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Research Article

The Third Logarithmic Coefficient for Certain Close-to-Convex Functions

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The logarithmic coefficients γ_n of a normalized analytic functions f are defined by $(\log f(z)/z) = 2\sum_{n=1}^{\infty} c_n z^n$. For certain close-to-convex functions $f(z) = z + a_2 z^2 + \cdots$, Cho et al. (on the third logarithmic coefficient in some subclasses of close-to-convex functions) has obtained the upper bound of the third logarithmic coefficient γ_3 when the second coefficient a_2 is real. In the present paper, the upper bound of the third logarithmic coefficient γ_3 is computed with no restriction on the second coefficient a_2.

1. Introduction and Preliminaries

Let $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathscr{A}$ be the set of all analytic normalized functions $f\colon \mathbb{D} \longrightarrow \mathbb{C}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathscr{S}$ be its subclass consisting of functions that are univalent in $\mathbb{D}$. Given a function $f \in \mathscr{S}$, the coefficients γ_n are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}\backslash\{0\},\ \log 1 := 0. \tag{2}$$

For example (see Figure 1), for the Koebe function k given by $k(z) = (z/(1-z)^2)$, the logarithmic coefficients $\gamma_n = (1/n)$ are as follows

$$\log \frac{k(z)}{z} = 2\sum_{n=1}^{\infty} \frac{1}{n} z^n. \tag{3}$$

The Milin conjecture ([1] and ([2] p. 155)) gives an inequality satisfied by the logarithmic coefficients. For $f \in \mathscr{S}$, the logarithmic coefficients satisfy

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \le 0. \tag{4}$$

The Milin conjecture was confirmed (e.g., ([2] p. 37), by Branges [3] and implies the famous Bieberbach conjecture that $|a_n| \le n$ for $f \in \mathscr{S}$. Sharp estimates for the class $\mathscr{S}$ are known only for the first two coefficients:

$$\begin{aligned} |\gamma_1| &\le 1,\\ |\gamma_2| &\le \frac{1}{2} + \frac{1}{e} = 0.635\ldots. \end{aligned} \tag{5}$$

Note that Obradović and Tuneski [4] obtained an upper bound of $|\gamma_3|$ for the class $\mathscr{S}$. The problem of estimating the modulus of the first three logarithmic coefficients is significantly studied for the subclasses of $\mathscr{S}$, and in some cases, sharp bounds are obtained. For instance, sharp estimates for the class of starlike functions $\mathscr{S}^*$ are given by the inequality $|\gamma_n| \le (1/n)$ holds for $n \in \mathbb{N}$ ([5], p. 42).

Furthermore, for $f \in \mathscr{S}\mathscr{S}^*$, the class of strongly starlike function of the order β, $(0 \le \beta \le 1)$, it holds that $|\gamma_n| \le (\beta/n)(n \in \mathbb{N})$ [6]. The bounds of γ_n for functions in subclasses of $\mathscr{S}$ have been widely studied in recent years. Sharp estimates for different subclasses are given in [6, 7] and ([5], p. 116) and [8], respectively, while nonsharp

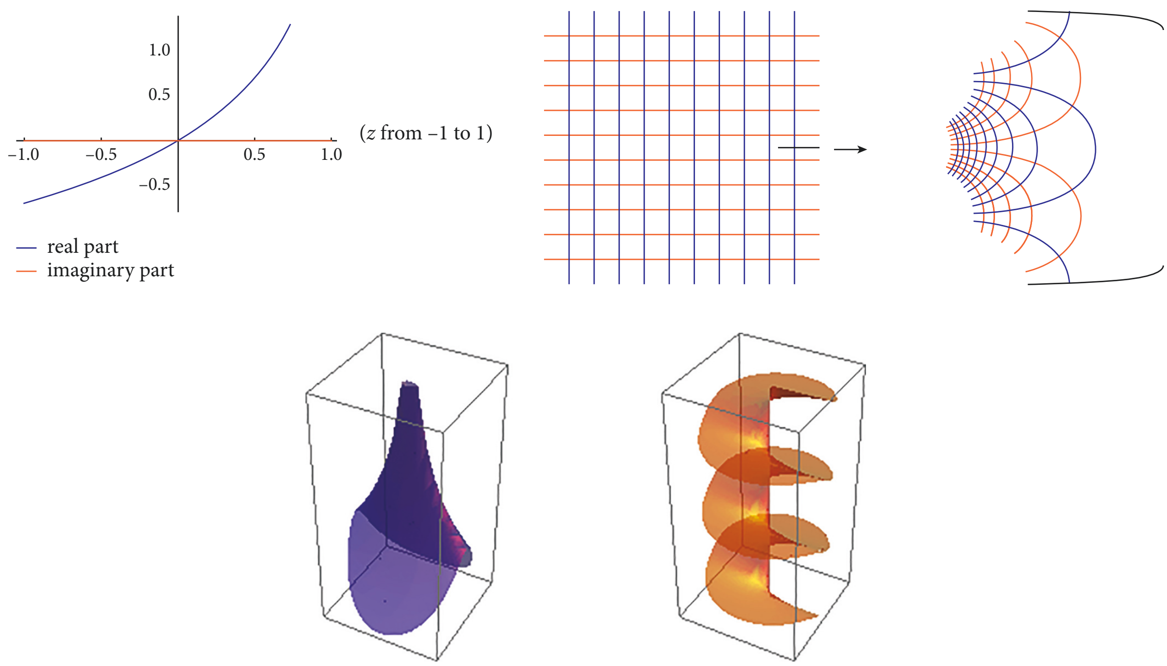

Figure 1: Plot of $\log(f(z)/z)$, $f(z) = (z/(1-z)^2)$.

estimates for the class of Bazilevic and close-to-convex are given in [9–11], respectively.

Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be the subclasses of $\mathcal{S}$ satisfying, respectively, the next conditions:

$$\begin{aligned} &\Re\{(1-z)f'(z)\} > 0, \quad z \in \mathbb{D},\\ &\Re\{(1-z^2)f'(z)\} > 0, \quad z \in \mathbb{D},\\ &\Re\{(1-z+z^2)f'(z)\} > 0, \quad z \in \mathbb{D}. \end{aligned} \tag{6}$$

Note that each class defined above is the subclass of the well-known class of close-to-convex functions; consequently, families $\mathcal{F}_i$, $i = 1, 2, 3$, contain only univalent functions ([2], Vol. II, p. 2). The sharp bounds of γ_1, γ_2 and partial results for γ_3 of the subclasses $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $\mathcal{S}$ were determined by Pranav Kumar and Vasudevarao [12].

Moreover, Cho et al. [13] computed the sharp upper bounds for the third logarithmic coefficient γ_3 of f when a_2 is a real number. Differentiating (1) and comparing the coefficients with (2), we get $\gamma_1 = (1/2)a_1$, $\gamma_2 = (1/2)(a_3 - (1/2)a^2)$, and

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{7}$$

The main aim of this paper is to determine the upper bound of the third logarithmic coefficient in the general case of a_2. The following lemma is needed to prove our main results.

Lemma 1 (see [14]). *Let $w(z) = c_1z + c_2z^2 + \cdots$ be a Schwarz function. Then*

$$\begin{aligned} &|c_1| \le 1,\\ &|c_2| \le 1 - |c_1|^2,\\ &|c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}. \end{aligned} \tag{8}$$

2. Main Results

Our main result is as follows:

Theorem 1. *Let $f \in \mathcal{F}_1$. Then*

$$|\gamma_3| \le \frac{15.75}{48} = 0.328125. \tag{9}$$

Proof. Since $f \in \mathcal{F}_1$, and for analytic function w in $\mathbb{D}$ with $w(0) = 0$ satisfying the formula

$$(1-z)f'(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2w(z) + 2w^2(z) + \cdots. \tag{10}$$

We obtain

$$w(z) = c_1z + c_2z^2 + \cdots. \tag{11}$$

Then, by using (10) along with (11) leads to

$$\begin{aligned} a_2 &= \frac{1}{2}\left(1+2c_1\right),\\ a_3 &= \frac{1}{3}\left(1+2c_1+2c_1^2+2c_2\right),\\ a_4 &= \frac{1}{4}\left(1+2c_1+2c_2+2c_3+2c_1^2+4c_1c_2+2c_1^3\right). \end{aligned} \tag{12}$$

From (7) and (12), we obtain

$$\gamma_3 = \frac{1}{48}\left(3+2c_1+4c_2+12c_3+8c_1c_2+4c_1^3\right). \tag{13}$$

In view of Lemma 1, we attain

$$\begin{aligned} 48\left|\gamma_3\right| &\le 3+2\left|c_1\right|+4\left|c_2\right|+12\left|c_3\right|+8\left|c_1\right|\left|c_2\right|+4\left|c_1\right|^3\\ &\le 3+2\left|c_1\right|+4\left|c_2\right|+12\left(1-\left|c_1\right|^2-\frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right)\\ &\quad+8\left|c_1\right|\left|c_2\right|+4\left|c_1\right|^3 =: f_1\left(\left|c_1\right|,\left|c_2\right|\right), \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_1(x,y) &= 3+2x+4y+12\left(1-x^2-\frac{y^2}{1+x}\right)\\ &\quad+8xy+4x^3,\\ &(x,y)\in E\colon 0\le x\le 1,\ 0\le y\le 1-x^2. \end{aligned} \tag{15}$$

The system

$$\begin{aligned} \frac{\partial f_1(x,y)}{\partial x} &= 2-24x+12\left(\frac{y}{1+x}\right)^2+8y+12x^2 = 0,\\ \frac{\partial f_1(x,y)}{\partial y} &= 4-\frac{24y}{1+x}+8x = 0, \end{aligned} \tag{16}$$

has a unique solution $(x_1, y_1) = ((1/4), (5/16)) \in E\backslash\partial E$ with

$$f_1\left(x_1,y_1\right) = 15.75. \tag{17}$$

The maximum value of f_1 is obtained when (x, y) is a point on the boundary of E. In view of this, we have

$$\begin{aligned} f_1(x,0) &= 15+2x-12x^2+4x^3\\ &\le 9+\frac{10\sqrt{30}}{9} = 15.08580\ldots,\\ f_1(0,y) &= 3+4y+12\left(1-y^2\right)\\ &= 15+4y-12y^2 \le \frac{46}{3} = 15.33\ldots, \end{aligned} \tag{18}$$

and

$$f_1\left(x,1-x^2\right) = 7+22x-4x^2-16x^3 \le 15.304035\ldots. \tag{19}$$

Using (14) and (17)–(19), we conclude the following outcome:

$$48\left|\gamma_3\right| \le 15.75, \quad \text{i.e., } \left|\gamma_3\right| \le 0.328125. \tag{20}$$

This completes the proof. □

Remark 1. If $f \in \mathcal{F}_1$, where $f''(0)$ is a real number, then we get the result in [13]

$$\left|\gamma_3\right| \le \frac{1}{288}\left(11+15\sqrt{30}\right) = 0.323466\ldots. \tag{21}$$

Theorem 2. *Let $f \in \mathcal{F}_2$. Then*

$$\left|\gamma_3\right| \le 0.258765\ldots. \tag{22}$$

Proof. Since $f \in \mathcal{F}_2$, then there exists an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and

$$\left(1-z^2\right)f'(z) = \frac{1+w(z)}{1-w(z)} = 1+2w(z)+2w^2(z)+\cdots. \tag{23}$$

The coefficients can be determined by comparing the information in (11) and (23)

$$\begin{aligned} a_2 &= c_1,\\ a_3 &= \frac{1}{3}\left(1+2c_2+2c_1^2\right),\\ a_4 &= \frac{1}{2}\left(c_1+c_3+2c_1c_2+c_1^3\right). \end{aligned} \tag{24}$$

From (7) and (24), we have the following conclusion:

$$\gamma_3 = \frac{1}{12}\left(c_1+3c_3+2c_1c_2+c_1^3\right). \tag{25}$$

Moreover, according to Lemma 1, we get the following inequality:

$$\begin{aligned} 12\left|\gamma_3\right| &\le \left|c_1\right|+3\left|c_3\right|+2\left|c_1\right|\left|c_2\right|+\left|c_1\right|^3\\ &\le \left|c_1\right|+3\left(1-\left|c_1\right|^2-\frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right)\\ &\quad+2\left|c_1\right|\left|c_2\right|+\left|c_1\right|^3 =: f_2\left(\left|c_1\right|,\left|c_2\right|\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} f_2(x,y) &= 3\left(1-x^2-\frac{y^2}{1+x}\right)+2xy+x+x^3,\\ &(x,y)\in E\colon 0\le x\le 1,\ 0\le y\le 1-x^2. \end{aligned} \tag{27}$$

From the system,

$$\frac{\partial f_2(x,y)}{\partial x} = 6x + 3\left(\frac{y}{1+x}\right)^2 + 2y + 1 + 3x^2 = 0,$$
$$\frac{\partial f_2(x,y)}{\partial y} = -\frac{6y}{1+x} + 2x = 0, \tag{28}$$

only one solution (x_2, y_2) lies in the interior of E, where

$$x_2 = \frac{4-\sqrt{7}}{6} = 0.22570\ldots,$$
$$y_2 = \frac{47-14\sqrt{7}}{108} = 0.092217\ldots, \tag{29}$$

and

$$f_2(x_2, y_2) = 3.10518\ldots. \tag{30}$$

On the boundary of E, we have the next property

$$f_2(x,0) = 3\left(1-x^2\right) + x + x^3 \le 2 + \frac{4}{9}\sqrt{6} = 3.08866, \quad \text{for } 0 \le x \le 1,$$
$$f_2(0,y) = 3\left(1-y^2\right) \le 3, \quad \text{for } 0 \le y \le 1,$$
$$f_2\left(x, 1-x^2\right) = 6x - 4x^3 \le 2\sqrt{2} = 2.82842\ldots. \tag{31}$$

Consequently, (26), (30), and (31) yield

$$12|\gamma_3| \le 3.10518\ldots, \quad \text{i.e., } |\gamma_3| \le 0.258765\ldots. \tag{32}$$

□

Remark 2. If $f \in \mathscr{F}_2$, where $f''(0)$ is a real number, then [13]

$$|\gamma_3| \le \frac{1}{972}\left(95 + 23\sqrt{46}\right) = 0.258223\ldots. \tag{33}$$

Theorem 3. *Let $f \in \mathscr{F}_3$. Then*

$$|\gamma_3| \le \frac{17.75}{48} = 0.36979\ldots. \tag{34}$$

Proof. Let $f \in \mathscr{F}_3$ and an analytic function w in $\mathbb{D}$ with $w(0) = 0$ such that

$$\left(1 - z + z^2\right) f'(z) = \frac{1+w(z)}{1-w(z)} = 1 + 2w(z) + 2w^2(z) + \cdots. \tag{35}$$

Substituting (11) into (35), we have

$$a_2 = \frac{1}{2}\left(1 + 2c_1\right),$$
$$a_3 = \frac{2}{3}\left(c_1 + c_2 + c_1^2\right), \tag{36}$$
$$a_4 = \frac{1}{4}\left(2c_2 + 2c_3 + 2c_1^2 + 2c_1^3 + 4c_1c_2 - 1\right).$$

By using (7) and (36), we obtain

$$\gamma_3 = \frac{1}{48}\left(-5 - 2c_1 + 4c_2 + 12c_3 + 8c_1c_2 + 4c_1^3\right). \tag{37}$$

According to Lemma 1, we conclude that

$$48|\gamma_3| \le 5 + 2|c_1| + 4|c_2| + 12|c_3| + 8|c_1||c_2| + 4|c_1|^3$$
$$\le 5 + 2|c_1| + 4|c_2| + 12\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) \tag{38}$$
$$+ 8|c_1||c_2| + 4|c_1|^3 =: f_3\left(|c_1|, |c_2|\right),$$

where

$$f_3(x,y) = 5 + 2x + 4y + 12\left(1 - x^2 - \frac{y^2}{1+x}\right) + 8xy + 4x^3,$$
$$(x,y) \in E\colon\ 0 \le x \le 1,\ 0 \le y \le 1 - x^2. \tag{39}$$

The system

$$\frac{\partial f_3(x,y)}{\partial x} = 2 - 24x + 12\left(\frac{y}{1+x}\right)^2 + 8y + 12x^2 = 0,$$
$$\frac{\partial f_3(x,y)}{\partial y} = 4 - \frac{24y}{1+x} + 8x = 0, \tag{40}$$

admits a unique solution $(x_3, y_3) = ((1/4), (5/16))$ in the interior of E such that

$$f_1(x_3, y_3) = 17.75. \tag{41}$$

On the boundary of E, the following cases are observed:

$$f_3(x,0) = 17 + 2x - 12x^2 + 4x^3 \le 11 + \frac{10\sqrt{30}}{9} = 17.08580\ldots,$$
$$f_3(0,y) = 3 + 4y + 12\left(1 - y^2\right) = 17 + 4y - 12y^2 \le \frac{46}{3} = 17.33\ldots, \tag{42}$$

and

$$f_3\left(x, 1-x^2\right) = 9 + 22x - 4x^2 - 20x^3 \le 16.56455\ldots. \tag{43}$$

Equations (38), (41)–(43) show that

$$\left|\gamma_3\right| \le \frac{17.75}{48} = 0.36979\ldots. \tag{44}$$

□

Remark 3. Let $f \in \mathcal{F}_3$, where $f''(0)$ is a real number. Then [13]

$$\left|\gamma_3\right| \le \frac{1}{7776}\left(743 + 131\sqrt{262}\right) = 0.368238\ldots. \tag{45}$$

Data Availability

No data were used in this study.

Disclosure

The author would like to declare that a preprint of this article has previously been published in [15].

Conflicts of Interest

The author declares that there are no conflicts of interest.

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