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Research Article

The (Multiplicative Degree-) Kirchhoff Index of Graphs Derived from the Cartesian Product of S_n and K_2

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It is well known that many topological indices have widespread use in lots of fields about scientific research, and the Kirchhoff index plays a major role in many different sectors over the years. Recently, Li et al. (Appl. Math. Comput. 382 (2020) 125335) proposed the problem of determining the Kirchhoff index and multiplicative degree-Kirchhoff index of graphs derived from $S_n \times K_2$, the Cartesian product of the star S_n, and the complete graph K_2. In the present study, we completely solve this problem, that is, the explicit closed-form formulae of the Kirchhoff index, multiplicative degree-Kirchhoff index, and number of spanning trees are obtained for some graphs derived from $S_n \times K_2$.

1. Introduction

In this study, we suppose that $G = (V, E)$ is a nontrivial simple and connected graph, where $V = \{v_1, v_2, \ldots, v_n\}$ and E are the vertex set and edge set of G, respectively. Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix, and $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ the degree matrix, where d_i is the degree of vertex v_i. Then, $L(G) = D(G) - A(G)$ is termed as the Laplacian matrix, and $\mathscr{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$ the normalized Laplacian matrix of graph G. It is easily seen that

$$(\mathscr{L}(G))_{ij} = \begin{cases} 1, & \text{if } i = j; \\ -\dfrac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j \text{ and } v_i v_j \in E; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $L(G)$ and $0 = \nu_1 < \nu_2 \leq \cdots \leq \nu_n$ the eigenvalues of $\mathscr{L}(G)$. The sets $Sp(L(G)) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ and $Sp(\mathscr{L}(G)) = \{\nu_1, \nu_2, \ldots, \nu_n\}$ are called the Laplacian spectrum and normalized Laplacian spectrum of G, respectively.

For two vertices v_i and v_j, the distance between them written as d_{ij} is the length of the shortest path linking them. The Wiener index [1] and Gutman index [2] of G are defined as $W(G) = \sum_{i<j} d_{ij}$ and $\operatorname{Gut}(G) = \sum_{i<j} d_i d_j d_{ij}$. For these two famous topological indices, one can refer to [3–10] and the references therein.

If regard each edge in $E(G)$ as an unit resistor, then for two vertices v_i and v_j, r_{ij} is represented as the effective resistance between them [11]. In the field of chemistry, resistance distance has been studied extensively and many profound results have been obtained. One of the most famous results is the Kirchhoff index, which is used to characterize the structure of a compound. The Kirchhoff index of G is written as $Kf(G) = \sum_{i<j} r_{ij}$. It is derived from molecular diagrams and is also a form used to numerically characterize molecules. Hitherto, the Kirchhoff index has been widely applied to mathematic, chemistry, physics, and so on. Later, the following relation between $Kf(G)$ and $Sp(L(G))$ was established by Zhu et al. [12] and Gutman and Mohar [13] independently.

Lemma 1 (See [12, 13]). *Let G be a simple graph of order $n \geq 2$. Then,*

$$Kf(G) = \sum_{i=2}^{n} \frac{1}{\mu_i}. \tag{2}$$

Similarly, Chen and Zhang [14] defined the multiplicative degree-Kirchhoff index of G as $Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}$. Moreover, the following relation between $Kf^*(G)$ and $Sp(\mathscr{L}(G))$ was confirmed.

Lemma 2 (See [14]). *Let G be a simple connected graph of order $n \geq 2$ and size m. Then,*

$$Kf^*(G) = 2m \sum_{i=2}^{n} \frac{1}{\nu_i}. \tag{3}$$

Nowadays, the (multiplicative degree-) Kirchhoff index has attracted a lot of attention from researchers over the few years. Furthermore, its closed-form formulae have been established depending on many kinds of graphs. For examples, the formulae of Kirchhoff index for cycles, circulant graphs, and composite graphs were obtained in [15, 16] and [17], respectively, and those of both indices for complete multipartite graphs were obtained in [18]. Besides, quite a few literature concerned the (multiplicative degree-) Kirchhoff index of polygon chains and their variants. Explicit expressions of the above index have been derived for linear polyomino chain [19], linear crossed polyomino chain [20], linear pentagonal chain [21], linear phenylenes [22, 23], cyclic phenylenes [24], Möbius phenylenes chain and cylinder phenylenes chain [25, 26], linear (n) phenylenes [27], generalized phenylenes [28, 29], linear hexagonal chain [30, 31], linear crossed hexagonal chain [32], Möbius hexagonal chain [33], periodic linear chains [34], linear octagonal chain [35], linear octagonal-quadrilateral chain [36], and linear crossed octagonal chain [37].

For two disjoint graphs G and H, the strong product of them is written as $G \otimes H$, that is, $V(G \otimes H) = V(G) \times V(H)$, and two distinct vertices (u_1, v_1) and (u_2, v_2) are contiguous. The Cartesian product of G and H, written as $G \times H$, is the graph with vertex set $V(G) \times V(H)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Figure 1 shows the graphs $S_n \otimes K_2$ and $S_n \times K_2$, where S_n and K_n denote the star and complete graph of order n, severally. Recently, Li et al. [38] determined the expressions of $Kf(S_r)$, $Kf^*(S_r)$, and $\tau(S_r)$, where S_r is a graph derived from $S_n \otimes K_2$ by randomly removing r vertical edges, and $\tau(G)$ denotes the number of spanning trees of a connected graph G. Finally, they proposed the problem of determining these three invariants for graphs derived from $S_n \times K_2$. In the present study, we completely solve this problem.

For convenience, we denote $S_n^2 = S_n \times K_2$. Then, $|V(S_n^2)| = 2n$ and $|E(S_n^2)| = 3n - 2$. Let $E' = \{ii' | i = 1, 2, \ldots, n\}$. $\mathscr{S}_{n,r}^2$ will denote the set of graphs derived from S_n^2 by discretionarily deleting r edges in E'. Obviously, the unique graph in $\mathscr{S}_{n,n}^2$ is disconnected; hence, we consider $\mathscr{S}_{n,r}^2$ for $0 \leq r \leq n-1$ only. Note also, $\mathscr{S}_{n,0}^2 = \{S_n^2\}$. In Section 2, some notations and known results are introduced, which will be applied to get our main results. In Section 3, explicit expressions of $Kf(S_n^2)$, $Kf^*(S_n^2)$, and $\tau(S_n^2)$ are obtained. Finally, $Kf(S_{n,r}^2)$ and $\tau(S_{n,r}^2)$ are determined in Section 4, where $S_{n,r}^2$ is an arbitrary graph in $\mathscr{S}_{n,r}^2$. Moreover, it is shown that $\lim_{n \longrightarrow +\infty} Kf(S_n^2)/W(S_n^2) = \lim_{n \longrightarrow +\infty} Kf(S_{n,r}^2)/W(S_{n,r}^2) = 8/15$ and $\lim_{n \longrightarrow +\infty} Kf^*(S_n^2)/\mathrm{Gut}(S_n^2) = 16/33$.

2. Preliminaries

In this section, we will introduce some basic concepts. These following celebrated definitions and fundamental lemmas can play a vital role in proving our consequences.

First, we mark the vertices of S_n^2 as in Figure 1; then, set $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{1', 2', \ldots, n'\}$. Therefore, we have

$$L(S_n^2) = \begin{pmatrix} L_{11}(S_n^2) & L_{12}(S_n^2) \\ L_{21}(S_n^2) & L_{22}(S_n^2) \end{pmatrix}, \quad \mathscr{L}(S_n^2) = \begin{pmatrix} \mathscr{L}_{11}(S_n^2) & \mathscr{L}_{12}(S_n^2) \\ \mathscr{L}_{21}(S_n^2) & \mathscr{L}_{22}(S_n^2) \end{pmatrix}, \tag{4}$$

where $L_{ij}(S_n^2)$ ($\mathscr{L}_{ij}(S_n^2)$) is the submatrix of $L(S_n^2)$ (respectively, $\mathscr{L}(S_n^2)$) whose rows (columns) correspond to the vertices in V_i (respectively V_j). It is easily seen that $L_{11}(S_n^2) = L_{22}(S_n^2)$, $L_{12}(S_n^2) = L_{21}(S_n^2)$, $\mathscr{L}_{11}(S_n^2) = \mathscr{L}_{22}(S_n^2)$, and $\mathscr{L}_{12}(S_n^2) = \mathscr{L}_{21}(S_n^2)$.

Let

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} I_n & \frac{1}{\sqrt{2}} I_n \\ \frac{1}{\sqrt{2}} I_n & -\frac{1}{\sqrt{2}} I_n \end{pmatrix}, \tag{5}$$

and then, we have

$$TL(S_n^2)T = \begin{pmatrix} L_A(S_n^2) & 0 \\ 0 & L_S(S_n^2) \end{pmatrix}, \quad T\mathscr{L}(S_n^2)T = \begin{pmatrix} \mathscr{L}_A(S_n^2) & 0 \\ 0 & \mathscr{L}_S(S_n^2) \end{pmatrix}, \tag{6}$$

where $L_A(S_n^2) = L_{11}(S_n^2) + L_{12}(S_n^2)$, $L_S(S_n^2) = L_{11}(S_n^2) - L_{12}(S_n^2)$, $\mathscr{L}_A(S_n^2) = \mathscr{L}_{11}(S_n^2) + \mathscr{L}_{12}(S_n^2)$, and $\mathscr{L}_S(S_n^2) = \mathscr{L}_{11}(S_n^2) - \mathscr{L}_{12}(S_n^2)$.

Based on the above arguments, by applying the technique used in [32, 39], we immediately have the following decomposition theorem, where $\Phi(B, \lambda) = |\lambda I - B|$ stands for the characteristic polynomial of B.

Lemma 3 *Let $L_A(S_n^2)$, $L_S(S_n^2)$, $\mathscr{L}_A(S_n^2)$, and $\mathscr{L}_S(S_n^2)$ be written as above. Thus, we obtain that*

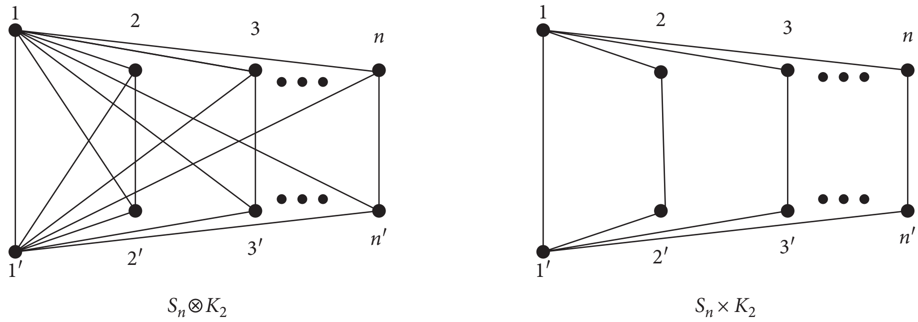

FIGURE 1: The graphs $S_n \otimes K_2$ and $S_n \times K_2$.

$$\begin{aligned}\Phi\left(L\left(S_n^2\right),\lambda\right) &= \Phi\left(L_A\left(S_n^2\right),\lambda\right)\Phi\left(L_S\left(S_n^2\right),\lambda\right),\\ \Phi\left(\mathscr{L}\left(S_n^2\right),\lambda\right) &= \Phi\left(\mathscr{L}_A\left(S_n^2\right),\lambda\right)\Phi\left(\mathscr{L}_S\left(S_n^2\right),\lambda\right).\end{aligned} \tag{7}$$

Lemma 4 (See [40]). *Assume that G is a connected graph with $n \geq 2$ vertices; then,*

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\mu_i. \tag{8}$$

3. Results for S_n^2

In this section, we will derive explicit expressions of $Kf(S_n^2)$, $Kf^*(S_n^2)$, and $\tau(S_n^2)$ as follows.

3.1. On $Kf(S_n^2)$ and $\tau(S_n^2)$. Obviously,

$$L_{11}\left(S_n^2\right) = \begin{pmatrix} n & -1 & -1 & \dots & -1\\ -1 & 2 & 0 & \dots & 0\\ -1 & 0 & 2 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ -1 & 0 & 0 & \dots & 2\end{pmatrix}_{n\times n},$$
$$L_{12}\left(S_n^2\right) = \begin{pmatrix} -1 & 0 & 0 & \dots & 0\\ 0 & -1 & 0 & \dots & 0\\ 0 & 0 & -1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & -1\end{pmatrix}_{n\times n}. \tag{9}$$

Hence,

$$L_A\left(S_n^2\right) = L_{11}\left(S_n^2\right) + L_{12}\left(S_n^2\right) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1\\ -1 & 1 & 0 & \dots & 0\\ -1 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ -1 & 0 & 0 & \dots & 1\end{pmatrix}_{n\times n}, \tag{10}$$

and we easily have $Sp(L_A(S_n^2)) = \{0, 1^{n-2}, n\}$, where a^k denotes the k successive a's.

Similarly, we have

$$L_S\left(S_n^2\right) = L_{11}\left(S_n^2\right) - L_{12}\left(S_n^2\right) = \begin{pmatrix} n+1 & -1 & -1 & \dots & -1\\ -1 & 3 & 0 & \dots & 0\\ -1 & 0 & 3 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ -1 & 0 & 0 & \dots & 3\end{pmatrix}_{n\times n}, \tag{11}$$

and get $Sp(L_S(S_n^2)) = \{2, 3^{n-2}, n+2\}$.

Hence, $Sp(L(S_n^2)) = \{0, 1^{n-2}, 2, 3^{n-2}, n, n+2\}$ from Lemma 3, and we get the following result.

Theorem 1. *Let $S_n^2 = S_n \times K_2$. Then,*

(1) $Kf(S_n^2) = 8n^3 + 3n^2 - 14n + 12/3n + 6$

(2) $\tau(S_n^2) = (n+2)\cdot 3^{n-2}$

(3) $lim_{n\longrightarrow+\infty}Kf(S_n^2)/W(S_n^2) = 8/15$

Proof. From Lemma 1, we have

$$Kf\left(S_n^2\right) = 2n\left[(n-2) + \frac{1}{2} + \frac{n-2}{3} + \frac{1}{n} + \frac{1}{n+2}\right] = \frac{8n^3 + 3n^2 - 14n + 12}{3(n+2)}. \tag{12}$$

From Lemma 4, we immediately have

$$\tau\left(S_n^2\right) = \frac{1}{2n}\cdot 2\cdot 3^{n-2}\cdot n\cdot(n+2) = (n+2)\cdot 3^{n-2}. \tag{13}$$

Finally, we end the proof by confirming that $W(S_n^2) = 5n^2 - 8n + 4$. Let $w_i = \sum_{j\in V(S_n^2)} d_{ij}$. Obviously, $w_i = 1\cdot n + 2(n-1) = 3n - 2$ if $i = 1, 1'$, and $w_i = 1 + 1 + 2(n-1) + 3(n-2) = 5n - 6$ otherwise. Hence,

$$W\left(S_n^2\right) = \frac{1}{2}\sum_{i\in V\left(S_n^2\right)} w_i = \frac{1}{2}\left[2(3n-2) + (2n-2)(5n-6)\right] = 5n^2 - 8n + 4. \tag{14}$$

□

3.2. On $Kf^*(S_n^2)$. Consequently, we will determine $Kf^*(S_n^2)$. Obviously,

$$\mathscr{L}_{11}\left(S_n^2\right) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2n}} & -\frac{1}{\sqrt{2n}} & \cdots & -\frac{1}{\sqrt{2n}} \\ -\frac{1}{\sqrt{2n}} & 1 & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{2n}} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\sqrt{2n}} & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n},$$

$$\mathscr{L}_{12}\left(S_n^2\right) = \begin{pmatrix} -\frac{1}{n} & 0 & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} \end{pmatrix}_{n\times n}. \tag{15}$$

Hence,

$$\mathscr{L}_A\left(S_n^2\right) = \mathscr{L}_{11}\left(S_n^2\right) + \mathscr{L}_{12}\left(S_n^2\right) = \begin{pmatrix} \frac{n-1}{n} & -\frac{1}{\sqrt{2n}} & -\frac{1}{\sqrt{2n}} & \cdots & -\frac{1}{\sqrt{2n}} \\ -\frac{1}{\sqrt{2n}} & \frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{2n}} & 0 & \frac{1}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\sqrt{2n}} & 0 & 0 & \cdots & \frac{1}{2} \end{pmatrix}_{n\times n}, \tag{16}$$

and we easily have $Sp(\mathscr{L}_A(S_n^2)) = \{0, (1/2)^{n-2}, 3n - 2/2n\}$.

Similarly, we have

$$\mathscr{L}_S\left(S_n^2\right) = \mathscr{L}_{11}\left(S_n^2\right) - \mathscr{L}_{12}\left(S_n^2\right) = \begin{pmatrix} \frac{n+1}{n} & -\frac{1}{\sqrt{2n}} & -\frac{1}{\sqrt{2n}} & \cdots & -\frac{1}{\sqrt{2n}} \\ -\frac{1}{\sqrt{2n}} & \frac{3}{2} & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{2n}} & 0 & \frac{3}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\sqrt{2n}} & 0 & 0 & \cdots & \frac{3}{2} \end{pmatrix}_{n\times n}, \tag{17}$$

and get $Sp(\mathscr{L}_S(S_n^2)) = \{2, (3/2)^{n-2}, n + 2/2n\}$.

Hence, $Sp(\mathscr{L}(S_n^2)) = \{0, (1/2)^{n-2}, n + 2/2n, 3n - 2/2n, (3/2)^{n-2}, 2\}$ from Lemma 3, and we immediately have the following result.

Theorem 2. *Let $S_n^2 = S_n \times K_2$. Then,*

(1) $Kf^*(S_n^2) = 48n^3 + 25n^2 - 180n + 116/3n + 6$

(2) $\lim_{n\longrightarrow+\infty} Kf^*(S_n^2)/Gut(S_n^2) = 16/33$

Proof. From Lemma 2, it is easily confirmed that

$$Kf^*\left(S_n^2\right) = 2(3n-2)\left[2n-4+\frac{2n}{n+2}+\frac{2n}{3n-2}+\frac{2(n-2)}{3}+\frac{1}{2}\right],$$
$$= \frac{48n^3+25n^2-180n+116}{3n+6}. \tag{18}$$

Now, let $g_i = \sum_{j\in V(S_n^2)} d_i d_j d_{ij}$. Obviously, if $i = 1, 1'$, then

$$g_i = n\cdot 2\cdot 1 + n\cdot 2\cdot 1\cdot (n-1) + n\cdot 2\cdot 2\cdot (n-1) = 7n^2-6n, \tag{19}$$

and otherwise,

$$g_i = 2\cdot n\cdot 1 + 2\cdot 2\cdot 1 + 2\cdot n\cdot 2 + 2\cdot 2\cdot 2\cdot (n-2) + 2\cdot 2\cdot 3\cdot (n-2) = 26n-36. \tag{20}$$

Hence,

$$\text{Gut}\left(S_n^2\right) = \frac{1}{2}\sum_{i\in V(S_n^2)} g_i = \frac{1}{2}\left[2\left(7n^2-6n\right)+(26n-36)(2n-2)\right] = 33n^2-68n+36, \tag{21}$$

and it follows that

$$\lim_{n\longrightarrow+\infty}\frac{Kf^*\left(S_n^2\right)}{\text{Gut}\left(S_n^2\right)} = \lim_{n\longrightarrow+\infty}\frac{48n^3+25n^2-180n+116}{(3n+6)\left(33n^2-68n+36\right)} = \frac{16}{33}, \tag{22}$$

which completes the proof. □

4. Results for Graphs in $\mathcal{S}_{n,r}^2$

Assume that $S_{n,r}^2$ is any graph in $\mathcal{S}_{n,r}^2$, $1\le r\le n-1$. We will carry out a computational study on $Kf(S_{n,r}^2)$ and $\tau(S_{n,r}^2)$ in this section.

We suppose that d_i is the degree of i in $S_{n,r}^2$. Therefore, $d_i = n$ or $n-1$ if $i = 1, 1'$, and $d_i = 1$ or 2 otherwise. We will compute $Sp(S_{n,r}^2)$ in the following.

Case 1. Edge $11' \notin (E_{n,r}^2)$. Then,

$$L_{11}\left(S_{n,r}^2\right) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & d_2 & 0 & \dots & 0 \\ -1 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & d_n \end{pmatrix},$$
$$L_{12}\left(S_{n,r}^2\right) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ 0 & 0 & t_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_n \end{pmatrix}, \tag{23}$$

where $t_i = 0$ if $d_i = 1$ and $t_i = 1$ if $d_i = 2$, $2\le i\le n$. Hence,

$$L_A\left(S_{n,r}^2\right) = L_{11}\left(S_{n,r}^2\right) + L_{12}\left(S_{n,r}^2\right) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}_{n\times n}, \tag{24}$$

and $Sp(L_A(S_{n,r}^2)) = \{0, 1^{n-2}, n\}$. On the other hand,

$$L_S(S_{n,r}^2) = L_{11}(S_{n,r}^2) - L_{12}(S_{n,r}^2) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & d_2 - t_2 & 0 & \dots & 0 \\ -1 & 0 & d_3 - t_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & d_n - t_n \end{pmatrix}, \tag{25}$$

where $d_i - t_i = 1$ if $d_i = 1$ and $d_i - t_i = 3$ if $d_i = 2$, $2 \leq i \leq n$. We will compute $Sp(L_S(S_{n,r}^2))$ in the following cases.

Case 1.1. $r = 1$. Then, $d_i - t_i = 3$, $2 \leq i \leq n$, and we easily have

$$Sp(L_S(S_{n,r}^2)) = \left\{ 3^{n-2}, \frac{n+2+\sqrt{n^2-4n+12}}{2}, \frac{n+2-\sqrt{n^2-4n+12}}{2} \right\}. \tag{26}$$

Case 1.2. $r \geq 2$. By direct calculations, we have

$$\Phi(L_S(S_{n,r}^2), \lambda) = \left[\lambda^3 - (n+3)\lambda^2 + 3n\lambda + 2r - 2n\right](\lambda - 1)^{r-2}(\lambda - 3)^{n-r-1}. \tag{27}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the three roots of $\lambda^3 - (n+3)\lambda^2 + 3n\lambda + 2r - 2n = 0$. Then, $Sp(L_S(S_{n,r}^2)) = \{1^{r-2}, 3^{n-r-1}, \lambda_1, \lambda_2, \lambda_3\}$, and it holds that $\lambda_1\lambda_2\lambda_3 = 2n - 2r$ and

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} = \frac{3n}{2n - 2r}, \tag{28}$$

Case 2. $11' \in (E_{n,r}^2)$. Then,

$$L_{11}(S_{n,r}^2) = \begin{pmatrix} n & -1 & -1 & \dots & -1 \\ -1 & d_2 & 0 & \dots & 0 \\ -1 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & d_n \end{pmatrix}, \quad L_{12}(S_{n,r}^2) = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ 0 & 0 & t_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_n \end{pmatrix}, \tag{29}$$

where $t_i = 0$ if $d_i = 1$ and $t_i = -1$ if $d_i = 2$, $2 \leq i \leq n$. Hence,

$$L_A(S_{n,r}^2) = L_{11}(S_{n,r}^2) + L_{12}(S_{n,r}^2) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}_{n \times n}, \tag{30}$$

and $Sp(L_A(S_{n,r}^2)) = \{0, 1^{n-2}, n\}$. On the other hand,

$$L_S(S_{n,r}^2) = L_{11}(S_{n,r}^2) - L_{12}(S_{n,r}^2) = \begin{pmatrix} n+1 & -1 & -1 & \ldots & -1 \\ -1 & d_2 - t_2 & 0 & \ldots & 0 \\ -1 & 0 & d_3 - t_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & 0 & \ldots & d_n - t_n \end{pmatrix}, \quad (31)$$

where $d_i - t_i = 1$ if $d_i = 1$ and $d_i - t_i = 3$ if $d_i = 2$, $2 \leq i \leq n$. By direct calculations, we have

$$\Phi(L_S(S_{n,r}^2), \lambda) = [\lambda^3 - (n+5)\lambda^2 + (3n+8)\lambda + 2r - 2n - 4](\lambda - 1)^{r-1}(\lambda - 3)^{n-r-2}. \quad (32)$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the three roots of $\lambda^3 - (n+5)\lambda^2 + (3n+8)\lambda + 2r - 2n - 4 = 0$. Then, $Sp(L_S(S_{n,r}^2)) = \{1^{r-1}, 3^{n-r-2}, \lambda_1, \lambda_2, \lambda_3\}$, and it holds that $\lambda_1\lambda_2\lambda_3 = 2n - 2r + 4$ and

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} = \frac{3n+8}{2n-2r+4}, \quad (33)$$

from Vieta's theorem.

Now, we are able to give the main result of this section.

Theorem 3. *If $S_{n,r}^2 \in \mathcal{S}_{n,r}^2$, $0 \leq r \leq n-1$, then*

(1) $Kf(S_{n,r}^2) = \{ (8n^3 - (4r+17)n^2 - (4r^2 - 26r - 6)n - 6r)/3(n-r)$, if $11' \notin E(S_{n,r}^2)$, $8n^3 - (4r-3)n^2 - (4r^2 - 30r + 14)n + 12 - 6r/3(n-r-2)$, if $11' \in E(S_{n,r}^2)$.

(2) $\tau(S_{n,r}^2) = \begin{cases} (n-r)\cdot 3^{n-r-1}, & \text{if } 11' \notin E(S_{n,r}^2), \\ (n-r+2)\cdot 3^{n-r+2}, & \text{if } 11' \in E(S_{n,r}^2), \end{cases}$

(3) $\lim_{n\longrightarrow+\infty} Kf(S_{n,r}^2)/W(S_{n,r}^2) = 8/15$

Proof. If $r = 0$, then $S_{n,r}^2 \cong S_n^2$, and the conclusion holds from Theorem 1. Hence, assume $r \geq 1$. We distinguish the following two cases.

Case 1. Edge $11' \notin (E_{n,r}^2)$.

Case 1.1. $r = 1$. Then,

$$Sp(L(S_{n,r}^2)) = \left\{0, 1^{n-2}, n, 3^{n-2}, \frac{n+2-\sqrt{n^2-4n+12}}{2}, \frac{n+2+\sqrt{n^2-4n+12}}{2}\right\}. \quad (34)$$

From Lemma 1, we have

$$\begin{aligned} Kf(S_{n,r}^2) &= 2n\left[n - 2 + \frac{1}{n} + \frac{n-2}{3} + \frac{2}{n+2-\sqrt{n^2-4n+12}} + \frac{2}{n+2+\sqrt{n^2-4n+12}}\right], \\ &= \frac{8n^3 - 21n^2 + 28n - 6}{3(n-1)} \\ &= \frac{8n^3 - (4r+17)n^2 - (4r^2 - 26r - 6)n - 6r}{3(n-r)}. \end{aligned} \quad (35)$$

Then, from Lemma 2, we have

$$\tau\left(S_{n,r}^2\right) = \frac{1}{2n}\left[n \cdot 3^{n-2} \cdot \frac{n+2-\sqrt{n^2-4n+12}}{2} \cdot \frac{n+2+\sqrt{n^2-4n+12}}{2}\right],$$
$$= (n-1)\cdot 3^{n-2}$$
$$= (n-r)\cdot 3^{n-r-1}. \tag{36}$$

Case 1.2. $r \geq 2$. Then, $Sp(L(S_{n,r}^2)) = \{0, 1^{n+r-4}, n, 3^{n-r-1}, \lambda_1, \lambda_2, \lambda_3\}$, where $\lambda_1\lambda_2\lambda_3 = 2n-2r$ and $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = 3n/(2n-2r)$. From Lemma 1, we have

$$Kf\left(S_{n,r}^2\right) = 2n\left[n+r-4+\frac{1}{n}+\frac{n-r-1}{3}+\frac{3n}{2n-2r}\right],$$
$$= \frac{8n^3-(4r+17)n^2-\left(4r^2-26r-6\right)n-6r}{3(n-r)}. \tag{37}$$

Then, from Lemma 2, we have

$$\tau\left(S_{n,r}^2\right) = \frac{n\cdot 3^{n-r-1}\cdot\lambda_1\cdot\lambda_2\cdot\lambda_3}{2n} = \frac{n\cdot 3^{n-r-1}\cdot(2n-2r)}{2n}$$
$$= (n-r)\cdot 3^{n-r-1}. \tag{38}$$

Case 2. Edge $11' \in (E_{n,r}^2)$. Then, $Sp(L(S_{n,r}^2)) = \{0, 1^{n+r-3}, n, 3^{n-r-2}, \lambda_1, \lambda_2, \lambda_3\}$, where $\lambda_1\lambda_2\lambda_3 = 2n-2r+4$ and $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = (3n+8)/(2n-2r+4)$. From Lemma 1, we have

$$Kf\left(S_{n,r}^2\right) = 2n\left[n+r-3+\frac{1}{n}+\frac{n-r-2}{3}+\frac{3n+8}{2n-2r+4}\right],$$
$$= \frac{8n^3-(4r-3)n^2-\left(4r^2-30r+14\right)n+12-6r}{3(n-r+2)}. \tag{39}$$

Then, from Lemma 2, we have

$$\tau\left(S_{n,r}^2\right) = \frac{n\cdot 3^{n-r-2}\cdot\lambda_1\cdot\lambda_2\cdot\lambda_3}{2n} = \frac{n\cdot 3^{n-r-2}\cdot(2n-2r+4)}{2n} = (n-r+2)\cdot 3^{n-r-2}. \tag{40}$$

Finally, it is straightforward to have $W(S_{n,r}^2) = W(S_n^2) + r = 5n^2 - 8n + r + 4$. Hence, in both cases, it holds that

$$\lim_{n\longrightarrow+\infty}\frac{Kf\left(S_{n,r}^2\right)}{W\left(S_{n,r}^2\right)} = \frac{8}{15}. \tag{41}$$

□

Data Availability

The data used to support this study are included within the article.

Disclosure

This study is also presented in arXiv (https://arxiv.org/abs/2007.10674, cited as [41]).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Jia-Bao Liu conceptualized the study, developed methodology, and wrote original draft. Xin-Bei Peng conceptualized the study and wrote original draft. Jiao-Jiao Gu collected resources. Wenshui Lin visualized, investigated, and conceptualized the study and collected resources.

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