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Research Article

An Expanded Concept and Graph of Zadeh's Algebraic Operations for 3-Dimensional Generalized Triangular Fuzzy Sets

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Algebraic operators for fuzzy sets have been studied in 2-dimensional space. We define the algebraic operators between two 3-dimensional fuzzy sets and calculate the operators for the generalized 3-dimensional triangular fuzzy sets. To facilitate an understanding of the results in 3D, the values of membership function are displayed in graphs using color density. We give various types of graphs of α-cuts.

1. Introduction

Many concepts defined by Zadeh have been developed in various fields of fuzzy theory and have been recently applied to fuzzy logic and fuzzy control [1–3]. The multivalued logic theorem was applied to fuzzy logic, and fuzzy operator theory was applied to fuzzy control [2, 4]. Theories on algebraic operators have been established by scholars, and studies on corresponding operators have been actively conducted [5–7]. One of the efficient ways to understand various operators is to apply them to a common, simple triangular fuzzy number [8–11].

With the number of uncertain factors increasing, fuzzy theories in 2-dimensional space have been researched [12–14]. Algebraic operators in 2-dimensional space are applied to many fields [15, 16]. When the results of the 3-dimensional algebraic operators on triangular fuzzy numbers are restricted to 2D, the results are consistent with those of 1D. Since the cases where the maximum value is not 1 are commonly found, the theories of general fuzzy sets can be more widely applied. Numerous studies on 2-dimensional triangular fuzzy sets which do not have the maximum value of 1 have been conducted [17–19].

In this paper, we define the algebraic operators between two 3-dimensional fuzzy sets and calculate the operators for the generalized 3-dimensional triangular fuzzy sets. To facilitate an understanding of the results in 3D, the values of membership function are displayed in graphs using color density. Various types of the graphs in this paper allow us to interpret the shape of the alpha cut, which may help promote further applications.

2. Preliminaries

We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 1 (see [20]). An α-cut of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} | \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \text{cl}\{x \in \mathbb{R} | \mu_A(x) > \alpha\}$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X | \mu_A(x) = \alpha\}$ is said to be the α-set of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} | \mu_A(x) > \alpha\}$, and $A^1 = A_1$.

Following Zadeh, Dubois, and Prade, the extension principle is defined as follows.

Definition 2 (see [21–23]). The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$, and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \quad * = +, -, \cdot, /. \tag{1}$$

Definition 3. A fuzzy set A with a membership function

$$\mu_A(x,y)=\begin{cases} h-\sqrt{\dfrac{(x-x_1)^2}{a^2}+\dfrac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2+a^2(y-y_1)^2\le a^2b^2h^2,\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

where $a,b>0$ and $0<h<1$ is called the generalized 2-dimensional triangular fuzzy set and denoted by $((a,x_1,h,b,y_1))^2$.

Theorem 1 (see [24]). *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha=\{(x,y)\in\mathbb{R}^2|\mu_A(x,y)=\alpha\}$ be the α-set of A. Then, for all $\alpha\in(0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0,2\pi]$ such that*

$$A^\alpha=\left\{\left(f_1^\alpha(t),f_2^\alpha(t)\right)\in\mathbb{R}^2|0\le t\le 2\pi\right\}. \tag{3}$$

We defined the parametric addition, parametric subtraction, parametric multiplication, and parametric division for two convex fuzzy sets A and B defined on $\mathbb{R}^2$ [25].

Theorem 2 (see [25]). *Let $A=((a_1,x_1,h_1,b_1,y_1))^2$ and $B=((a_2,x_2,h_2,b_2,y_2))^2$ be two generalized 2-dimensional triangular fuzzy sets. If $0<h_1<h_2<1$, then we have the following.*

(1) For $0<\alpha<h_1$, the α-set of $A(+)_pB$ is

$$\begin{aligned}\left(A(+)_pB\right)^\alpha=\Bigg\{(x,y)\in\mathbb{R}^2\Bigg|&\left(\frac{x-x_1-x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\right)^2\\&+\left(\frac{y-y_1-y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\right)^2=1\Bigg\}.\end{aligned} \tag{4}$$

Furthermore, we have

$$\begin{aligned}\left(A(+)_pB\right)^0&=\left\{(x,y)\in\mathbb{R}^2\left|\left(\frac{x-x_1-x_2}{a_1h_1+a_2h_2}\right)^2+\left(\frac{y-y_1-y_2}{b_1h_1+b_2h_2}\right)^2=1\right.\right\},\\ \left(A(+)_pB\right)^{h_1}&=\left\{(x,y)\in\mathbb{R}^2\left|\left(\frac{x-x_1-x_2}{a_2(h_2-h_1)}\right)^2+\left(\frac{y-y_1-y_2}{b_2(h_2-h_1)}\right)^2=1\right.\right\},\\ \left(A(+)_pB\right)^\alpha&=\varnothing,\quad h_1<\alpha\le h_2.\end{aligned} \tag{5}$$

(2) For $0<\alpha<h_1$, the α-set of $A(-)_pB$ is

$$\begin{aligned}\left(A(-)_pB\right)^\alpha=\Bigg\{(x,y)\in\mathbb{R}^2\Bigg|&\left(\frac{x-x_1+x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\right)^2\\&+\left(\frac{y-y_1+y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\right)^2=1\Bigg\}.\end{aligned} \tag{6}$$

Furthermore, we have

$$\left(A(-)_pB\right)^0 = \left\{(x,y)\in\mathbb{R}^2 \Big| \left(\frac{x-x_1+x_2}{a_1h_1+a_2h_2}\right)^2 + \left(\frac{y-y_1+y_2}{b_1h_1+b_2h_2}\right)^2 = 1\right\},$$
$$\left(A(-)_pB\right)^{h_1} = \left\{(x,y)\in\mathbb{R}^2 \Big| \left(\frac{x-x_1+x_2}{a_2(h_2-h_1)}\right)^2 + \left(\frac{y-y_1+y_2}{b_2(h_2-h_1)}\right)^2 = 1\right\}, \quad (7)$$
$$\left(A(-)_pB\right)^{\alpha} = \varnothing, \quad h_1<\alpha\le h_2.$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0\le t\le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos t + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t, \quad 0<\alpha<h_1,$$
$$y_\alpha(t) = y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t, \quad 0<\alpha<h_1. \quad (8)$$

Furthermore, we have

$$x_0(t) = x_1x_2 + (x_1a_2h_2 + x_2a_1h_1)\cos t + a_1a_2h_1h_2\cos^2 t,$$
$$x_{h_1}(t) = x_1x_2 + x_1a_2(h_2-h_1)\cos t,$$
$$y_0(t) = y_1y_2 + (y_1b_2h_2 + y_2b_1h_1)\sin t + b_1b_2h_1h_2\sin^2 t,$$
$$y_{h_1}(t) = y_1y_2 + y_1b_2(h_2-h_1)\sin t,$$
$$\left(A(\cdot)_pB\right)^\alpha = \varnothing, \quad h_1<\alpha\le h_2. \quad (9)$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0\le t\le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1+a_1(h_1-\alpha)\cos t}{x_2-a_2(h_2-\alpha)\cos t},$$
$$y_\alpha(t) = \frac{y_1+b_1(h_1-\alpha)\sin t}{y_2-b_2(h_2-\alpha)\sin t}, \quad 0<\alpha<h_1. \quad (10)$$

Furthermore, we have

$$x_0(t) = \frac{x_1+a_1h_1\cos t}{x_2-a_2h_2\cos t},$$
$$x_{h_1}(t) = \frac{x_1}{x_2-a_2(h_2-h_1)\cos t},$$
$$y_0(t) = \frac{y_1+b_1h_1\sin t}{y_2-b_2h_2\sin t}, \quad (11)$$
$$y_{h_1}(t) = \frac{y_1}{y_2-b_2(h_2-h_1)\sin t},$$
$$\left(A(/)_pB\right)^\alpha = \varnothing, \quad h_1<\alpha\le h_2.$$

3. A Generalized 3-Dimensional Triangular Fuzzy Set

Definition 4. A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} h-\sqrt{\dfrac{(x-x_1)^2}{a^2}+\dfrac{(y-y_1)^2}{b^2}+\dfrac{(z-z_1)^2}{c^2}}, & \text{if } b^2c^2(x-x_1)^2+c^2a^2(y-y_1)^2+a^2b^2(z-z_1)^2\le a^2b^2c^2h^2, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

where $a,b,c>0$ and $0<h<1$ is called the generalized 3-dimensional triangular fuzzy set and denoted by $((h,a,x_1,b,y_1,c,z_1))^3$.

For $0<\alpha<h$, the α-cut A_α of a generalized 3-dimensional triangular fuzzy set $A=(h,a,x_1,b,y_1,c,z_1)^3$ is the following set:

$$A_\alpha = \left\{(x,y,z)\in\mathbb{R}^3 \Big| \left(\frac{x-x_1}{a(h-\alpha)}\right)^2 + \left(\frac{y-y_1}{b(h-\alpha)}\right)^2 + \left(\frac{z-z_1}{c(h-\alpha)}\right)^2 \le 1\right\}. \quad (13)$$

Definition 5. A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called convex fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \left\{ (x, y, z) \in \mathbb{R}^3 | \mu_A (x, y, z) \geq \alpha \right\}, \tag{14}$$

are convex subsets in $\mathbb{R}^3$.

Definition 6 (see [26]). Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^3$.

(1) Parametric addition $A(+)_pB$:

$$\left(A(+)_pB\right)^\alpha = \left\{ \left(f_1^\alpha (s) + g_1^\alpha (s), f_2^\alpha (s,t) + g_2^\alpha (s,t), f_3^\alpha (s,t) + g_3^\alpha (s,t)\right) \in \mathbb{R}^3, 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \tag{15}$$

(2) Parametric subtraction $A(-)_pB$:

$$\begin{aligned} \left(A(-)_pB\right)^\alpha &= \{ (f_1^\alpha (s) - g_1^\alpha (s+\pi), f_2^\alpha (s,t) - g_2^\alpha (s+\pi, t), \\ &\quad f_3^\alpha (s,t) - g_3^\alpha (s+\pi,t)) \in \mathbb{R}^3 | 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \Big\}, \\ (A(-)_pB)^\alpha &= \{ (f_1^\alpha (s) - g_1^\alpha (s-\pi), f_2^\alpha (s,t) - g_2^\alpha (s-\pi, t), \\ &\quad f_3^\alpha (s,t) - g_3^\alpha (s-\pi,t)) \in \mathbb{R}^3 | \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \Big\}. \end{aligned} \tag{16}$$

(3) Parametric multiplication $A(\cdot)_pB$:

$$\left(A(\cdot)_pB\right)^\alpha = \left\{ \left(f_1^\alpha (s) \cdot g_1^\alpha (s), f_2^\alpha (s,t) \cdot g_2^\alpha (s,t), f_3^\alpha (s,t) \cdot g_3^\alpha (s,t)\right) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \tag{17}$$

(4) Parametric division $A(/)_pB$:

$$\begin{aligned} \left(A(/)_pB\right)^\alpha &= \left\{ \left(\frac{f_1^\alpha (s)}{g_1^\alpha (s+\pi)}, \frac{f_2^\alpha (s,t)}{g_2^\alpha (s+\pi,t)}, \frac{f_3^\alpha (s,t)}{g_3^\alpha (s+\pi,t)} \right) \in \mathbb{R}^3 | 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}, \\ \left(A(/)_pB\right)^\alpha &= \left\{ \left(\frac{f_1^\alpha (s)}{g_1^\alpha (s-\pi)}, \frac{f_2^\alpha (s,t)}{g_2^\alpha (s-\pi,t)}, \frac{f_3^\alpha (s,t)}{g_3^\alpha (s-\pi,t)} \right) \in \mathbb{R}^3 | \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \end{aligned} \tag{18}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \longrightarrow 0^+} (A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \longrightarrow 1^-} (A(*)_pB)^\alpha$, where $* = +, -, \cdot, /$.

Theorem 3. *Let $A = ((h_1, a_1, x_1, b_1, y_1, c_1, z_1))^3$ and $B = ((h_2, a_2, x_2, b_2, y_2, c_2, z_2))^3$ be two generalized 3-dimensional triangular fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the following.*

(1) For $0 < \alpha < h_1$, the α-set of $A(+)_pB$ is

$$\begin{aligned} \left(A(+)_pB\right)^\alpha = \Bigg\{ (x,y,z) \in \mathbb{R}^3 | &\left(\frac{x - x_1 - x_2}{a_1 (h_1 - \alpha) + a_2 (h_2 - \alpha)} \right)^2 \\ &+ \left(\frac{y - y_1 - y_2}{b_1 (h_1 - \alpha) + b_2 (h_2 - \alpha)} \right)^2 + \left(\frac{z - z_1 - z_2}{c_1 (h_1 - \alpha) + c_2 (h_2 - \alpha)} \right)^2 = 1 \Bigg\}. \end{aligned} \tag{19}$$

(2) For $0 < \alpha < h_1$, the α-set of $A(-)_pB$ is

$$\left(A(-)_pB\right)^\alpha = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\right)^2 + \left(\frac{z - z_1 + z_2}{c_1(h_1 - \alpha) + c_2(h_2 - \alpha)}\right)^2 = 1\right\}. \tag{20}$$

(3) For $0 < \alpha < h_1$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

$$\begin{aligned} x_\alpha(s) &= x_1x_2 + (x_1a_2(h_2 - \alpha) + x_2a_1(h_1 - \alpha))\cos s + a_1a_2(h_1 - \alpha)(h_2 - \alpha)\cos^2 s, \\ y_\alpha(s,t) &= y_1y_2 + (y_1b_2(h_2 - \alpha) + y_2b_1(h_1 - \alpha))\sin s \cos t + b_1b_2(h_1 - \alpha)(h_2 - \alpha)\sin^2 s \cos^2 t, \\ z_\alpha(s,t) &= z_1z_2 + (z_1c_2(h_2 - \alpha) + z_2c_1(h_1 - \alpha))\sin s \sin t + c_1c_2(h_1 - \alpha)(h_2 - \alpha)\sin^2 s \sin^2 t. \end{aligned} \tag{21}$$

(4) For $0 < \alpha < h_1$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

$$\begin{aligned} x_\alpha(s) &= \frac{x_1 + a_1(h_1 - \alpha)\cos s}{x_2 - a_2(h_2 - \alpha)\cos s}, \\ y_\alpha(s.t) &= \frac{y_1 + b_1(h_1 - \alpha)\sin s \cos t}{y_2 - b_2(h_2 - \alpha)\sin s \cos t}, \\ z_\alpha(s.t) &= \frac{z_1 + c_1(h_1 - \alpha)\sin s \sin t}{z_2 - c_2(h_2 - \alpha)\sin s \sin t}. \end{aligned} \tag{22}$$

Furthermore, we have

$$\begin{aligned} \left(A(*)_pB\right)^0 &= \lim_{\alpha \longrightarrow 0^+} \left(A(*)_pB\right)^\alpha, \quad * = +, -, \cdot, /, \\ \left(A(*)_pB\right)^{h_1} &= \lim_{\alpha \longrightarrow h_1^-} \left(A(*)_pB\right)^\alpha, \quad * = +, -, \cdot, /. \end{aligned} \tag{23}$$

If $h_1 < \alpha \le h_2$, by Zadeh's max-min principle operations, we get

$$\left(A(*)_pB\right)^\alpha = \varnothing, \quad * = +, -, \cdot, /. \tag{24}$$

Proof. Since A and B are continuous convex fuzzy sets defined on $\mathbb{R}^3$, by Theorem 3.3 in [26], there exists $f_1^\alpha(s), g_1^\alpha(s) f_i^\alpha(s,t), g_i^\alpha(s,t), (i = 2, 3)$ such that

$$\begin{aligned} A^\alpha &= \left\{\left(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\}, \\ B^\alpha &= \left\{\left(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\}. \end{aligned} \tag{25}$$

Since $A = ((h_1, a_1, x_1, b_1, y_1, c_1, z_1))^3$ and $B = ((h_2, a_2, x_2, b_2, y_2, c_2, z_2))^3$, we have

$$\begin{aligned} f_1^\alpha(s) &= x_1 + a_1(h_1 - \alpha)\cos s, \\ f_2^\alpha(s,t) &= y_1 + b_1(h_1 - \alpha)\sin s \cos t, \\ f_3^\alpha(s,t) &= z_1 + c_1(h_1 - \alpha)\sin s \sin t, \quad \text{if } 0 \le \alpha \le h_1, \\ g_1^\alpha(s) &= x_2 + a_2(h_2 - \alpha)\cos s, \\ g_2^\alpha(s,t) &= y_2 + b_2(h_2 - \alpha)\sin s \cos t, \\ g_3^\alpha(s,t) &= z_2 + c_2(h_2 - \alpha)\sin s \sin t, \quad \text{if } 0 \le \alpha \le h_2. \end{aligned} \tag{26}$$

(1) If $0 < \alpha < h_1$, since

$$\begin{aligned} f_1^\alpha(s) + g_1^\alpha(s) &= x_1 + x_2 + (a_1(h_1 - \alpha) + a_2(h_2 - \alpha))\cos s, \\ f_2^\alpha(s,t) + g_2^\alpha(s,t) &= y_1 + y_2 + (b_1(h_1 - \alpha) + b_2(h_2 - \alpha))\sin s \cos t, \\ f_3^\alpha(s,t) + g_3^\alpha(s,t) &= z_1 + z_2 + (c_1(h_1 - \alpha) + c_2(h_2 - \alpha))\sin s \sin t, \end{aligned} \tag{27}$$

we have

$$\left(A(+)_pB\right)^{\alpha} = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{x-x_1-x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\right)^2 + \left(\frac{y-y_1-y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\right)^2 + \left(\frac{z-z_1-z_2}{c_1(h_1-\alpha)+c_2(h_2-\alpha)}\right)^2 = 1\right\}. \tag{28}$$

Furthermore, we have

$$\begin{aligned}
\left(A(+)_pB\right)^{0} &= \left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{x-x_1-x_2}{a_1h_1+a_2h_2}\right)^2 + \left(\frac{y-y_1-y_2}{b_1h_1+b_2h_2}\right)^2 + \left(\frac{z-z_1-z_2}{c_1h_1+c_2h_2}\right)^2 = 1\right\},\\
\left(A(+)_pB\right)^{h_1} &= \left\{(x,y) \in \mathbb{R}^2 \middle| \left(\frac{x-x_1-x_2}{a_2(h_2-h_1)}\right)^2 + \left(\frac{y-y_1-y_2}{b_2(h_2-h_1)}\right)^2 + \left(\frac{z-z_1-z_2}{c_2(h_2-h_1)}\right)^2 = 1\right\},\\
\left(A(+)_pB\right)^{\alpha} &= \varnothing, \quad h_1 < \alpha \leq h_2.
\end{aligned} \tag{29}$$

(2) If $0 \leq s \leq \pi, -(\pi/2) \leq t \leq (\pi/2)$ and $0 < \alpha < h_1$,

$$\begin{aligned}
f_1^{\alpha}(s) - g_1^{\alpha}(s+\pi) &= x_1 - x_2 + \left(a_1(h_1-\alpha) + a_2(h_2-\alpha)\right)\cos s,\\
f_2^{\alpha}(s,t) - g_2^{\alpha}(s+\pi,t) &= y_1 - y_2 + \left(b_1(h_1-\alpha) + b_2(h_2-\alpha)\right)\sin s \cos t,\\
f_3^{\alpha}(s,t) - g_3^{\alpha}(s+\pi,t) &= z_1 - z_2 + \left(c_1(h_1-\alpha) + c_2(h_2-\alpha)\right)\sin s \sin t.
\end{aligned} \tag{30}$$

In the case of $\pi \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)$, we have

$$\begin{aligned}
f_1^{\alpha}(s) - g_1^{\alpha}(s-\pi) &= f_1^{\alpha}(s) - g_1^{\alpha}(s+\pi),\\
f_2^{\alpha}(s,t) - g_2^{\alpha}(s-\pi,t) &= f_2^{\alpha}(s,t) - g_2^{\alpha}(s+\pi,t),\\
f_3^{\alpha}(s,t) - g_3^{\alpha}(s-\pi,t) &= f_3^{\alpha}(s,t) - g_3^{\alpha}(s+\pi,t).
\end{aligned} \tag{31}$$

Thus,

$$\left(A(-)_pB\right)^{\alpha} = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{x-x_1+x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\right)^2 + \left(\frac{y-y_1+y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\right)^2 + \left(\frac{z-z_1+z_2}{c_1(h_1-\alpha)+c_2(h_2-\alpha)}\right)^2 = 1\right\}. \tag{32}$$

Furthermore, we have

$$\begin{aligned}
\left(A(-)_pB\right)^0 &= \left\{(x,y,z)\in\mathbb{R}^3 \middle| \left(\frac{x-x_1+x_2}{a_1h_1+a_2h_2}\right)^2+\left(\frac{y-y_1+y_2}{b_1h_1+b_2h_2}\right)^2+\left(\frac{z-z_1+z_2}{c_1h_1+c_2h_2}\right)^2=1\right\},\\
\left(A(-)_pB\right)^{h_1} &= \left\{(x,y,z)\in\mathbb{R}^3 \middle| \left(\frac{x-x_1+x_2}{a_2(h_2-h_1)}\right)^2+\left(\frac{y-y_1+y_2}{b_2(h_2-h_1)}\right)^2+\left(\frac{z-z_1+z_2}{c_2(h_2-h_1)}\right)^2=1\right\},\\
\left(A(-)_pB\right)^{\alpha} &= \varnothing,\quad h_1<\alpha\le h_2.
\end{aligned}\tag{33}$$

(3) Let $(A(\cdot)_pB)^\alpha=\{(x_\alpha(s),y_\alpha(s,t),z_\alpha(s,t))|0\le s\le 2\pi,-(\pi/2)\le t\le(\pi/2)\}$. From the definitions of $f_1^\alpha(s),g_1^\alpha(s),f_i^\alpha(s,t)$, and $g_i^\alpha(s,t)\,(i=2,3)$, we have

$$\begin{aligned}
x_\alpha(s) &= x_1x_2+\left(x_1a_2(h_2-\alpha)+x_2a_1(h_1-\alpha)\right)\cos s+a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 s,\quad 0<\alpha<h_1,\\
y_\alpha(s,t) &= y_1y_2+\left(y_1b_2(h_2-\alpha)+y_2b_1(h_1-\alpha)\right)\sin s\cos t+b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\cos^2 t,\quad 0<\alpha<h_1.\\
z_\alpha(s,t) &= z_1z_2+\left(z_1c_2(h_2-\alpha)+z_2c_1(h_1-\alpha)\right)\sin s\sin t+c_1c_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 t,\quad 0<\alpha<h_1.
\end{aligned}\tag{34}$$

Furthermore, we have

$$\begin{aligned}
x_0(s) &= x_1x_2+\left(x_1a_2h_2+x_2a_1h_1\right)\cos s+a_1a_2h_1h_2\cos^2 s,\\
x_{h_1}(s) &= x_1x_2+x_1a_2(h_2-h_1)\cos s,\\
y_0(s,t) &= y_1y_2+\left(y_1b_2h_2+y_2b_1h_1\right)\sin s\cos t+b_1b_2h_1h_2\sin^2 s\cos^2 t,\\
y_{h_1}(s,t) &= y_1y_2+y_1b_2(h_2-h_1)\sin s\cos t,\\
z_0(s,t) &= z_1z_2+\left(z_1c_2h_2+z_2c_1h_1\right)\sin s\sin t+c_1c_2h_1h_2\sin^2 s\sin^2 t,\\
z_{h_1}(s,t) &= z_1z_2+z_1c_2(h_2-h_1)\sin s\sin t,\\
\left(A(\cdot)_pB\right)^\alpha &= \varnothing,\quad h_1<\alpha\le h_2.
\end{aligned}\tag{35}$$

(4) Let $(A(/)_pB)^\alpha=\{(x_\alpha(s),y_\alpha(s,t),z_\alpha(s,t))|0\le s\le 2\pi,-(\pi/2)\le t\le(\pi/2)\}$. Similarly, if $0<\alpha<h_1$,

$$\begin{aligned}
x_\alpha(s) &= \frac{x_1+a_1(h_1-\alpha)\cos s}{x_2-a_2(h_2-\alpha)\cos s},\\
y_\alpha(s.t) &= \frac{y_1+b_1(h_1-\alpha)\sin s\cos t}{y_2-b_2(h_2-\alpha)\sin s\cos t},\\
z_\alpha(s.t) &= \frac{z_1+c_1(h_1-\alpha)\sin s\sin t}{z_2-c_2(h_2-\alpha)\sin s\sin t}.
\end{aligned}\tag{36}$$

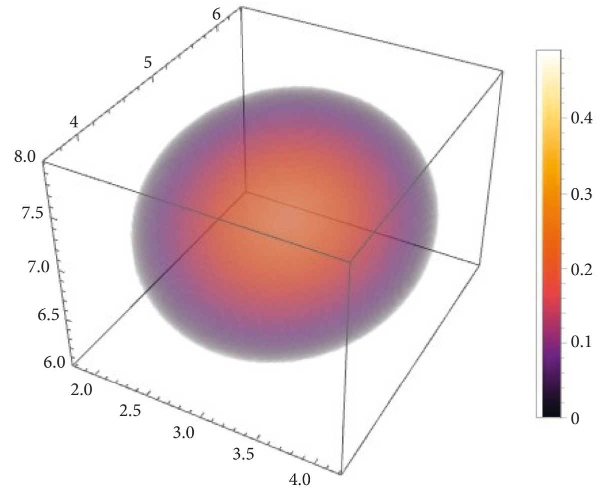

Figure 1: A.

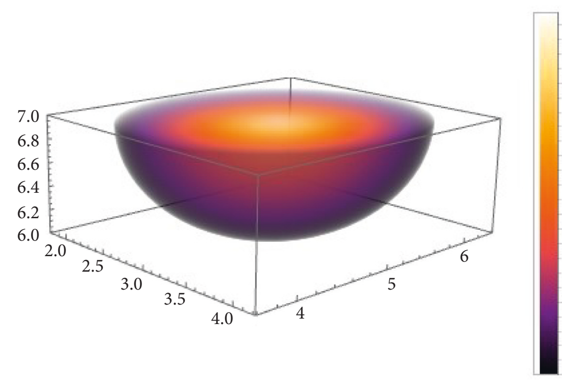

Figure 2: $A2$.

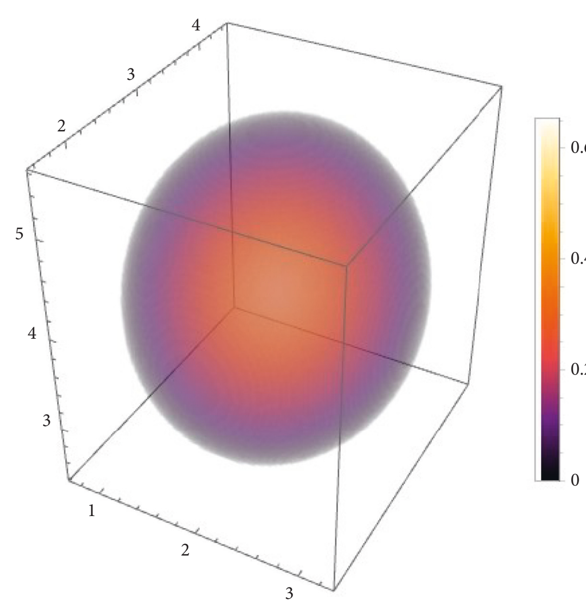

Figure 3: B.

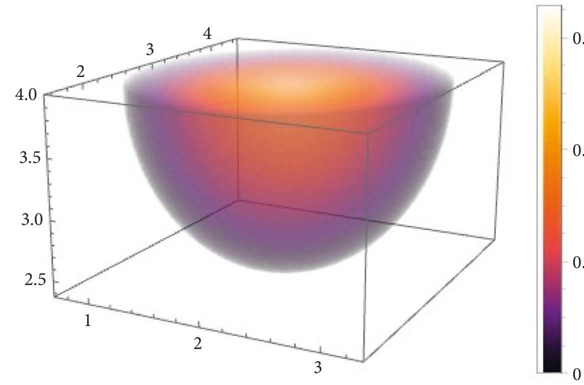

Figure 4: $B2$.

Furthermore, we have

$$\begin{aligned}
x_0(s) &= \frac{x_1 + a_1 h_1 \cos s}{x_2 - a_2 h_2 \cos s},\\
x_{h_1}(s) &= \frac{x_1}{x_2 - a_2(h_2 - h_1)\cos s},\\
y_0(s,t) &= \frac{y_1 + b_1 h_1 \sin s \cos t}{y_2 - b_2 h_2 \sin s \cos t},\\
y_{h_1}(s,t) &= \frac{y_1}{y_2 - b_2(h_2 - h_1)\sin s \cos t},\\
z_0(s,t) &= \frac{z_1 + c_1 h_1 \sin s \sin t}{z_2 - c_2 h_2 \sin s \sin t},\\
z_{h_1}(s,t) &= \frac{z_1}{z_2 - c_2(h_2 - h_1)\sin s \sin t},
\end{aligned} \tag{37}$$

$$\left(A(/)_p B\right)^\alpha = \varnothing, \quad h_1 < \alpha \le h_2.$$

The proof is complete. □

Example 1. Let $A = ((1/2), 6, 3, 8, 5, 4, 7)^3$ and $B = ((2/3), 4, 2, 5, 3, 6, 4)^3$. Then, by Theorem 3, we have the following.

(1) For $0 < \alpha < (1/2)$, the α-set of $A(+)_p B$ is

$$\left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{3x-15}{17-30\alpha}\right)^2 + \left(\frac{3y-24}{22-39\alpha}\right)^2 + \left(\frac{z-11}{6-10\alpha}\right)^2 = 1\right\}. \tag{38}$$

(2) For $0 < \alpha < (1/2)$, the α-set of $A(-)_p B$ is

$$\left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{3x-3}{17-30\alpha}\right)^2 + \left(\frac{3y-6}{22-39\alpha}\right)^2 + \left(\frac{z-3}{6-10\alpha}\right)^2 = 1\right\}. \tag{39}$$

(3) For $0 < \alpha < (1/2)$, $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

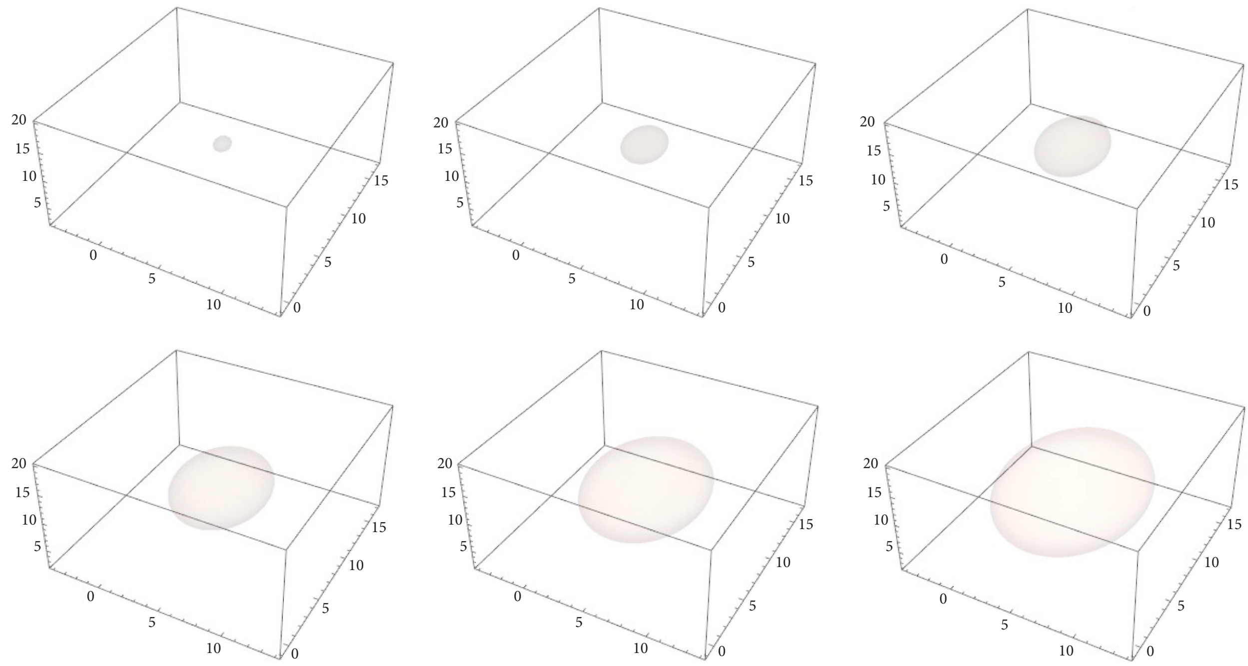

Figure 5: $(A+B), 0-\text{cut} \sim 1/2-\text{cut}$.

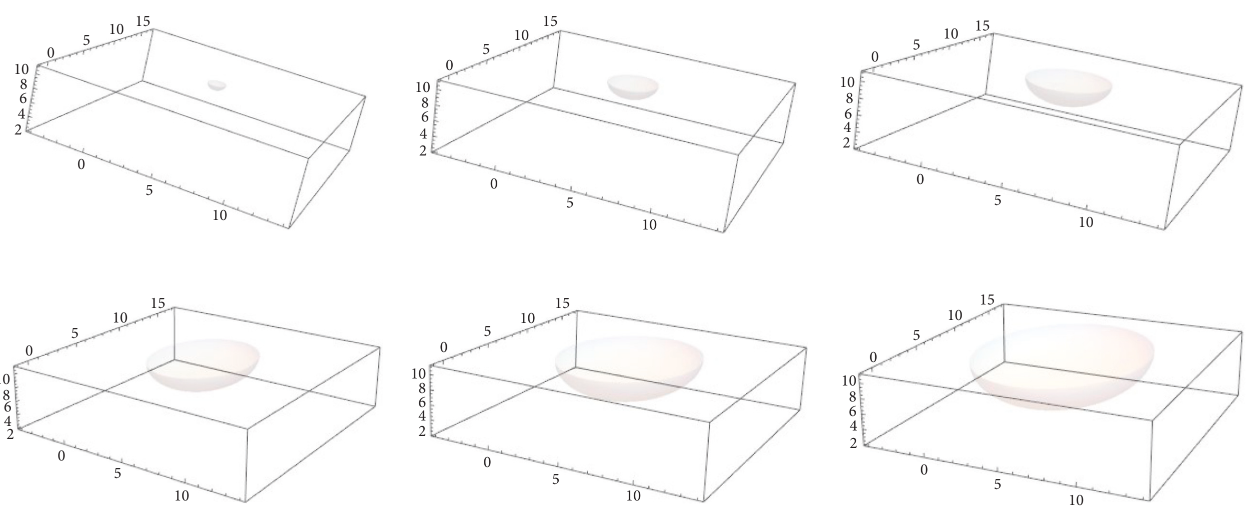

Figure 6: $(A+B)/2, 0-\text{cut} \sim 1/2-\text{cut}$.

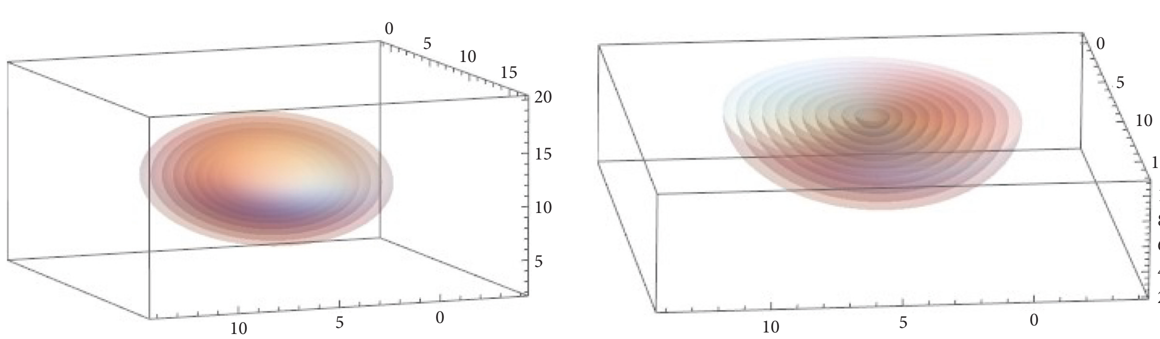

Figure 7: $A+B$, $(A+B)/2$.

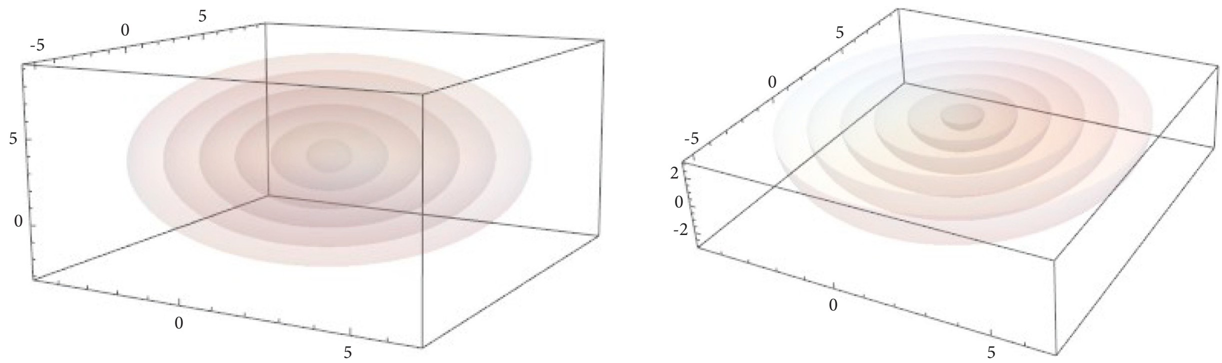

Figure 8: $A - B$, $(A - B)/2$.

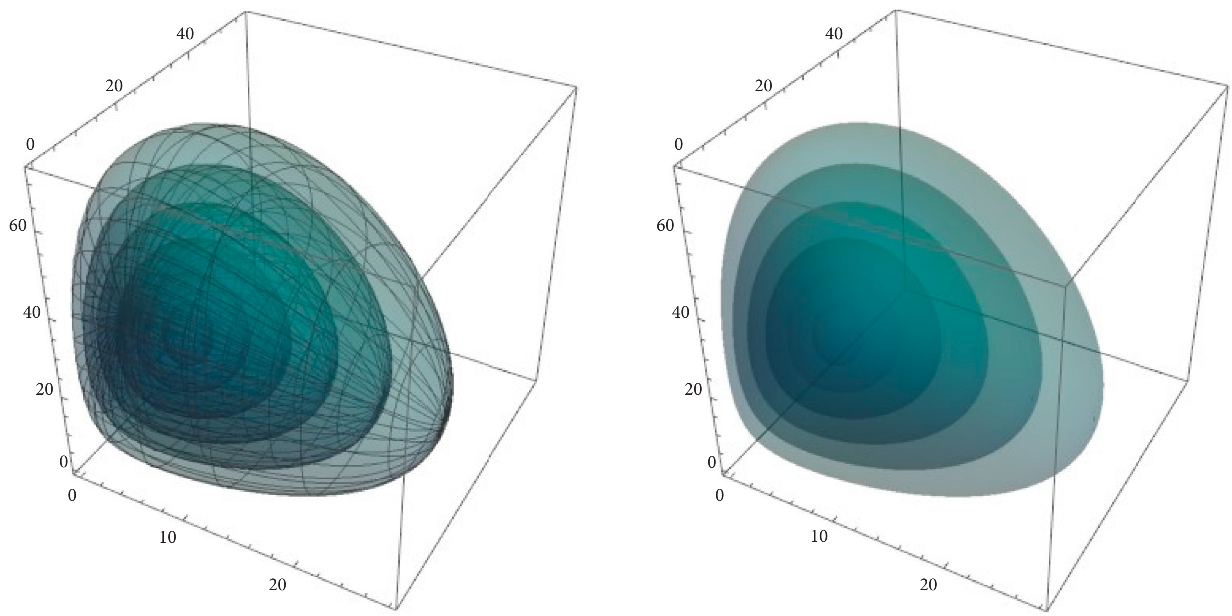

Figure 9: AB.

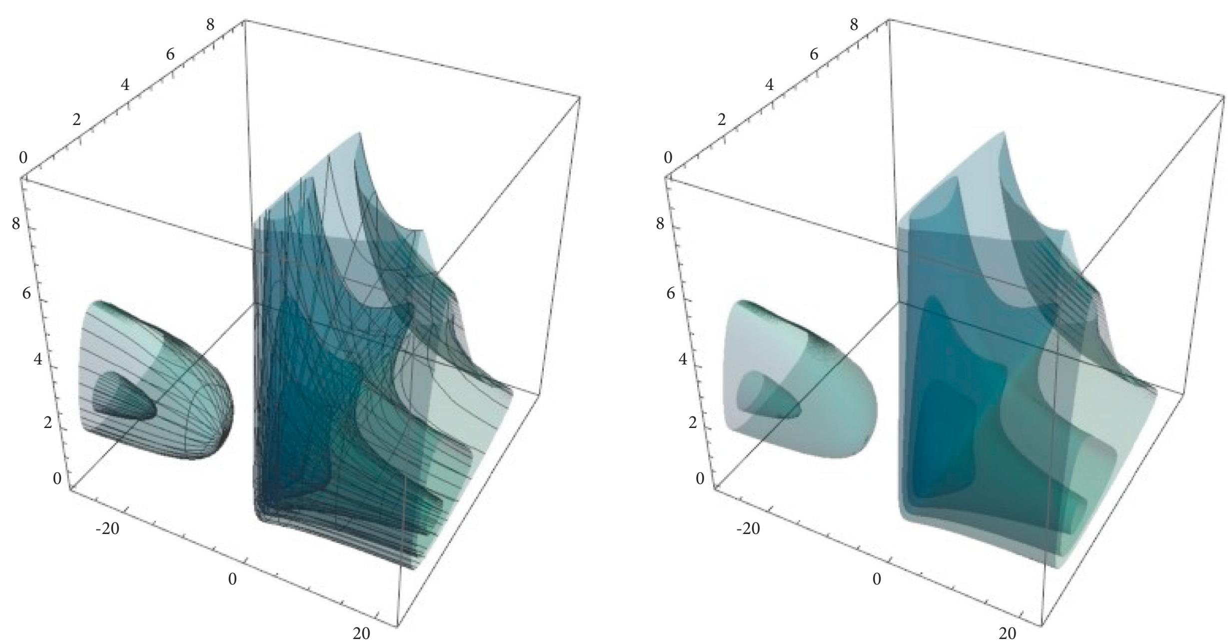

Figure 10: A/B.

$$x_\alpha(s) = 6 + (14 - 24\alpha)\cos s + 4(1 - 2\alpha)(2 - 3\alpha)\cos^2 s,$$
$$y_\alpha(s,t) = 15 + \left(\frac{86}{3} - 49\alpha\right)\sin s \cos t + 20(1 - 2\alpha)\left(\frac{2}{3} - \alpha\right)\sin^2 s \cos^2 t, \tag{40}$$
$$z_\alpha(s,t) = 28 + (36 - 58\alpha)\sin s \sin t + 4(1 - 2\alpha)(2 - 3\alpha)\sin^2 s \sin^2 t.$$

(4) For $0 < \alpha < (1/2)$, $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

$$x_\alpha(s) = \frac{9 + 9(1 - 2\alpha)\cos s}{6 - 4(2 - 3\alpha)\cos s},$$
$$y_\alpha(s,t) = \frac{15 + 12(1 - 2\alpha)\sin s \cos t}{9 - 15(2 - 3\alpha)\sin s \cos t}, \tag{41}$$
$$z_\alpha(s,t) = \frac{7 + 2(1 - 2\alpha)\sin s \sin t}{4 - 2(2 - 3\alpha)\sin s \sin t}.$$

4. Conclusion

When there are several uncertain elements, definitions and theories of the algebraic operator on fuzzy sets defined in 3-dimensional space are required and they should be natural extensions of the one-dimensional and 2-dimensional concepts. In Definition 6, we defined the algebraic operator as an extension of the 2-dimensional case in 3-dimensional space. In Theorem 3, we calculated the algebraic operators on the generalized triangular fuzzy set. Using this theorem, we obtained the results for the 3-dimensional generalized triangular fuzzy sets A and B in Example 1, and the graphs were presented.

Graphs for results obtained in the example were drawn and the Mathematica commands were presented for each. Each graph of A and B was cut with a plane passing through its central axis, and A2 and B2 show the result of the cut (Figures 1–4).

```
(: pre-run :)
PP = 100; IS = 360; legend = Placed[BarLegend[Automatic, LegendMarkerSize -> 300], Right];
```

Among the four results of the algebraic operator, six graphs from 0 cut to 1/2 cut for plus were presented (Figures 5 and 6). Also, with the commands introduced in the paper, researchers will be able to obtain cuts or information of their need.

```
(* (A + B) 1~6 *)
p = 17; q = 22; r = 6;
g[a_]: = ContourPlot3D[((3x – 15)/(p – 30a))^2 + ((3y – 24)/(q – 39a))^2 + ((z – 1)/(r – 10a))^2 = = 1,{x, –4, 14}, {y, –1, 17}, {z, 2, 20}, ContourStyle -> Directive [RGBColor[1 – a, 1 – a, 1 – a], Opacity[0.1]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5}, BoundaryStyle -> None, ImageSize -> IS];
tg = Column[Table[g[a] ,{a, 0, 0.5, 0.1}]];
Show[tg]
```

```
(* (A + B)/2 1~6 *)
p = 17; q = 22; r = 6;
g[a_]: = ContourPlot3D[((3x – 15)/(p – 30a))^2 + ((3y – 24)/(q – 39a))^2 + ((z – 11)/(r – 10a))^2 = = 1,{x,–4, 14},{y, –1, 17},{z, 2, 11}, ContourStyle -> Directive[RGBColor[1 – ,1 – a, 1 – a], Opacity [0.1]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5/2}, BoundaryStyle -> None, ImageSize -> IS];
tg = Column[Table[g[a], {a, 0, 0.5, 0.05}]];
Show[tg].
```

```
(* A + B *)
p = 17; q = 22; r = 6;
g[a_]: = ContourPlot3D[((3x – 15)/(p – 30a))^2 + ((3y – 24)/(q – 9a))^2 + ((z – 11)/(r – 10a))^2 = = 1, {x, –4, 14}, {y, –1, 17}, {z, 2, 20}, ContourStyle -> Directive[RGBColor[1 – a, 1 – a, 1 – a], Opacity [0.2]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5}, BoundaryStyle -> None, ImageSize -> IS, ViewPoint -> {p, 2q, 1.2r}];
tg = Table[g[a], {a, 0, 0.5, 0.05}];
Show[tg]
```

```
(* (A + B)/2 *)
p = 17; q = 22; r = 6;
g[a_]: = ContourPlot3D[((3x – 15)/(p – 30a))^2 + ((3y – 24)/(q – 39a))^2 + ((z – 11)/(r – 10a))^2 = = 1,{x, –4, 14},{y, –1, 17},{z, 2, 11}, ContourStyle -> Directive[RGBColor[1 – a, 1 – a, 1 – a], Opacity [0.2]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5/2}, BoundaryStyle -> None, ImageSize -> IS, ViewPoint -> {p, 2q, 1.2r}];
tg = Table[g[a],{a, 0, 0.5, 0.05}];
Show[tg]
```

```
(* A – B *)
p = 17; q = 22; r = 6;
g1[a_]: = ContourPlot3D[((3x – 3)/(p – 30a))^2 + ((3y – 6)/(q – 39a))^2 + ((z – 3)/(r – 10a))^2 = = 1, {x, 1 – (p – 30a)/3, 1 + (p – 30a)/3}, {y, 2 – (q – 39a)/3, 2 + (q – 39a)/3}, {z, 3 – (r – 10a), 3 + (r – 10a)}, ContourStyle -> Directive[RGBColor [1 – a, 1 – a, 1 – a], Opacity[0.1]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5},BoundaryStyle -> None, ImageSize -> IS];
tg = Table[g1[a], {a, 0, 0.5, 0.1}];
Show[tg]
```

```
( * (A − B)/2 * )
p = 17; q = 22; r = 6;
g1[a_]: = ContourPlot3D[((3x − 3)/(p − 30a))^2 + ((3y − 6)/(q − 39a))^2 + ((z − 3)/(r − 10a))^2 = = 1, {x, 1 − (p − 30a)/3, 1 + (p − 30a)/3}, {y, 2 − (q − 39a)/3, 2 + (q − 9a)/3}, {z, 3 − (r − 10a), 3}, ContourStyle -> Directive[RGBColor[1 − a, 1 − a, 1 − a], Opacity [0.1]], Mesh -> None, PlotPoints -> PP, BoxRatios -> {1, 1, 0.5/2}, BoundaryStyle -> None, ImageSize -> IS];
tg = Table[g1[a], {a, 0, 0.5, 0.1}];
Show[tg].
```

In general, it is not easy to express the results defined in 3D in graphs. Mathematica commands in this paper will help facilitate better visual understanding. Algebraic operators extended in 3D were defined, and they were calculated for a general trigonometric fuzzy set. Also, the resultant graphs can be specifically applied to fuzzy logic and fuzzy control (Figures 7–10). In sum, this paper expanded the results of two-dimensional cases and will provide the theoretical base for further development of 4-dimensional cases.

Data Availability

No data were used to support this study.

Disclosure

Some of the 2D results have been published previously [17].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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References

[1] M. M. Gupta, T. Meitzler, Z. G. Hou, K. K. Garg, A. M. G. Solo, and L. A. Zadeh, “Real applications of fuzzy logic,” *Advances in fuzzy systems*, vol. 2013, Article ID 581879, 3 pages, 2013.

[2] A. M. Mustafa, Z. Gong, and M. Osman, “Fuzzy optimal control problem of several variables,” *Advances in mathematical physics*, vol. 2019, Article ID 2182640, 12 pages, 2019.

[3] Z. Wenqiang, “Topic recommendation system using personalized fuzzy logic interest set,” *Journal of Intelligent & Fuzzy Systems*, vol. 40, no. 2, pp. 2891–2901, 2021.

[4] J. Ruan, X. Wang, C. Yue, G. Chen, and M. Kim, “Optimization models and algorithms for operation and control with advanced information technologies,” *Scientific Programming*, vol. 2017, Article ID 4706714, 2 pages, 2017.

[5] M. P. A. Santim, M. C. M. Teixeira, W. A. de Souza, R. Cardim, and E. Assunção, “Design of a takagi-sugeno fuzzy regulator for a set of operation points,” *Mathematical Problems in Engineering*, vol. 2012, Article ID 731298, 17 pages, 2012.

[6] S. A. M. Mohsenalhosseini1 and H. Mazaheri, “Approximate fixed point theorems in fuzzy norm spaces for an operator,” *Advances in Fuzzy Systems*, vol. 2013, Article ID 613604, 8 pages, 2013.

[7] P. Chansangiam, “A survey on operator monotonicity, operator convexity, and operator means,” *International Journal of Analysis*, vol. 2015, Article ID 649839, 8 pages, 2015.

[8] X. Zhang, W. Ma, and L. Chen, “New similarity of triangular fuzzy number and its application,” *The Scientific World Journal*, vol. 2014, Article ID 215047, 7 pages, 2014.

[9] X. Cao, Z. Xing, Y. Sun, and S. Yin, “A novel dynamic multicriteria decision-making approach for low-carbon supplier selection of low-carbon buildings based on interval-valued triangular fuzzy numbers,” *Advances in Civil Engineering*, vol. 2018, Article ID 7456830, 16 pages, 2018.

[10] Y. S. Yun, “Graphic representation of a dimensional expansion of triangular fuzzy number,” *Journal of Mathematics*, vol. 2021, Article ID 1970553, 11 pages, 2021.

[11] N. Khan, N. Yaqoob, M. Shams, Y. U. Gaba, and M. Riaz, “Solution of linear and quadratic equations based on triangular linear diophantine fuzzy numbers,” *Journal of Function Spaces*, vol. 2021, Article ID 8475863, 14 pages, 2021.

[12] J. Li and J. S. Chiou, “Two-dimensional fuzzy sliding mode control of a field-sensed magnetic suspension system,” *Mathematical Problems in Engineering*, vol. 2014, Article ID 386796, 9 pages, 2014.

[13] H. Nouriani and R. Ezzati, “Numerical solution of two-dimensional linear fuzzy fredholm integral equations by the fuzzy Lagrange interpolation,” *Advances in Fuzzy Systems*, vol. 2018, Article ID 5405124, 8 pages, 2018.

[14] Y. S. Yun, “Parametric operations for two 2-dimensional trapezoidal fuzzy sets,” *Journal of Algebra and Applied Mathematics*, vol. 18, no. 1, pp. 27–41, 2020.

[15] I. M. Krichever, “Two-dimensional algebraic-geometric operators with self-consistent potentials,” *Functional Analysis and Its Applications*, vol. 28, pp. 21–32, 1994.

[16] Y. S. Yun, “An algebraic operations for two generalized 2-dimensional quadratic fuzzy sets,” *Journal of the Chungcheong Mathematical Society*, vol. 31, no. 4, pp. 379–386, 2018.

[17] H. S. Ko, *Parametric operations for 2-dimensional fuzzy sets*, PhD Thesis, Jeju National University, Jeju City, Republic of Korea, 2018.

[18] A. Ebadian, F. Farahrooz, and A. Khajehnasiri, “On the convergence of two-dimensional fuzzy Volterra-Fredholm integral equations by using Picard method,” *Applications and Applied Mathematics*, vol. 11, no. 2, pp. 585–598, 2016.

[19] R. Chourasia, V. Saxena, N. Yadala, and C. H. Rhee, “Visualization of two-dimensional interval type-2 fuzzy membership functions using general type-2 fuzzy membership functions,” in *Proceedings of the 2017 Joint 17th World Congress of International Fuzzy Systems Association and 9th International Conference on Soft Computing and Intelligent Systems (IFSA-SCIS)*, Otsu, Japan, June 2017.

[20] H. J. Zimmermann, *Fuzzy Set Theory and its Applications*, Kluwer-Nijhoff Publishing, Leiden, Netherlands, 1985.

[21] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning-I,” *Information Sciences*, vol. 8, pp. 199–249, 1975.

[22] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning-II,” *Information Sciences*vol. 8, pp. 301–357, 1975.

[23] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning-III,” *Information Sciences*, vol. 9, pp. 43–80, 1975.
[24] J. Byun and Y. S. Yun, “Parametric operations for two fuzzy numbers,” *Communications of Korean Mathematical Society*, vol. 28, no. 3, pp. 635–642, 2013.
[25] C. Kim and Y. S. Yun, “Parametric operations for generalized 2-dimensional triangular fuzzy sets,” *International Journal of Mathematical Analysis*, vol. 11, no. 4, pp. 189–197, 2017.
[26] Y. S. Yun, “A Zadeh’s max-min composition operator for 3-dimensional triangular fuzzy number,” *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 3, pp. 3783–3793, 2020.