utilizing hereditary classes, another equivalent definition of T_0-space is obtained.

Theorem 8. *A μ-proximity space $(\mathcal{X}, \delta_\mu)$ is T_0 iff for any two distinct points x, y of X, $\delta_\mu[x] \neq \delta_\mu[y]$.*

Proof. Let x, y be any two distinct points in a T_0-space $(\mathcal{X}, \delta_\mu)$; then, $x\overline{\delta}_\mu y$. In view of Definition 5, $y \notin \sigma_\mu(x)$ or $x \notin \sigma_\mu(y)$. Suppose $y \notin \sigma_\mu(x)$, which gives $y \in \delta_\mu[x]$, but from (3) of Definition 2$y \notin \delta_\mu[y]$. Consequently, $\delta_\mu[x] \neq \delta_\mu[y]$. Conversely, let $x, y \in \mathcal{X}$ and $x \neq y$ with $x\delta_\mu y$. Suppose that $\delta_\mu[x] \neq \delta_\mu[y]$; then, there is a subset G of $\mathcal{X}$ s.t. $G \in \delta_\mu[x]$ and $G \notin \delta_\mu[y]$. Then, $G \notin \sigma_\mu(x)$ and $G \in \sigma_\mu(y)$. Hence, $x\overline{\delta}_\mu G$ but $y\delta_\mu G$. Since $x\delta_\mu y$ and $y\delta_\mu G$, then in view of (3) of Proposition 2, $x\delta_\mu G$. It is a contradiction. Thus, $(\mathcal{X}, \delta_\mu)$ is T_0-space. □

Lemma 6. *Let $(\mathcal{X}, \mu)$ be a μ-normal GTS. Then, $c_\mu(G) \cap c_\mu(K) = \varnothing \Leftrightarrow c_\mu(G)$, $c_\mu(K)$ are functionally distinguishable.*

Proof. Suppose $c_\mu(G) \cap c_\mu(K) = \varnothing$; then, by Urysohn's lemma, $c_\mu(G)$ and $c_\mu(K)$ are functionally distinguishable. In the other side, suppose $c_\mu(G) \cap c_\mu(K) \neq \varnothing$. Then, there exists $p \in \mathcal{X}$ s.t. $p \in c_\mu(G) \cap c_\mu(K)$. Since there is no function f s.t. $f(p)$ has distinct values at p; hence, $c_\mu(G)$ and $c_\mu(K)$ are not functionally distinguishable. □

Theorem 9. *Let $(\mathcal{X}, \mu)$ be a μ-normal GTS. For any subsets G and $K$of$\mathcal{X}$, the relation δ_μ on $\mathcal{X}$ given by $G \in \delta_\mu[K] \Leftrightarrow c_\mu(G) \cap c_\mu(K) = \varnothing$ is a compatible μ-proximity on $\mathcal{X}$.*

Proof. According to Lemma 6, $c_\mu(G) \cap c_\mu(K) = \varnothing$ iff $c_\mu(G)$ and $c_\mu(K)$ are functionally distinguishable. From the features of a μ-continuous function, G and K are functionally distinguishable iff $c_\mu(G)$ and $c_\mu(K)$ are functionally distinguishable. So, $G \in \delta_\mu[K]$ iff $c_\mu(G) \cap c_\mu(K) = \varnothing$ iff G and K are functionally distinguishable. By Urysohn's lemma, every μ-normal GTS is μ-completely regular; then, from Theorem 2.11 of [26], the relation δ_μ is a compatible μ-proximity on $\mathcal{X}$. □

Theorem 10. *If $(\mathcal{X}, \mu)$ is a μ-completely regular, GTS has a compatib1e μ-proximity δ_μ defined by*

$$G \in \delta_\mu[K] \Leftrightarrow c_\mu(G) \cap c_\mu(K) = \varnothing \tag{4}$$

Then, $(\mathcal{X}, \mu)$ is μ-normal GTS.

Proof. Suppose $c_\mu(G), c_\mu(K)$ are disjoint μ-closed sets; then, $G \in \delta_\mu[K]$. Hence, there exists $E \subseteq \mathcal{X}$ s.t. $G \in \delta_\mu[\mathcal{X} \backslash E]$ and $E \in \delta_\mu[K]$. From Corollary 2.7 of [26] and Definition 7, $G \subseteq \mathcal{X} \backslash c_\mu(\mathcal{X} \backslash E) = i_\mu(E)$ and $K \subseteq i_\mu(\mathcal{X} E)$. Since $i_\mu(E)$ and $i_\mu(\mathcal{X} \backslash E)$ are disjoint μ-open sets, so $(\mathcal{X}, \mu)$ is μ-normal. □

5. On Basic μ-Proximity with Hereditary Class

Definition 10. A relation δ_μ is called $O\mu$-proximity on $\mathcal{X}$ if it is a basic μ-proximity on $\mathcal{X}$, and it satisfies the following condition:

$$G\delta_\mu A \text{ and } a\delta_\mu K \forall a \in A \Rightarrow G\delta_\mu K. \tag{5}$$

Example 6. In one of the schools, suppose that G, A, K be parents' council, set of students, and set of teachers, respectively. Evidently, δ_μ satisfies $O\mu$-proximity axioms on $\mathcal{X}$, see Figures 1 and 2.

Definition 11. A relation δ_μ is called $R\mu$-proximity on $\mathcal{X}$ if it is a basic μ-proximity on $\mathcal{X}$, and it satisfies the following condition:

$$x\overline{\delta}_\mu K \Rightarrow \text{ there exists } E \subseteq \mathcal{X} \text{ s.t. } x\overline{\delta}_\mu E \text{ and } (\mathcal{X} E)\overline{\delta}_\mu K. \tag{6}$$

Theorem 11. *Let δ_μ be a basic μ -proximity on $\mathcal{X}$. Then, $c_\mu(K) = \cap\{A \mid (\mathcal{X} \backslash A) \in \delta_\mu[K]\}$.*

Proof. Suppose that $L = \cap\{A \mid (\mathcal{X} \backslash A) \in \delta_\mu[K]\}$. We shall prove $L = c_\mu(K)$. Let $x \notin c_\mu(K)$; then, $x \in \delta_\mu[K]$. Hence, $x \notin L$, so $L \subseteq c_\mu(K)$. On the contrary, let $x \notin L$; then, there is a subset A of $\mathcal{X}$ s.t. $x \notin A$ and $(\mathcal{X} \backslash A) \in \delta_\mu[K]$. According to (1) of Lemma 3, $x \in \delta_\mu[K]$. Thus, $x \notin c_\mu(K)$, so $c_\mu(K) \subseteq L$. Hence, $L = c_\mu(K)$ □

Theorem 12. *Let $\delta_\mu \in \wp(\mathcal{X})$. Then, the following are equivalent:*

(1) δ_μ is a μ-proximity on $\mathcal{X}$

(2) If$G \in \delta_\mu[K]$, then $\mathcal{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$

(3) If$G \in \mathcal{N}_\mu(K)$, then there exists $E \in \mathcal{N}_\mu(K)$s.t. $G \in \mathcal{N}_\mu(E)$

Proof

$(1) \Rightarrow (2)$ Let $G \in \delta_\mu[K]$. In view of (1) and (4) of Definition 7, there exists $E \subseteq \mathcal{X}$ s.t. $(\mathcal{X} \backslash E) \in \delta_\mu[G]$ and $E \in \delta_\mu[K]$. Hence, by (2) of Theorem 5, $E \in \mathcal{N}_\mu(G) \cap \delta_\mu[K]$. Consequently, $\mathcal{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$.

$(2) \Rightarrow (3)$ Let $G \in \mathcal{N}_\mu(K)$; then, $(\mathcal{X} \backslash G) \in \delta_\mu[K]$. Hence, $\mathcal{N}_\mu(\mathcal{X} \backslash G) \cap \delta_\mu[K] \neq \varnothing$ which implies that there exists a set A s.t. $A \in \mathcal{N}_\mu(\mathcal{X} \backslash G)$ and $A \in \delta_\mu[K]$. In view of (2) of Lemma 1 and (2) of Theorem 5, $G \in \mathcal{N}_\mu(\mathcal{X} \backslash A)$ and $(\mathcal{X} \backslash A) \in \mathcal{N}_\mu(K)$. Put $(\mathcal{X}/A)$; then, (3) holds.

$(3) \Rightarrow (1)$ Let $G \in \delta_\mu[K]$; then, $(\mathcal{X} \backslash G) \in \mathcal{N}_\mu(K)$. According to (3), then there exists $E \in \mathcal{N}_\mu(K)$ s.t. $(\mathcal{X} \backslash G) \in \mathcal{N}_\mu(E)$. Therefore, $G \in \delta_\mu[E]$ and $(\mathcal{X} \backslash E) \in \delta_\mu[K]$, so δ_μ is a μ-proximity on $\mathcal{X}$. □

Theorem 13. *Let $\delta_\mu \in \wp(\mathcal{X})$. Then, the following are equivalent:*

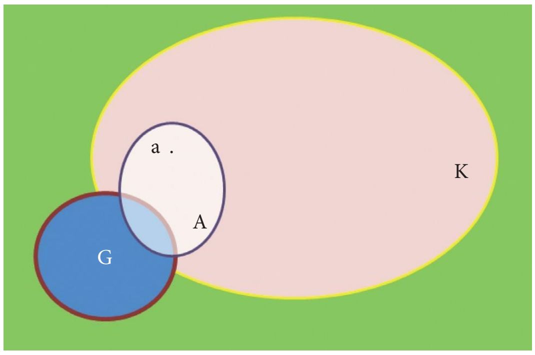

FIGURE 1: $G\delta_\mu A$ and $a\delta_\mu K \forall a \in A \Rightarrow G\delta_\mu K$.

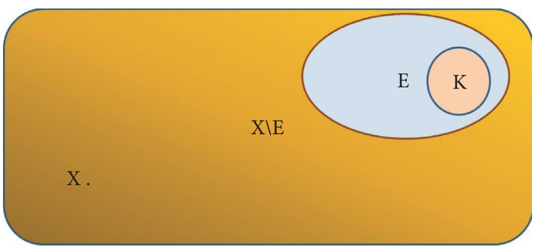

FIGURE 2: $x\bar{\delta}_\mu K \Rightarrow x\bar{\delta}_\mu E$ and $(\mathscr{X} \setminus E)\bar{\delta}_\mu K$, for some $E \subseteq \mathscr{X}$.

(1) δ_μ *is a* $R\mu$*-proximity on* $\mathscr{X}$

(2) *If* $x \in \delta_\mu[K]$, *then* $\mathscr{N}_\mu(x) \cap \delta_\mu[K] \neq \varnothing$

(3) *If* $K \in \mathscr{N}_\mu(x)$, *then there exists* $E \in \mathscr{N}_\mu(x)$ *s.t.* $K \in \mathscr{N}_\mu(E)$

Proof

(1) $\Rightarrow$ (2) Let $x \in \delta_\mu[K]$. From Definition 11, there exists $E \subseteq \mathscr{X}$ s.t. $x \in \delta_\mu[\mathscr{X} \backslash E]$ and $E \in \delta_\mu[K]$. So, $(\mathscr{X} \backslash E) \in \delta_\mu[x]$ and $E \in \delta_\mu[K]$. Hence, $E \in \mathscr{N}_\mu(x) \cap \delta_\mu[K]$. Consequently, $\mathscr{N}_\mu(x) \cap \delta_\mu[K] \neq \varnothing$.

(2) $\Rightarrow$ (3) Let $K \in \mathscr{N}_\mu(x)$; then, $(\mathscr{X} \backslash K) \in \delta_\mu[x]$. Hence, $\mathscr{N}_\mu(\mathscr{X} \backslash K) \cap \delta_\mu[x] \neq \varnothing$ which implies that there exists a set A of $\mathscr{X}$ s.t. $A \in \mathscr{N}_\mu(\mathscr{X} \backslash K)$ and $A \in \delta_\mu[x]$. In view of (2) of Lemma 1 and (2) of Theorem 5, $K \in \mathscr{N}_\mu(\mathscr{X} \backslash A)$ and $(\mathscr{X} \backslash A) \in \mathscr{N}_\mu(x)$. Put $(\mathscr{X}/A)$; then, (3) holds.

(3) $\Rightarrow$ (1) Let $x \in \delta_\mu[K]$; then, $(\mathscr{X} \backslash K) \in \mathscr{N}_\mu(x)$. According to (3), then there exists $E \in \mathscr{N}_\mu(x)$ s.t. $(\mathscr{X} \backslash K) \in \mathscr{N}_\mu(E)$. Therefore, $K \in \delta_\mu[E]$ and $(\mathscr{X} \backslash E) \in \delta_\mu[x]$, so δ_μ is a $R\mu$-proximity on $\mathscr{X}$. □

Definition 12. Let $\mathscr{H}$ be a hereditary class on a basic μ-proximity space $(\mathscr{X}, \delta_\mu)$. A mapping $\lambda: \wp(\mathscr{X}) \times \mathbb{H} \longrightarrow \mathbb{H}$ is called a hereditary class operator on $\mathscr{X}$ if it identifies to each pair $(\delta_\mu, \mathscr{H})$, a hereditary class $\lambda(\delta_\mu, \mathscr{H})$ on $\mathscr{X}$, satisfying the following conditions: $\lambda(\delta_\mu, \mathscr{H}_1) \subseteq \lambda(\delta_\mu, \mathscr{H}_2)$ whenever $\mathscr{H}_1 \subseteq \mathscr{H}_2$, for every $\mathscr{H}_1$, $\mathscr{H}_2 \in \mathbb{H}$.

Definition 13. Let λ be a hereditary class operator on $\mathscr{X}$. Then, a basic μ-proximity δ_μ on $\mathscr{X}$ is called a λ-μ-proximity if, for every $G \subseteq \mathscr{X}$, $\delta_\mu[G] \subseteq \lambda(\delta_\mu, \delta_\mu[G])$. The family of all λ-μ-proximities is denoted by $\mu\mathscr{P}_\lambda$.

In the next definition, several kinds of hereditary class operators are listed.

Definition 14. For a set $\mathscr{X}$, for all $\delta_\mu \in \wp(\mathscr{X})$ and $\mathscr{H} \in \mathbb{H}$, we define

(1) $e(\delta_\mu, \mathscr{H}) = \mathscr{H}$

(2) $\lambda_0(\delta_\mu, \mathscr{H}) = \{G \subseteq \mathscr{X} \mid \mathscr{N}_\mu(G) \cap \mathscr{H} \neq \varnothing\}$

(3) $\lambda_1(\delta_\mu, \mathscr{H}) = \{G \subseteq \mathscr{X} \mid c_{\delta_\mu}(G) \in \mathscr{H}\}$

(4) $\lambda_2(\delta_\mu, \mathscr{H}) = \{G \subseteq \mathscr{X} \mid \{x\} \in \delta_\mu[G] \cup \mathscr{H}, \forall x \in \mathscr{X}\}$

(5) $\lambda_3(\delta_\mu, \mathscr{H}) = \{G \subseteq \mathscr{X} \mid \mathscr{N}_\mu(\{a\}) \cap \mathscr{H} \neq \varnothing, \forall a \in G\}$

When there is no ambiguity, we will write λ_i for $\lambda_i(\delta_\mu, \mathscr{H})$), where $i = 0, 1, 2, 3$.

Theorem 14. *For all* $\delta_\mu \in \wp(\mathscr{X})$ *and* $\mathscr{H} \in \mathbb{H}$ *and for* $\lambda \in \{e, \lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, *we have that* λ *is a hereditary class operator on* $\mathscr{X}$.

Proof

(1) It is understandable; $e(\delta_\mu, \mathscr{H}) = \mathscr{H}$ is a hereditary class operator on $\mathscr{X}$.

(2) Suppose that $\delta_\mu \in \wp(\mathscr{X})$ and $\mathscr{H} \in \mathbb{H}$. Let $G \in \lambda_0 = \lambda_0(\delta_\mu, \mathscr{H})$; then, $\mathscr{N}_\mu(G) \cap \mathscr{H} \neq \varnothing$. If $A \subseteq G$, then according to Lemma 1, $\mathscr{N}_\mu(A) \cap \mathscr{H} \neq \varnothing$ and so $A \in \lambda_0$. Hence, λ_0 is a hereditary class. Now, let $\mathscr{H}_1 \subseteq \mathscr{H}_2$ and $G \in \lambda_0(\delta_\mu, \mathscr{H}_1)$; then, $\mathscr{N}_\mu(G) \cap \mathscr{H}_1 \neq \varnothing$. Therefore, $\mathscr{N}_\mu(G) \cap \mathscr{H}_2 \neq \varnothing$ and so $G \in \lambda_0(\delta_\mu, \mathscr{H}_2)$. Consequently, λ_0 is a hereditary class operator on $\mathscr{X}$.

(3) By using $\delta_\mu$closure operator properties, λ_1 is a hereditary class operator on $\mathscr{X}$.

(4) In view of Lemma 4 (1), λ_2 is a hereditary class operator on $\mathscr{X}$.

(5) By similar manner, λ_3 is a hereditary class operator on $\mathscr{X}$. □

Theorem 15. *Let* λ *be a hereditary class operator. If* $I = \{\lambda \mid \lambda(\delta_\mu, \cap_{r\in\Lambda}\mathscr{H}_r) = \cap_{r\in\Lambda}\lambda (\delta_\mu, \mathscr{H}_r), \delta_\mu \in \wp(\mathscr{X}), \mathscr{H}_r \in \mathbb{H}, \mathscr{H}_r \in \mathbb{H}, r \in \Lambda\}$, *thene,* $\lambda_1, \lambda_2 \in I$.

Proof. Straightforward. □

Corollary 2. *Let* λ *be a hereditary class operator. If* $\tilde{I} = \{\lambda \mid \lambda(\delta_\mu, \mathscr{H}_1 \cap \mathscr{H}_2) = \lambda(\delta_\mu, \mathscr{H}_1) \cap \lambda(\delta_\mu, \mathscr{H}_2), \delta_\mu \in \wp(\mathscr{X}), \mathscr{H}_1, \mathscr{H}_2 \in \mathbb{H}\}$, *thene,* $\lambda_1, \lambda_2 \in \tilde{I}$

Proof. The proof is obvious by using $I \subseteq \tilde{I}$. □

Remark 5. The following example illustrates that $\lambda_0, \lambda_3 \notin \widetilde{I}$, in general, if $\mathscr{H}_1, \mathscr{H}_2$ are hereditary classes.

Example 7. In Example 2, suppose $\mathscr{H}_1 = \{\varnothing, \{a\}\}$ and $\mathscr{H}_2 = \{\varnothing, \{b\}\}$. Then, $\lambda_0(\delta_\mu, \mathscr{H}_1 \cap \mathscr{H}_2) \neq \lambda_0(\delta_\mu, \mathscr{H}_1) \cap \lambda_0(\delta_\mu, \mathscr{H}_2)$ and $\lambda_3(\delta_\mu, \mathscr{H}_1 \cap \mathscr{H}_2) \neq \lambda_3(\delta_\mu, \mathscr{H}_1) \cap \lambda_3(\delta_\mu, \mathscr{H}_2)$. Hence, $\lambda_0, \lambda_3 \notin \widetilde{I}$

Theorem 16. *Letλ be a hereditary class operator. If* $T = \left\{\lambda \mid \lambda(\delta_{\mu_1}, \mathscr{H}) = \lambda(\delta_{\mu_2}\mathscr{H}) \text{with}\, c_{\delta_{\mu_1}}(.) = c_{\delta_{\mu_2}}(.), \delta_{\mu_1}, \delta_{\mu_2} \in \wp(\mathscr{X}), \mathscr{H} \in \mathbb{H}\right\}$, *thene,*$\lambda_1$,$\lambda_2$,$\lambda_3 \in T$.

Proof. We shall prove only for λ_2 and the rest of the proof is similar. Let $G \in \lambda_2(\delta_{\mu_1}, \mathscr{H})$; then, $\{x\} \in \delta_{\mu_1}[G] \cup \mathscr{H}, \forall x \in \mathscr{X}$. Hence, $G \in \delta_{\mu_1}[x]$. Since $c_{\delta_{\mu_1}}(.) = c_{\delta_{\mu_2}}(.)$, then, by using Theorem 7, $\delta_{\mu_1}[x] = \delta_{\mu_2}[x]$. Consequently, $G \in \delta_{\mu_2}[x]$ and so $G \in \lambda_2(\delta_{\mu_2}, \mathscr{H})$, i.e., $\lambda_2(\delta_{\mu_1}, \mathscr{H}) \subseteq \lambda_2(\delta_{\mu_2}, \mathscr{H})$. By the same manner, we can prove $\lambda_2(\delta_{\mu_2}, \mathscr{H}) \subseteq \lambda_2(\delta_{\mu_1}, \mathscr{H})$. It follows that $\lambda_2 \in T$.

From Definition 3, one can deduce the following results. □

Lemma 7. *Let $\delta_{\mu_1}, \delta_{\mu_2}$ be two μ-proximities on a set $\mathscr{X}$ and $G \subseteq \mathscr{X}$. If$\delta_{\mu_1} \prec \delta_{\mu_2}$, then*

(1) $c_{\delta_{\mu_2}}(G) \subseteq c_{\delta_{\mu_1}}(G)$
(2) $\mathscr{N}_{\mu_1}(G) \subseteq \mathscr{N}_{\mu_2}(G)$

Theorem 17. *Let λ be a hereditary class operator. If* $U = \left\{\lambda \mid \lambda(\delta_{\mu_1}, \mathscr{H}) \subseteq \lambda(\delta_{\mu_2}\mathscr{H}) \text{whenever} \delta_{\mu_1} \prec \delta_{\mu_2}, \mathscr{H} \in \mathbb{H}\right\}$, *then e,*$\lambda_0$,$\lambda_1$,$\lambda_2$,$\lambda_3 \in U$.

Proof. Let $G \in \lambda_0(\delta_{\mu_1}, \mathscr{H})$; then, $\mathscr{N}_{\mu_1}(G) \cap \mathscr{H} \neq \varnothing$. Since $\delta_{\mu_1} \prec \delta_{\mu_2}$, hence by Lemma 7 (2), $\mathscr{N}_{\mu_1}(G) \subseteq \mathscr{N}_{\mu_2}(G)$. Consequently, $\mathscr{N}_{\mu_2}(G) \cap \mathscr{H} \neq \varnothing$. So, $G \in \lambda_0(\delta_{\mu_2}, \mathscr{H})$, i.e., $\lambda_2(\delta_{\mu_1}, \mathscr{H}) \subseteq \lambda_2(\delta_{\mu_2}, \mathscr{H})$. It follows that $\lambda_2 \in U$. The rest of the proof is similar. □

Theorem 18. *$(\mathscr{X}, \delta_\mu)$ is a λ_0 - μ -proximity space iff $\mathscr{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$, for every $G \in \delta_\mu[K]$.*

Proof. $(\Rightarrow)$ Let $G \in \delta_\mu[K]$. Since δ_μ is a λ_0-μ-proximity on $\mathscr{X}$ and $\delta_\mu[K]$ is a hereditary class on $\mathscr{X}$, then $G \in \lambda_0(\delta_\mu, \delta_\mu[K])$. Hence, by Theorem 12, $\mathscr{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$. $(\Leftarrow)$ Let $G \in \delta_\mu[K]$; then, $\mathscr{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$, which implies that $G \in \lambda_0(\delta_\mu, \delta_\mu[K])$. Consequently, $\delta_\mu[K] \subseteq \lambda_0(\delta_\mu, \delta_\mu[K])$, so δ_μ is λ_0-μ-proximity.

In view of Theorems 12 and 18, the next corollary is verified. □

Corollary 3. *δ_μ is a μ -proximity on $\mathscr{X}$ iff it isλ_0-μ-proximity.*

Theorem 19. *Let $\delta_\mu \in \wp(\mathscr{X})$ and $\mathscr{H} \in \mathbb{H}$; then,* $\lambda_0\delta_\mu, \cup_{K \in \mathscr{H}} \delta_\mu[\mathscr{X} \backslash K]$.

Proof. Suppose that $G \in \lambda_0(\delta_\mu, \mathscr{H})$. Then, $\mathscr{N}_\mu(G) \cap \mathscr{H} \neq \varnothing$. So, there exists $K \in \mathscr{H}$ s.t. $K \in \mathscr{N}_\mu(G)$, i.e., $\mathscr{X} \backslash K \in \delta_\mu[G]$, which leads to $K \in \mathscr{H}$ s.t. $G \in \delta_\mu[\mathscr{X} \backslash K]$. Consequently, $G \in \cup_{K \in \mathscr{H}} \delta_\mu[\mathscr{X} \backslash K]$, so $\lambda_0(\delta_\mu, \mathscr{H}) \subseteq \cup_{K \in \mathscr{H}} \delta_\mu[\mathscr{X} \backslash K]$. By the same manner, $\cup_{K \in \mathscr{H}} \delta_\mu[\mathscr{X} \backslash K] \subseteq \lambda_0(\delta_\mu, \mathscr{H})$ is obtained. □

Theorem 20. *Let $\delta_\mu \in \wp(\mathscr{X})$. If$\delta_\mu \in \mu\mathscr{P}_{\lambda 1}$, then c_{δ_μ} is a μ-closure operator.*

Proof. Certainly, from c_{δ_μ} properties, c_{δ_μ} operator is monotone and extensive. So, we shall prove c_{δ_μ} is an idempotent operator. Obviously, $c_{\delta_\mu}(G) \subseteq c_{\delta_\mu} c_{\delta_\mu}(G)$. Let $G \subseteq \mathscr{X}$ and $x \in c_{\delta_\mu} c_{\delta_\mu}(G)$; then, by Remark 4 (3), $c_{\delta_\mu}(G) \notin \delta_\mu[x]$. Since $\delta_\mu[x]$ is a hereditary class on $\mathscr{X}$, so from Definition 14, $G \notin \lambda_1(\delta_\mu, \delta_\mu[x])$. Since $\delta_\mu \in \mu\mathscr{P}_{\lambda 1}$, i.e., δ_μ on $\mathscr{X}$ is an λ_1-μ-proximity or $\delta_\mu[x] \subseteq \lambda_1(\delta_\mu, \delta_\mu[x])$; then, $G \notin \delta_\mu[x]$, so $x \in c_{\delta_\mu}(G)$. Consequently, $c_{\delta_\mu} c_{\delta_\mu}(G) \subseteq c_{\delta_\mu}(G)$. So, $c_{\delta_\mu} c_{\delta_\mu}(G) = c_{\delta_\mu}(G)$. Hence, c_{δ_μ} is a μ-closure operator. □

Theorem 21. *Let $\delta_\mu \in \wp(\mathscr{X})$. Then, δ_μ is a λ_1-μ-proximity iff $c_{\delta_\mu}(G) \in \delta_\mu[K]$, for every $G \in \delta_\mu[K]$.*

Proof. . Let $G \in \delta_\mu[K]$. Since δ_μ is a λ_1 proximity. Then, $G \in \lambda_1(\delta_\mu, \delta_\mu[K])$ and so $c_{\delta_\mu}(G) \in \delta_\mu[K]$. Conversely, let $K \subseteq \mathscr{X}$ and $G \in \delta_\mu[K]$; then, $c_{\delta_\mu}(G) \in \delta_\mu[K]$. Therefore, $G \in \lambda_1(\delta_\mu, \delta_\mu[H])$, so $\delta_\mu[K] \subseteq \lambda_1(\delta_\mu, \delta_\mu[K])$. Hence, δ_μ is a λ_1-μ-proximity. □

Theorem 22. *Let $\delta_\mu \in \wp(\mathscr{X})$. Then, δ_μ is an $O\mu$-proximity on $\mathscr{X}$ iff it is λ_1-μ-proximity.*

Proof. $(\Rightarrow)$ Let $G \subseteq \mathscr{X}$ and $K \notin \lambda_1(\delta_\mu, \delta_\mu[G])$; then, $c_{\delta_\mu}(K) \notin \delta_\mu[G]$. In view of Definition 8, $G \notin \delta_\mu[c_{\delta_\mu}(K)]$. From the definition of $c_{\delta_\mu}(K)$, then $x \notin \delta_\mu[K], \forall x \in c_{\delta_\mu}(K)$. So, $G \notin \delta_\mu[c_{\delta_\mu}(K)]$, $x \notin \delta_\mu[K]$, and $\forall x \in c_{\delta_\mu}(K)$. Since δ_μ is an $O\mu$-proximity relation on $\mathscr{X}$, then $G \notin \delta_\mu[K]$. So, δ_μ is λ_1-μ-proximity.

$(\Leftarrow)$ Let $G \notin \delta_\mu[A]$ and $a \notin \delta_\mu[K] \forall a \in A$. Obviously, $A \subseteq c_{\delta_\mu}(K)$, so $G \notin \delta_\mu[c_{\delta_\mu}(K)]$. Hence, $c_{\delta_\mu}(K) \notin \delta_\mu[G]$. Since δ_μ is a λ_1-μ-proximity relation on $\mathscr{X}$, then $K \notin \delta_\mu[G]$, i.e., $G \notin \delta_\mu[K]$, which induces to δ_μ is an $O\mu$-proximity on $\mathscr{X}$. □

Theorem 23. *Let $\delta_\mu \in \wp(\mathscr{X})$ and $\mathscr{H} \in \mathbb{H}$. If $\lambda \in \{e, \lambda_0, \lambda_1, \lambda_3\}$, then $\lambda(\delta_\mu, \mathscr{H}) \subseteq \mathscr{H}$.*

Proof. We prove only for λ_0The rest of the proof follow directly from definitions of e and λ_1.

Let $G \in \lambda_0(\delta_\mu, \mathscr{H})$; then, $\mathscr{N}_\mu(G) \cap \mathscr{H} \neq \varnothing$. Hence, there exists $E \in \mathscr{H}$ s.t. $E \in \mathscr{N}_\mu(G)$. Since $E \in \mathscr{N}_\mu(G)$, then $G \subseteq E \in \mathscr{H}$, so $G \in \mathscr{H}$, i.e., $\lambda_0(\delta_\mu, \mathscr{H}) \subseteq \mathscr{H}$. □

Theorem 24. *Let $\delta_\mu \in \wp(\mathscr{X})$ and $x \in \mathscr{X}$. Then, $\delta_\mu \in \mu\mathscr{P}_{\lambda 2}iff(G \in \delta_\mu[K] \Rightarrow (G \in \delta_\mu[x] \text{or} K \in \delta_\mu[x]))$.*

Proof. $(\Rightarrow)$ Let $G \in \delta_\mu[K]$. Since $\delta_\mu \in \mu\mathscr{P}_{\lambda 2}$, then $G \in \lambda_2(\delta_\mu, \delta_\mu[K])$, so $\{x\} \in \delta_\mu[G] \cup \delta_\mu[K]$, $\forall x \in \mathscr{X}$. Thus, $G \in \delta_\mu[x]$ or $K \in \delta_\mu[x]$. $(\Leftarrow)$ Let $K \in \delta_\mu[G]$ and $x \in \mathscr{X}$; then, $K \in \delta_\mu[x]$ or $G \in \delta_\mu[x]$. Hence, $\{x\}\delta_\mu[G] \cup \delta_\mu[K]$, $\forall x \in \mathscr{X}$; it follows that $K \in h_2(\delta_\mu, \delta_\mu[G])$, $\forall G \subseteq \mathscr{X}$. Hence, $\delta_\mu[G] \subseteq \lambda_2(\delta_\mu, \delta_\mu[G])$. Consequently, $\delta_\mu \in \mu\mathscr{P}_{\lambda 2}$. □

Theorem 25. *$(\mathcal{X}, \delta_\mu)$ is a λ_3 - μ -proximity space iff $\mathcal{N}_\mu(x) \cap \delta_\mu[K] \neq \varnothing$, for every $x \in \delta_\mu[K]$.*

Proof. ($\Rightarrow$) Let $x \in \delta_\mu[K]$. Since δ_μ is a λ_3-μ-proximity on $\mathcal{X}$ and $\delta_\mu[K]$ is a hereditary class on $\mathcal{X}$, then $x \in \lambda_3(\delta_\mu, \delta_\mu[K])$. Hence, by Theorem 13, $\mathcal{N}_\mu(x) \cap \delta_\mu[K] \neq \varnothing$. ($\Leftarrow$) Let $x \in \delta_\mu[K]$, then $\mathcal{N}_\mu(x) \cap \delta_\mu[K] \neq \varnothing$, which implies that $x \in \lambda_0(\delta_\mu, \delta_\mu[K])$. Consequently, $\delta_\mu[K] \subseteq \lambda_3(\delta_\mu, \delta_\mu[K])$, so δ_μ is λ_3-μ-proximity. □

Corollary 4. *δ_μ is a $R\mu$ -proximity on $\mathcal{X}$ iff it isλ_3-μ-proximity.*

Theorem 26. *For all $\delta_\mu \in \wp(\mathcal{X})$ and for all $\mathcal{H} \in \mathbb{H}$, we have*

(1) $\mu\mathcal{P}_{\lambda 0} \subseteq \mu\mathcal{P}_{\lambda 1} \cap \mu\mathcal{P}_{\lambda 3}$
(2) $\mu\mathcal{P}_{\lambda 1} \subseteq \mu\mathcal{P}_{\lambda 2}$

Proof

(1) Let $\delta_\mu \in \mu\mathcal{P}_{\lambda 0}$, i.e., δ_μ is a λ_0-μ-proximity on $\mathcal{X}$. Suppose that $B \in \delta_\mu[G]$; then, $\mathcal{N}_\mu(B) \cap \delta_\mu[G] \neq \varnothing$. Hence, there exists $F \subseteq \mathcal{X}$ s.t. $F \in \mathcal{N}_\mu(B)$ and $F \in \delta_\mu[G]$ and so $\mathcal{X} \backslash F \in \delta_\mu[B]$. According to Theorem 11, $c_{\delta_\mu}(B) \subseteq F$. Since $F \in \delta_\mu[G]$, then $c_{\delta_\mu}(B) \in \delta_\mu[G]$. Hence, $B \in \lambda_1(\delta_\mu, \delta_\mu[G])$. Consequently, $\delta_\mu \in \mu\mathcal{P}_{\lambda 1}$ and so $\mu\mathcal{P}_{\lambda 0} \subseteq \mu\mathcal{P}_{\lambda 1}$. Also, let $\delta_\mu \in \mu\mathcal{P}_{\lambda 0}$. Suppose $G \in \delta_\mu[K]$; then, $\mathcal{N}_\mu(G) \cap \delta_\mu[K] \neq \varnothing$, so $\mathcal{N}_\mu(a) \cap \delta_\mu[K] \neq \varnothing$, for every $a \in G$. Hence, $G \in \lambda_3(\delta_\mu, \delta_\mu[H])$. Hence, $\delta_\mu[K] \subseteq \lambda_3(\delta_\mu, \delta_\mu[H])$. Consequently, $\delta_\mu \in \mu\mathcal{P}_{\lambda 3}$. So, $\mu\mathcal{P}_{\lambda 0} \subseteq \mu\mathcal{P}_{\lambda 3}$.

(2) Let $\delta_\mu \in \mu\mathcal{P}_{\lambda 1}$ and let $G \in \delta_\mu[K]$. Then, by Theorem 21, $c_{\delta_\mu}(G) \in \delta_\mu[K]$. We claim that $G \in \lambda_2(\delta_\mu, \delta_\mu[H])$. Suppose $G \notin h_2(\delta_\mu, \delta_\mu[H])$; then, there exists $x \in X$ s.t. $\{x\} \notin \delta_\mu[G]$ and $\{x\} \notin \delta_\mu[K]$; then, $x \in c_{\delta_\mu}(G)$, $\{x\} \notin \delta_\mu[K]$. However, $\{x\} \subseteq c_{\delta_\mu}(G)$ and $\delta_\mu[K] \in \mathbb{H}$, so $c_{\delta_\mu}(G) \notin \delta_\mu[K]$, a contradiction. Hence, $G \in h_2(\delta_\mu, \delta_\mu[H])$. It follows that $\delta_\mu[K] \subseteq h_2(\delta_\mu, \delta_\mu[H])$. Consequently, $\delta_\mu \in \mu\mathcal{P}_{h2}$ and so $\mu\mathcal{P}_{h1} \subseteq \mu\mathcal{P}_{h2}$. □

Figure 3: A bit more, punch, 1845 [29].

Theorem 27. *Let λ be a hereditary class operator. If $E = \{\lambda \mid \lambda(\delta_\mu, \mathcal{H}) \subseteq \lambda(\delta_\mu, \lambda(\delta_\mu, \mathcal{H})), \delta_\mu \in \mu\mathcal{P}_\lambda, \mathcal{H} \in \mathbb{H}\}$, then e , λ_0 , λ_1 , $\lambda_2 \in E$.*

Proof. Let $G \in \lambda_0(\delta_\mu, \mathcal{H})$; then, $\mathcal{N}_\mu(G) \cap \mathcal{H} \neq \varnothing$ which implies that there exists $F \in \mathcal{H}$ s.t. $F \in \mathcal{N}_\mu(G)$. Since $\delta_\mu \in \mu\mathcal{P}_{\lambda_0}$, then, by Theorem 18, there exists $B \in \mathcal{N}_\mu(G)$ s.t. $F \in \mathcal{N}_\mu(B)$ and so $\mathcal{N}_\mu(B) \cap \mathcal{H} \neq \varnothing$. So, $B \in \lambda_0(\delta_\mu, \mathcal{H})$. However, $B \in \mathcal{N}_\mu(G)$; thus, $\mathcal{N}_\mu(G) \cap \lambda_0(\delta_\mu, \mathcal{H}) \neq \varnothing$. Hence, $G \in \lambda_0(\delta_\mu, \lambda_0(\delta_\mu, \mathcal{H}))$. Consequently, $\lambda_0(\delta_\mu, \mathcal{H}) \subseteq \lambda_0(\delta_\mu, \lambda_0(\delta_\mu, \mathcal{H}))$. It follows that $\lambda_0 \in E$.

Next, let $\delta_\mu \in \mu\mathcal{P}_{\lambda_1}$ and let $G \in \lambda_1(\delta_\mu, \mathcal{H})$. Then, $c_{\delta_\mu}(G) \in \mathcal{H}$ and so $c_{\delta_\mu}(G) = c_{\delta_\mu} c_{\delta_\mu}(G) \in \mathcal{H}$ (by Theorem 20). Hence, $c_{\delta_\mu}(G) \in \lambda_1(\delta_\mu, \mathcal{H})$. Hence, $G \in \lambda_1(\delta_\mu, \lambda_1(\delta_\mu, \mathcal{H}))$. Consequently, $\lambda_1(\delta_\mu, \mathcal{H}) \subseteq \lambda_1(\delta_\mu, \lambda_1(\delta_\mu, \mathcal{H}))$. It follows that $\lambda_1 \in E$.

Now, let $G \notin \lambda_2(\delta_\mu, \lambda_2(\delta_\mu, \mathcal{H}))$. Since $\delta_\mu \in \mu\mathcal{P}_{\lambda_2}$, hence, there exists $x \in \mathcal{X}$ s.t. $\{x\} \notin \delta_\mu[G] \cup \lambda_2(\delta_\mu, \mathcal{H})$. This implies that $\{x\} \notin \delta_\mu[G]$ and $\{x\} \notin \lambda_2(\delta_\mu, \mathcal{H})$. It follows that there exists $y \in X$ s.t. $\{y\} \notin \delta_\mu[x] \cup \mathcal{H}$. Then, $y \notin \delta_\mu[x]$ and $y \notin \mathcal{H}$. Since $\{x\} \notin \delta_\mu[G]$, i.e., $G \notin \delta_\mu[x]$, then in view of Theorem 24, $\{y\} \notin \delta_\mu[G] \cup \mathcal{H}$. Hence, $G \notin \lambda_2(\delta_\mu, \mathcal{H})$. Hence, $\lambda_2(\delta_\mu, \mathcal{H}) \subseteq \lambda_2(\delta_\mu, \lambda_2(\delta_\mu, \mathcal{H}))$. Consequently, $\lambda_2 \in E$.

Finally, let $G \in \lambda_3(\delta_\mu, \mathcal{H})$. Since $\delta_\mu \in \mu\mathcal{P}_{\lambda_3}$, then $\mathcal{N}_\mu(a) \cap \mathcal{H} \neq \varnothing$, for every $a \in G$, which implies that there exists $F \in \mathcal{H}$ s.t. $F \in \mathcal{N}_\mu(a)$, for every $a \in G$. Therefore, in view of Theorem 13 and Theorem 25, there exists $B \in \mathcal{N}_{\mu}(a)$ s.t. $F \in \mathcal{N}_{\mu}(B)$, for every $a \in G$. Hence, $\mathcal{N}_{\mu}(B) \cap \mathcal{H} \neq \emptyset$, so $\mathcal{N}_{\mu}(x) \cap \mathcal{H} \neq \emptyset$, for every $x \in B$. It follows that$B \in \lambda_3 \delta_{\mu}$.Since$B \in \mathcal{N}_{\mu}(a)$, so$\mathcal{N}_{\mu}(a) \cap \lambda_3 \delta_{\mu} \neq \emptyset$. Consequently, $G \in \lambda_3(\delta_{\mu}, \lambda_3(\delta_{\mu}, \mathcal{H}))$.Hence, $\lambda_3(\delta_{\mu}, \mathcal{H}) \subseteq \lambda_3(\delta_{\mu}, \lambda_3(\delta_{\mu}, \mathcal{H}))$ and so $\lambda_3 \in E$. □

6. Application

Near sets in mathematics are either spatially close or descriptively close. The classical idea of the nearness of sets is spatial, where sets are near, as long as the sets possess joint elements. Descriptively close sets consist of organs that have matching descriptions, i.e., the set $\mathcal{X}$ with descriptively close sets δ include some of sets that consist of elements, in which every element of them have position and measurable attributes as colour or frequency of apparition.

In the next section, we will display an application about spatially close using $\delta_{\mu}[.]$ idea.

Remark 6. Obviously, a point μ-cluster $\sigma_{\mu}(x)$ is spatial nearness collection for any point x.

Example 8. In Example 2 of [29], suppose that $\mathcal{X}$ is the set of points in the picture (see Figure 3). Let $G, K \subseteq \mathcal{X}$ be the set of points in the knights' horse and set of points in the suspended knight, respectively. $G \in \delta_{\mu}[K]$, since there is no common element between $c_{\mu}(G)$ and $c_{\mu}(K)$. So, the subsets G, K are spatially nonnear sets.

7. Conclusion

In this work, we have introduced the concept of μ-clusters to study μ-proximity (or μ-nearness) spaces and investigated main properties. Also, we have defined other types of μ-proximity called $R\mu$-proximity and $O\mu$-proximity on $\mathcal{X}$. Furthermore, we have presented descriptive near sets as a tool of solving classification and pattern recognition problems emerging from disjoint sets; hence, a new approach to basic μ-proximity structures, which depend on the realization of the structures in the theory of hereditary classes, has been introduced.

Finally, we hope this article helps to enrich the near set theory and opens up a door for researchers to conduct further studies in this interesting theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] S. A. Naimpally and J. F. Peters, "Topology with applications," *Topological Spaces via Near and Far*, World Scientific, Singapore, 2013.

[2] J. F. Peters, "Topology of digital images - visual pattern discovery in proximity spaces," *Intelligent Systems Reference Library*, Vol. 63, Springer, , Berlin, Germany, 2014.

[3] J. Peters and S. Naimpally, "Applications of near sets," *Notices of the American Mathematical Society*, vol. 59, no. 4, pp. 536–542, 2012.

[4] J. Peters and S. Naimpally, "Approach spaces for near families," *General Mathematics Notes*, vol. 2, no. 1, pp. 159–164, 2011.

[5] A. E. Hassanien, A. Abraham, J. F. Peters, G. Schaefer, and C. Henry, "Rough sets and near sets in medical imaging: a review," *IEEE Transactions on Information Technology in Biomedicine: A Publication of the IEEE Engineering in Medicine and Biology Society*, vol. 13, pp. 955–68, 2009.

[6] S. K. Pal and J. F. Peters, *Rough Fuzzy Image Analysis: Foundations and Methodologies*, CRC Press, Boca Raton, FL, USA, 2010.

[7] S. Gupta and K. Patnaik, "Enhancing performance of face recognition systems by using near set approach for selecting facial features," *Journal of Theoretical and Applied Information Technology*, vol. 4, no. 5, pp. 433–441, 2008.

[8] J. F. Peters, A. Skowron, and J. Stepaniuk, "Nearness of visual objects. Application of rough sets in proximity spaces," *Fundamenta Informaticae*, vol. 128, no. 1–2, pp. 159–176, 2013.

[9] S. Tiwari and P. K. Singh, "Ech rough proximity spaces," *Matematicki Vesnik*, vol. 72, no. 1, pp. 6–16, 2020.

[10] M. F. Worboys, "Nearness relations in environmental space," *International Journal of Geographical Information Science*, vol. 15, no. 7, pp. 633–651, 2001.

[11] T. M. Al-shami, "An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application," *Information Sciences*, vol. 569, pp. 110–124, 2021.

[12] T. M. Al-shami, "Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets," *Soft Computing*, vol. 25, no. 23, pp. 14449–14460, 2021.

[13] T. M. Al-shami and D. Ciucci, "Subset neighborhood rough sets," *Knowledge-Based Systems*, vol. 237, Article ID 107868, 2022.

[14] J. F. Peters, S. Shahfar, S. Ramanna, and T. Szturm, "Biologically-inspired adaptive learning: a near set approach," in *Proceedings of the Frontiers in the Convergence of Bioscience and Information Technologies*, Jeju, Korea, October 2007.

[15] J. F. Peters and P. Wasilewski, "Foundations of near sets," *Information Sciences*, vol. 179, no. 18, pp. 3091–3109, 2009.

[16] S. A. Naimpally and B. D. Warrack, "Proximity spaces," *Cambridge Tracts in Mathematics and Mathematical Physics*, Vol. 59, Cambridge University Press, Cambridge, England, 1970.

[17] W. J. Thron, "Proximity structures and grills," *Mathematische Annalen*, vol. 206, pp. 35–62, 1973.

[18] S. Leader, "On clusters in proximity spaces," *Fundamenta Mathematicae*, vol. 47, no. 2, pp. 205–213, 1959.

[19] M. Lodato, "On topologically induced generalized proximity relations. II," *Pacific Journal of Mathematics*, vol. 17, no. 1, pp. 131–135, 1966.

[20] R. A. Hosny and O. A. Tantawy, "New proximities from old via ideals," *Acta Mathematica Hungarica*, vol. 110, no. 1–2, pp. 37–50, 2006.

[21] R. A. Hosny, "Relations and applications on proximity structures," *Gen*, vol. 11, no. 1, pp. 24–40, 2012.

[22] A. Kandil, S. A. El-Sheikh, M. M. Yakout, and S. A. Hazza, "Proximity structures and ideals," *Matematicki Vesnik*, vol. 67, no. 2, pp. 130–142, 2015.

[23] A. Kandil, O. A. Tantawy, S. A. El-Sheikh, and A. Zakaria, “New structures of proximity spaces,” *Information Sciences Letters*, vol. 3, pp. 85–89, 2014.

[24] A. Kandil, O. A. Tantawy, S. A. El-Sheikh, and A. Zakaria, “Multiset proximity spaces,” *Journal of the Egyptian Mathematical Society*, vol. 24, no. 4, pp. 562–567, 2016.

[25] V. A. Efremovi, “The geometry of proximity I (in Russian),” *Matematicheskii Sbornik*, vol. 31, no. 73, pp. 189–200, 1952.

[26] M. N. Mukherjee, D. Mandal, and D. Dipankar, “Proximity structure on generalized topological spaces,” *Afrika Matematika*, vol. 30, pp. 91–100, 2019.

[27] E. D. Yildirim, “–Proximity structure via hereditary classes,” *Maejo International Journal of Science and Technology*, vol. 15, no. 2, pp. 129–136, 2021.

[28] A. Császár, “Modification of generalized topologies via hereditary classes,” *Acta Mathematica Hungarica*, vol. 115, no. 1–2, pp. 29–36, 2007.

[29] J. F. Peters, “Near sets: an introduction,” *Mathematics in Computer Science*, vol. 7, no. 1, pp. 3–9, 2013.