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Research Article

On Cauchy–Pompeiu's Operator in Octant Ring

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In this study, we have determined Cauchy–Pompeiu representation formula in octant unit ring R_4 by modulation the main Cauchy–Pompeiu representation; this modulation was obtained by applying the parqueting-reflection method. Moreover, the boundedness of the modulated Cauchy–Pompeiu's operator in R_4 is proved by applying Schmitz's inequality.

1. Introduction

Many studies have investigated the modulation of Cauchy–Pompeiu representation formula in different particular domains, e.g., the ring [1], half-disk and half-ring [2], quarter-ring [3], half-hexagon [4], and lens and lune [5], to obtain an explicit solution to the Schwartz problem [6]:

$$\begin{aligned} \partial_{\bar{z}} w &= f \text{ on } D, \\ w &= f \text{ on } \partial D. \end{aligned} \tag{1}$$

The Cauchy–Pompeiu representation formula is the fundamental tool for solving Schwartz problem which just has to be properly modified.

The main Cauchy–Pompeiu representation formula is given as follows.

Theorem 1 (Cauchy–Pompeiu representation [1]). *Any function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ for a regular complex domain $D \subset \mathbb{C}$ can be represented as follows:*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}(*),$$

$$\text{Or } w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}(**). \tag{2}$$

The area integral appearing in the complex Cauchy–Pompeiu representation $(*)$ defines a weakly singular integral operator T; we call it as "the Cauchy–Pompeiu's operator" [1]:

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad f \in L_1(D;\mathbb{C}). \tag{3}$$

Its properties have been studied by Vekua [7]. The operator T helps in solving Schwartz problem because it holds

$$\partial_{\bar{z}} Tf = f, \tag{4}$$

in a weak sense (the derivative in Sobolev space [6]).

When Cauchy–Pompeiu representation formula is modulated, the operator T is also modulated.

In this study, we modulate the Cauchy–Pompeiu representation formula in octant unit ring by using the parqueting-reflection method; then, we define the modulated T operator in R_4, and we prove the boundedness of this operator.

2. The Octant Unit Ring

We define the octant unit ring as follows (Figure 1):

$$R_4 = \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1, 0 < \arg(z) < \frac{\pi}{4}\right\}. \tag{5}$$

First, we will find the reflections of the domain R_4 at its boundaries.

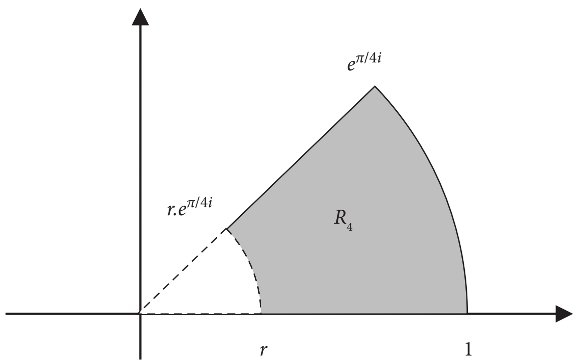

Figure 1: Octant unit disk.

Let $\partial_1 R_4$ be the segment $[r, 1]$, $\partial_2 R_4$ be the circular arc $[1, e^{\pi/4i}]$: $\tau \longrightarrow e^{\tau i}$, where $\tau \in [0, \pi/4]$, $\partial_3 R_4$ be the segment $[e^{\pi/4i}, r.e^{\pi/4i}]$, and $\partial_4 R_4$ be the circular arc $[r.e^{\pi/4i}, r]$: $\tau \longrightarrow r.e^{\tau i}$, where it is $\tau \in [\pi/4, 0]$.

Let $z \in R_4$; then,

(1) The point z is reflected at the segment $\partial_3 R_4$ onto $i\overline{z}$.

(2) The points $z, i\overline{z}$ are reflected at y-axis onto $-\overline{z}, iz$, respectively.

(3) The points $z, i\overline{z}, iz, -\overline{z}$ are reflected at x-axis onto $\overline{z}, -iz, -i\overline{z}, -z$, respectively.

We obtain the following eight sets:

$$\begin{aligned}
R_{4,0} &= R_4,\\
R_{4,1} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{\pi}{4} < \arg z < \frac{\pi}{2}\right\} = \{i\overline{z};\quad z \in R_4\},\\
R_{4,2} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{\pi}{2} < \arg z < \frac{3\pi}{4}\right\} = \{iz;\quad z \in R_4\},\\
R_{4,3} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{3\pi}{4} < \arg z < \pi\right\} = \{-\overline{z};\quad z \in R_4\},\\
R_{4,4} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \pi < \arg z < \frac{5\pi}{4}\right\} = \{-z;\quad z \in R_4\},\\
R_{4,5} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{5\pi}{4} < \arg z < \frac{3\pi}{2}\right\} = \{-i\overline{z};\quad z \in R_4\},\\
R_{4,6} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{3\pi}{2} < \arg z < \frac{7\pi}{4}\right\} = \{-iz;\quad z \in R_4\},\\
R_{4,7} &= \left\{z \in \mathbb{C};\quad 0 < r < |z| < 1,\ \frac{7\pi}{4} < \arg z < 2\pi\right\} = \{\overline{z};\quad z \in R_4\}.
\end{aligned} \tag{6}$$

We can easily notice that (see Figure 2)

$$\bigcup_{k=0}^{7} \overline{R_{4,k}} = \overline{R} = \{z \in \mathbb{C};\quad 0 < r \leq |z| \leq 1\}. \tag{7}$$

The points $z, i\overline{z}, iz, -\overline{z}, -z, -i\overline{z}, -iz, \overline{z}$ are reflected at $\partial\mathbb{D}$ onto the points:

$$\frac{1}{\overline{z}}, \frac{i}{z}, \frac{i}{\overline{z}}, -\frac{1}{z}, -\frac{1}{\overline{z}}, -\frac{i}{z}, -\frac{i}{\overline{z}}, \frac{1}{z}. \tag{8}$$

The points $z, i\overline{z}, iz, -\overline{z}, -z, -i\overline{z}, -iz, \overline{z}$ are reflected at $\partial\mathbb{D}_r$ onto the points:

$$\frac{r^2}{\overline{z}}, \frac{ir^2}{z}, \frac{ir^2}{\overline{z}}, -\frac{r^2}{z}, -\frac{r^2}{\overline{z}}, -\frac{ir^2}{z}, -\frac{ir^2}{\overline{z}}, \frac{r^2}{z}, \tag{9}$$

where $\mathbb{D}_r = \{z \in \mathbb{C};\quad |z| < r < 1\}$

Point (7) is reflected at $\partial\mathbb{D}_r$ onto the points:

$$r^2 z, ir^2\overline{z}, ir^2 z, -r^2\overline{z}, -r^2 z, -ir^2\overline{z}, -ir^2 z, r^2\overline{z}. \tag{10}$$

Point (10) is reflected at $\partial\mathbb{D}$ onto

$$\frac{1}{r^2\overline{z}}, \frac{i}{r^2 z}, \frac{i}{r^2\overline{z}}, -\frac{1}{r^2 z}, -\frac{1}{r^2\overline{z}}, -\frac{i}{r^2 z}, -\frac{i}{r^2\overline{z}}, \frac{1}{r^2 z}. \tag{11}$$

Point (11) is reflected at $\partial\mathbb{D}_r$ onto

$$r^4 z, ir^4\overline{z}, ir^4 z, -r^4\overline{z}, -r^4 z, -ir^4\overline{z}, -ir^4 z, r^4\overline{z}. \tag{12}$$

When continuing that way, we obtain

$$\pm r^{2k} z,\ \pm r^{2k}\overline{z},\ \pm ir^{2k} z,\ \pm ir^{2k}\overline{z},\ \pm\frac{1}{r^{2k} z},\ \pm\frac{1}{r^{2k}\overline{z}},\ \pm\frac{i}{r^{2k} z},\ \pm\frac{i}{r^{2k}\overline{z}}, \tag{13}$$

where $k \in \mathbb{N}$.

Point (9) is reflected at $\partial\mathbb{D}$ onto

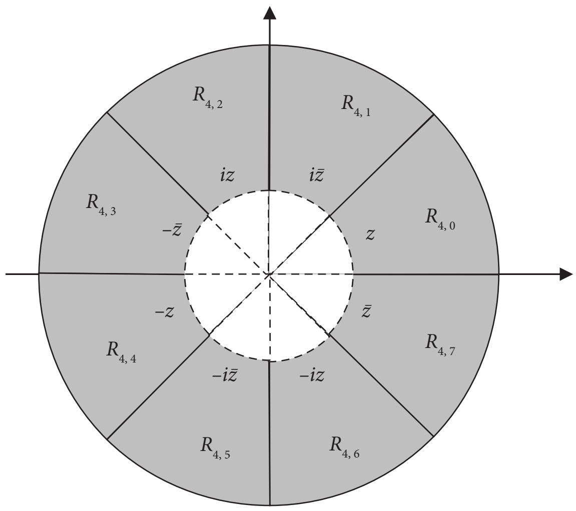

FIGURE 2: Unit ring.

$$\frac{z}{r^2}, \frac{i\overline{z}}{r^2}, \frac{iz}{r^2}, -\frac{\overline{z}}{r^2}, -\frac{z}{r^2}, -\frac{i\overline{z}}{r^2}, -\frac{iz}{r^2}, \frac{\overline{z}}{r^2}. \tag{14}$$

Points (14) are reflected at $\partial\mathbb{D}_r$ onto

$$\frac{r^4}{\overline{z}}, \frac{ir^4}{z}, \frac{ir^4}{\overline{z}}, -\frac{r^4}{z}, -\frac{r^4}{\overline{z}}, -\frac{ir^4}{z}, -\frac{ir^4}{\overline{z}}, \frac{r^4}{z}. \tag{15}$$

When continuing that way, we obtain

$$\pm\frac{z}{r^{2k}}, \pm\frac{\overline{z}}{r^{2k}}, \pm\frac{iz}{r^{2k}}, \pm\frac{i\overline{z}}{r^{2k}}, \pm\frac{r^{2k}}{z}, \pm\frac{r^{2k}}{\overline{z}}, \pm\frac{ir^{2k}}{z}, \pm\frac{ir^{2k}}{\overline{z}}, \tag{16}$$

where $k \in \mathbb{N}$

Each point of sets (13) and (16) belong to a different domain D_m; this domains parquets the complex plane:

$$\mathbb{C} = \bigcup_{m\in I} \overline{D_m}, \quad I \text{ is countable set.} \tag{17}$$

3. Main Result

In this section, we have determined the Cauchy–Pompeiu representation formula in octant unit ring R_4 by applying Theorem 1 at points (13) and (16).

Theorem 2. *Any function $w \in C^1(R_4;\mathbb{C}) \cap C(\overline{R_4};\mathbb{C})$ can be represented as follows:*

$$\begin{aligned}
w(z) = {} & \frac{2}{\pi i}\int_{\substack{|\zeta|=1\\ Re\zeta,\ Im\zeta>0}} \operatorname{Re}w(\zeta)\left[\frac{\zeta^4+z^4}{\zeta^4-z^4} - \frac{\overline{\zeta}^4+z^4}{\overline{\zeta}^4-z^4} + 2\sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4}{r^{8k}\zeta^4-z^4} - \frac{z^4}{r^{8k}z^4-\zeta^4} + \frac{z^4}{r^{8k}z^4-\overline{\zeta}^4} - \frac{\overline{\zeta}^4}{r^{8k}\overline{\zeta}^4-z^4}\right)\right]\cdot\frac{d\zeta}{\zeta} \\
& - \frac{2}{\pi i}\int_{\substack{|\zeta|=r\\ Re\zeta,\ Im\zeta>0}} \operatorname{Re}w(\zeta)\left[\frac{\zeta^4+z^4}{\zeta^4-z^4} - \frac{\overline{\zeta}^4+z^4}{\overline{\zeta}^4-z^4} + 2\sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4}{r^{8k}\zeta^4-z^4} - \frac{z^4}{r^{8k}z^4-\zeta^4} + \frac{z^4}{r^{8k}z^4-\overline{\zeta}^4} - \frac{\overline{\zeta}^4}{r^{8k}\overline{\zeta}^4-z^4}\right)\right]\cdot\frac{d\zeta}{\zeta} \\
& + \frac{4}{\pi i}\int_r^1 \operatorname{Re}w(\zeta)\left[\frac{t^3}{t^4-z^4} - \frac{t^3z^4}{1-t^4z^4} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{t^3}{r^{8k}t^4-z^4} - \frac{z^4}{t\left(r^{8k}z^4-t^4\right)} + \frac{t^3z^4}{r^{8k}t^4z^4-1} - \frac{1}{t\left(r^{8k}-t^4z^4\right)}\right)\right]dt \\
& + \frac{4}{\pi i}\int_1^r \operatorname{Re}w\left(te^{\pi/4i}\right)\left[\frac{t^3}{t^4+z^4} + \frac{t^3z^4}{t^4z^4+1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{t^3}{r^{8k}t^4+z^4} - \frac{z^4}{r^{8k}z^4+t^4} + \frac{t^3z^4}{r^{8k}t^4z^4-1} - \frac{1}{r^{8k}+t^4z^4}\right)\right]dt \\
& + \frac{4}{\pi}\int_{\substack{|\zeta|=1\\ Re\zeta,\ Im\zeta>0}} \operatorname{Im}w(\zeta). \\
& \frac{d\zeta}{\zeta} - \frac{4}{\pi}\int_{R_4}\left\{w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3}{\zeta^4-z^4} + \frac{\zeta^3z^4}{\zeta^4z^4-1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4-z^4} - \frac{z^4}{\zeta\left(r^{8k}z^4-\zeta^4\right)} + \frac{\zeta^3z^4}{r^{8k}\zeta^4z^4-1} - \frac{1}{\zeta\left(r^{8k}-\zeta^4z^4\right)}\right)\right]\right. \\
& \left. - \overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{\overline{\zeta}^3}{\overline{\zeta}^4-z^4} + \frac{\overline{\zeta}^3z^4}{\overline{\zeta}^4z^4-1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4-z^4} - \frac{z^4}{\overline{\zeta}\left(r^{8k}z^4-\overline{\zeta}^4\right)} + \frac{\overline{\zeta}^3z^4}{r^{8k}\overline{\zeta}^3z^4-1} - \frac{1}{\overline{\zeta}\left(r^{8k}-\overline{\zeta}^4z^4\right)}\right)\right]\right\}d\xi d\eta,
\end{aligned} \tag{18}$$

where $\zeta = \xi + i\eta$, *and* $z \in R_4$.

Proof. Using Theorem 1 for the points $\pm r^{2k}z$, $\pm z/r^{2k}$, for every $k \in \mathbb{N}$, we find

$$w(z) = \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{19}$$

$$0 = \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\frac{d\zeta}{\zeta + z} - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta + z}, \tag{20}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta - r^{2k}z}\right)d\zeta - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta - r^{2k}z}\right)d\xi d\eta,\\ 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta + r^{2k}z}\right)d\zeta - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta + r^{2k}z}\right)d\xi d\eta,\\ 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta - z/r^{2k}}\right)d\zeta - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta - z/r^{2k}}\right)d\xi d\eta, \end{aligned} \tag{21}$$

$$0 = \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta + z/r^{2k}}\right)d\zeta - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\sum_{k=1}^{\infty}\left(\frac{1}{\zeta + z/r^{2k}}\right)d\xi d\eta. \tag{22}$$

When adding (19)–(22) and fixing the terms, we obtain

$$\begin{aligned} w(z) = {}& \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta}{\zeta^2 - z^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 - r^{4k}z^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 - z^2}\right)\right]d\zeta\\ & - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\left[\frac{2\zeta}{\zeta^2 - z^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 - r^{4k}z^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 - z^2}\right)\right]d\xi d\eta. \end{aligned} \tag{23}$$

Similarly, when using Theorem 1 for the points, $\pm ir^{2k}z$, $\pm iz/r^{2k}$, for $k \in \mathbb{N}$, and fixing the terms, we find

$$\begin{aligned} 0 = {}& \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta}{\zeta^2 + z^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 + r^{4k}z^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 + z^2}\right)\right]d\zeta\\ & - \frac{1}{\pi}\int_{R_4} w_{\bar\zeta}(\zeta)\left[\frac{2\zeta}{\zeta^2 + z^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 + r^{4k}z^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 + z^2}\right)\right]d\xi d\eta. \end{aligned} \tag{24}$$

When using Theorem 1 for the points, $\pm 1/r^{2k}z$, $\pm r^{2k}/z$, $\pm i/r^{2k}z$, $\pm ir^{2k}/z$, for $k \in \mathbb{N}$, and fixing the terms we also find

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta z^2}{\zeta^2 z^2 - 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta z^2 r^{4k}}{r^{4k}\zeta^2 z^2 - 1} + \frac{2\zeta z^2}{\zeta^2 z^2 - r^{4k}}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta z^2}{\zeta^2 z^2 - 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta z^2 r^{4k}}{r^{4k}\zeta^2 z^2 - 1} + \frac{2\zeta z^2}{\zeta^2 z^2 - r^{4k}}\right)\right]d\xi d\eta, \end{aligned} \tag{25}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta z^2}{\zeta^2 z^2 + 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta z^2 r^{4k}}{r^{4k}\zeta^2 z^2 + 1} + \frac{2\zeta z^2}{\zeta^2 z^2 + r^{4k}}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta z^2}{\zeta^2 z^2 + 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta z^2 r^{4k}}{r^{4k}\zeta^2 z^2 + 1} + \frac{2\zeta z^2}{\zeta^2 z^2 + r^{4k}}\right)\right]d\xi d\eta. \end{aligned} \tag{26}$$

Finally, when using Theorem 1 for the points, we obtain

$$\pm r^{2k}\overline{z},\ \pm \frac{\overline{z}}{r^{2k}},\ \pm ir^{2k}\overline{z},\ \pm \frac{i\overline{z}}{r^{2k}},\ \pm \frac{1}{r^{2k}\overline{z}},\ \pm \frac{r^{2k}}{\overline{z}},\ \pm \frac{i}{r^{2k}\overline{z}},\ \pm \frac{ir^{2k}}{\overline{z}}. \tag{27}$$

For $k \in \mathbb{N}$, we find

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta}{\zeta^2 - \overline{z}^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 - r^{4k}\overline{z}^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 - \overline{z}^2}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta}{\zeta^2 - \overline{z}^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 - r^{4k}\overline{z}^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 - \overline{z}^2}\right)\right]d\xi d\eta, \end{aligned} \tag{28}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta}{\zeta^2 + \overline{z}^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 + r^{4k}\overline{z}^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 + \overline{z}^2}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta}{\zeta^2 + \overline{z}^2} + \sum_{k=1}^{\infty}\left(\frac{2\zeta}{\zeta^2 + r^{4k}\overline{z}^2} + \frac{2\zeta r^{4k}}{r^{4k}\zeta^2 + \overline{z}^2}\right)\right]d\xi d\eta, \end{aligned} \tag{29}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta\overline{z}^2 r^{4k}}{r^{4k}\zeta^2\overline{z}^2 - 1} + \frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - r^{4k}}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta\overline{z}^2 r^{4k}}{r^{4k}\zeta^2\overline{z}^2 - 1} + \frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - r^{4k}}\right)\right]d\xi d\eta, \end{aligned} \tag{30}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 + 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta\overline{z}^2 r^{4k}}{r^{4k}\zeta^2\overline{z}^2 + 1} + \frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 + r^{4k}}\right)\right]d\zeta \\ &\quad - \frac{1}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 + 1} + \sum_{k=1}^{\infty}\left(\frac{2\zeta\overline{z}^2 r^{4k}}{r^{4k}\zeta^2\overline{z}^2 + 1} + \frac{2\zeta\overline{z}^2}{\zeta^2\overline{z}^2 + r^{4k}}\right)\right]d\xi d\eta. \end{aligned} \tag{31}$$

Since $0 \notin R_4$, we can rewrite (23)–(31) as follows:

$$\begin{aligned} w(z) = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2}{\zeta^2 - z^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2}{r^{4k}\zeta^2 - z^2} + \frac{z^2}{\zeta^2 - r^{4k}z^2}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta}{\zeta^2 - z^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta}{r^{4k}\zeta^2 - z^2} + \frac{z^2}{\zeta\left(\zeta^2 - r^{4k}z^2\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{32}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2}{\zeta^2 + z^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2}{r^{4k}\zeta^2 + z^2} - \frac{z^2}{\zeta^2 + r^{4k}z^2}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta}{\zeta^2 + z^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta}{r^{4k}\zeta^2 - z^2} - \frac{z^2}{\zeta\left(\zeta^2 - r^{4k}z^2\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{33}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2 z^2}{\zeta^2 z^2 - 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2 z^2}{r^{4k}\zeta^2 z^2 - 1} + \frac{1}{\zeta^2 z^2 - r^{4k}}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta z^2}{\zeta^2 z^2 - 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta z^2}{r^{4k}\zeta^2 z^2 - 1} + \frac{1}{\zeta\left(\zeta^2 z^2 - r^{4k}\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{34}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2 z^2}{\zeta^2 z^2 + 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2 z^2}{r^{4k}\zeta^2 z^2 + 1} - \frac{1}{\zeta^2 z^2 + r^{4k}}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta z^2}{\zeta^2 z^2 + 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta z^2}{r^{4k}\zeta^2 z^2 + 1} - \frac{1}{\zeta\left(\zeta^2 z^2 + r^{4k}\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{35}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2}{\zeta^2 - \overline{z}^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2}{r^{4k}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta^2 - r^{4k}\overline{z}^2}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta}{\zeta^2 - \overline{z}^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta}{r^{4k}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta\left(\zeta^2 - r^{4k}\overline{z}^2\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{36}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2}{\zeta^2 + \overline{z}^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2}{r^{4k}\zeta^2 + \overline{z}^2} - \frac{\overline{z}^2}{\zeta^2 + r^{4k}\overline{z}^2}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta}{\zeta^2 + \overline{z}^2} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta}{r^{4k}\zeta^2 + \overline{z}^2} - \frac{\overline{z}^2}{\zeta\left(\zeta^2 + r^{4k}\overline{z}^2\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{37}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2 \overline{z}^2}{\zeta^2 \overline{z}^2 - 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2 \overline{z}^2}{r^{4k}\zeta^2 \overline{z}^2 - 1} + \frac{1}{\zeta^2 \overline{z}^2 - r^{4k}}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta \overline{z}^2}{\zeta^2 \overline{z}^2 - 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta \overline{z}^2}{r^{4k}\zeta^2 \overline{z}^2 - 1} + \frac{1}{\zeta\left(\zeta^2 \overline{z}^2 - r^{4k}\right)}\right)\right]d\xi d\eta, \end{aligned} \tag{38}$$

$$\begin{aligned} 0 = {} & \frac{1}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^2 \overline{z}^2}{\zeta^2 \overline{z}^2 + 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta^2 \overline{z}^2}{r^{4k}\zeta^2 \overline{z}^2 + 1} - \frac{1}{\zeta^2 \overline{z}^2 + r^{4k}}\right)\right]\frac{d\zeta}{\zeta} \\ & - \frac{2}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta \overline{z}^2}{\zeta^2 \overline{z}^2 + 1} + \sum_{k=1}^{\infty} r^{4k}\left(\frac{\zeta \overline{z}^2}{r^{4k}\zeta^2 \overline{z}^2 + 1} - \frac{1}{\zeta\left(\zeta^2 \overline{z}^2 + r^{4k}\right)}\right)\right]d\xi d\eta. \end{aligned} \tag{39}$$

Adding (32) and (33) gives

$$\begin{aligned} w(z) = \frac{2}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^4}{\zeta^4 - z^4} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta^4 - r^{8k}z^4}\right)\right]\frac{d\zeta}{\zeta} \\ - \frac{4}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3}{\zeta^4 - z^4} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)}\right)\right] d\xi d\eta. \end{aligned} \tag{40}$$

Adding (36) and (37) gives

$$\begin{aligned} 0 = \frac{2}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^4}{\zeta^4 - \overline{z}^4} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4}{r^{8k}\zeta^4 - \overline{z}^4} + \frac{\overline{z}^4}{\zeta^4 - r^{8k}\overline{z}^4}\right)\right]\frac{d\zeta}{\zeta} \\ - \frac{4}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3}{\zeta^4 - \overline{z}^4} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4 - \overline{z}^4} + \frac{\overline{z}^4}{\zeta\left(\zeta^4 - r^{8k}\overline{z}^4\right)}\right)\right] d\xi d\eta. \end{aligned} \tag{41}$$

Adding (34) and (35) gives

$$\begin{aligned} 0 = \frac{2}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^4 z^4}{z^4\zeta^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4 z^4}{r^{8k}z^4\zeta^4 - 1} + \frac{1}{z^4\zeta^4 - r^{8k}}\right)\right]\frac{d\zeta}{\zeta} \\ - \frac{4}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3 z^4}{\zeta^4 z^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1} + \frac{1}{\zeta\left(\zeta^4 z^4 - r^{8k}\right)}\right)\right] d\xi d\eta. \end{aligned} \tag{42}$$

Adding (38) and (39) gives

$$\begin{aligned} 0 = \frac{2}{\pi i}\int_{\partial R_4} w(\zeta)\left[\frac{\zeta^4\overline{z}^4}{\overline{z}^4\zeta^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^4\overline{z}^4}{r^{8k}\overline{z}^4\zeta^4 - 1} + \frac{1}{\overline{z}^4\zeta^4 - r^{8k}}\right)\right]\frac{d\zeta}{\zeta} \\ - \frac{4}{\pi}\int_{R_4} w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3\overline{z}^4}{\zeta^4\overline{z}^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3\overline{z}^4}{r^{8k}\zeta^4\overline{z}^4 - 1} + \frac{1}{\zeta\left(\zeta^4\overline{z}^4 - r^{8k}\right)}\right)\right] d\xi d\eta. \end{aligned} \tag{43}$$

When taking the complex conjugate of (41) and (43), we can obtain

$$\begin{aligned}
0 &= \frac{2}{\pi i}\int_{\substack{|\zeta|=1\\ Re\zeta,\quad Im\zeta>0}} \overline{w(\zeta)}\left[\frac{1}{1-\zeta^4 z^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{1}{r^{8k}-\zeta^4 z^4}+\frac{z^4\zeta^4}{1-r^{8k}z^4}\right)\right]\frac{d\zeta}{\zeta}\\
&+\frac{2}{\pi i}\int_{\substack{|\zeta|=r\\ Re\zeta,\quad Im\zeta>0}} \overline{w(\zeta)}\left[\frac{r^8}{r^8-\zeta^4 z^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{r^8}{r^{8(k+1)}-\zeta^4 z^4}+\frac{\zeta^4 z^4}{r^8-r^{8k}\zeta^4 z^4}\right)\right]\frac{d\zeta}{\zeta}\\
&-\frac{2}{\pi i}\int_r^1 \overline{w(t)}\left[\frac{t^4}{t^4-z^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{t^4}{r^{8k}t^4-z^4}+\frac{z^4}{t^4-r^{8k}z^4}\right)\right]\frac{dt}{t}\\
&-\frac{2}{\pi i}\int_1^r \overline{w\left(t.e^{\pi/4i}\right)}\left[\frac{t^4}{t^4+z^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{t^4}{r^{8k}t^4+z^4}-\frac{z^4}{t^4+r^{8k}z^4}\right)\right]\frac{dt}{t}\\
&-\frac{4}{\pi}\int_{R_4} \overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{\overline{\zeta}^3}{\overline{\zeta}^4-z^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4-z^4}+\frac{z^4}{\overline{\zeta}\left(\overline{\zeta}^4-r^{8k}z^4\right)}\right)\right]d\xi d\eta,
\end{aligned} \tag{44}$$

$$\begin{aligned}
0 &= \frac{2}{\pi i}\int_{\substack{|\zeta|=1\\ Re\zeta,\quad Im\zeta>0}} \overline{w(\zeta)}\left[\frac{z^4}{z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{z^4}{r^{8k}z^4-\zeta^4}+\frac{\zeta^4}{z^4-r^{8k}\zeta^4}\right)\right]\frac{d\zeta}{\zeta}\\
&+\frac{2}{\pi i}\int_{\substack{|\zeta|=r\\ Re\zeta,\quad Im\zeta>0}} \overline{w(\zeta)}\left[\frac{r^8 z^4}{r^8 z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{r^8 z^4}{r^{8(k+1)}z^4-\zeta^4}+\frac{\zeta^4}{z^4 r^8-r^{8k}\zeta^4}\right)\right]\frac{d\zeta}{\zeta}\\
&-\frac{2}{\pi i}\int_r^1 \overline{w(t)}\left[\frac{t^4 z^4}{z^4 t^4-1}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{t^4 z^4}{r^{8k}z^4 t^4-1}+\frac{1}{z^4 t^4-r^{8k}}\right)\right]\frac{dt}{t}\\
&-\frac{2}{\pi i}\int_1^r \overline{w\left(t.e^{\pi/4i}\right)}\left[\frac{t^4 z^4}{z^4 t^4+1}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{t^4 z^4}{r^{8k}z^4 t^4+1}-\frac{1}{z^4 t^4+r^{8k}}\right)\right]\frac{dt}{t}\\
&-\frac{4}{\pi}\int_{R_4} \overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{\overline{\zeta}^3 z^4}{\overline{\zeta}^4 z^4-1}+\sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3 z^4}{r^{8k}\overline{\zeta}^4 z^4-1}+\frac{1}{\overline{\zeta}\left(\overline{\zeta}^4 z^4-r^{8k}\right)}\right)\right]d\xi d\eta.
\end{aligned} \tag{45}$$

When adding (40)–(42) and then subtracting (44) and (45) from the result, we can obtain

$$\begin{aligned}
w(z) = {} & \frac{2}{\pi i}\int_{\substack{|\zeta|=1\\ Re\zeta,\quad Im\zeta>0}} \Bigg\{ w(\zeta)\Bigg[\frac{\zeta^4}{\zeta^4-z^4}+\frac{\zeta^4 z^4}{\zeta^4 z^4-1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{\zeta^4}{r^{8k}\zeta^4-z^4}+\frac{z^4}{\zeta^4-r^{8k}z^4}+\frac{\zeta^4 z^4}{r^{8k}\zeta^4 z^4-1}+\frac{1}{z^4\zeta^4-r^{8k}}\Bigg)\Bigg] \\
& -\overline{w(\zeta)}\Bigg[\frac{1}{1-\zeta^4 z^4}+\frac{z^4}{z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{\zeta^4}{z^4-r^{8k}\zeta^4}+\frac{z^4}{r^{8k}z^4-\zeta^4}+\frac{\zeta^4 z^4}{1-r^{8k}\zeta^4 z^4}+\frac{1}{r^{8k}-z^4\zeta^4}\Bigg)\Bigg]\Bigg\}\frac{d\zeta}{\zeta} \\
& +\frac{2}{\pi i}\int_{\substack{|\zeta|=r\\ Re\zeta,\quad Im\zeta>0}} \Bigg\{ w(\zeta)\Bigg[\frac{\zeta^4}{\zeta^4-z^4}+\frac{\zeta^4 z^4}{\zeta^4 z^4-1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{\zeta^4}{r^{8k}\zeta^4-z^4}+\frac{z^4}{\zeta^4-r^{8k}z^4}+\frac{\zeta^4 z^4}{r^{8k}\zeta^4 z^4-1}+\frac{1}{z^4\zeta^4-r^{8k}}\Bigg)\Bigg] \\
& -\overline{w(\zeta)}\Bigg[\frac{r^8}{r^8-\zeta^4 z^4}+\frac{r^8 z^4}{r^8 z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{r^8}{r^{8(k+1)}-\zeta^4 z^4}+\frac{z^4\zeta^4}{r^8-r^{8k}z^4\zeta^4}+\frac{r^8 z^4}{r^{8(k+1)}z^4-\zeta^4}+\frac{\zeta^4}{r^8 z^4-r^{8k}\zeta^4}\Bigg)\Bigg]\Bigg\}\frac{d\zeta}{\zeta} \\
& +\frac{2}{\pi i}\int_r^1 \Bigg\{ w(t)\Bigg[\frac{t^4}{t^4-z^4}+\frac{t^4 z^4}{t^4 z^4-1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{t^4}{r^{8k}t^4-z^4}+\frac{z^4}{t^4-r^{8k}z^4}+\frac{t^4 z^4}{r^{8k}t^4 z^4-1}+\frac{1}{z^4 t^4-r^{8k}}\Bigg)\Bigg] \\
& -\overline{w(t)}\Bigg[\frac{t^4 z^4}{t^4 z^4-1}+\frac{t^4}{t^4-z^4}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{t^4}{r^{8k}t^4-z^4}+\frac{z^4}{t^4-r^{8k}z^4}+\frac{t^4 z^4}{r^{8k}t^4 z^4-1}+\frac{1}{t^4 z^4-r^{8k}}\Bigg)\Bigg]\Bigg\}\frac{dt}{t} \\
& +\frac{2}{\pi i}\int_1^r \Bigg\{ w\big(t.e^{\pi/4i}\big)\Bigg[\frac{t^4}{t^4+z^4}+\frac{t^4 z^4}{t^4 z^4+1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{t^4}{r^{8k}t^4+z^4}-\frac{z^4}{t^4+r^{8k}z^4}+\frac{t^4 z^4}{r^{8k}t^4 z^4+1}-\frac{1}{z^4 t^4+r^{8k}}\Bigg)\Bigg] \\
& -\overline{w\big(t.e^{\pi/4i}\big)}\Bigg[\frac{t^4}{t^4+z^4}+\frac{t^4 z^4}{t^4 z^4+1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{t^4}{r^{8k}t^4+z^4}-\frac{z^4}{t^4+r^{8k}z^4}+\frac{t^4 z^4}{r^{8k}t^4 z^4+1}-\frac{1}{t^4 z^4+r^{8k}}\Bigg)\Bigg]\Bigg\}\frac{dt}{t} \\
& -\frac{4}{\pi}\int_{R_4} \Bigg\{ w_{\overline{\zeta}}(\zeta)\Bigg[\frac{\zeta^3}{\zeta^4-z^4}+\frac{\zeta^3 z^4}{z^4\zeta^4-1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{\zeta^3}{r^{8k}\zeta^4-z^4}+\frac{z^4}{\zeta\big(\zeta^4-r^{8k}z^4\big)}+\frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4-1}+\frac{1}{\zeta\big(z^4\zeta^4-r^{8k}\big)}\Bigg)\Bigg] \\
& -\overline{w_{\overline{\zeta}}(\zeta)}\Bigg[\frac{\overline{\zeta}^3}{\overline{\zeta}^4-z^4}+\frac{\overline{\zeta}^3 z^4}{\overline{\zeta}^4 z^4-1}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4-z^4}+\frac{z^4}{\overline{\zeta}\big(\overline{\zeta}^4-r^{8k}z^4\big)}+\frac{\overline{\zeta}^3 z^4}{r^{8k}\overline{\zeta}^4 z^4-1}+\frac{1}{\overline{\zeta}\big(\overline{\zeta}^4 z^4-r^{8k}\big)}\Bigg)\Bigg]\Bigg\} d\xi d\eta.
\end{aligned} \tag{46}$$

We notice

$$\begin{aligned}
& \frac{r^8}{r^8-\zeta^4 z^4}+\frac{r^8 z^4}{r^8 z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{r^8}{r^{8(k+1)}-\zeta^4 z^4}+\frac{z^4\zeta^4}{r^8-r^{8k}z^4\zeta^4}+\frac{r^8 z^4}{r^{8(k+1)}z^4-\zeta^4}+\frac{\zeta^4}{r^8 z^4-r^{8k}\zeta^4}\Bigg) \\
& =\frac{z^4\zeta^4}{1-z^4\zeta^4}+\frac{\zeta^4}{z^4-\zeta^4}+\sum_{k=1}^{\infty} r^{8k}\Bigg(\frac{1}{r^{8k}-\zeta^4 z^4}+\frac{z^4\zeta^4}{r^8-r^{8k}z^4\zeta^4}+\frac{z^4}{r^{8k}z^4-\zeta^4}+\frac{\zeta^4}{r^8 z^4-r^{8k}\zeta^4}\Bigg).
\end{aligned} \tag{47}$$

By putting $w(\zeta)=\mathrm{Re}[w(\zeta)]+i.\mathrm{Im}[w(\zeta)]$ on (3.26) and fixing the terms, we end up the proof.

The Cauchy–Pompeiu's operator in R_4 is the area integral in the modulated Cauchy–Pompeiu representation in R_4; it is given by the following formula:

$$T_{R_4}f(z) = -\frac{4}{\pi}$$

$$\int_{R_4}\left\{w_{\overline{\zeta}}(\zeta)\left[\frac{\zeta^3}{\zeta^4 - z^4} + \frac{\zeta^3 z^4}{z^4\zeta^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)} + \frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1} + \frac{1}{\zeta\left(z^4\zeta^4 - r^{8k}\right)}\right)\right]\right.$$

$$\left. - \overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{\overline{\zeta}^3}{\overline{\zeta}^4 - z^4} + \frac{\overline{\zeta}^3 z^4}{\overline{\zeta}^4 z^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4 - z^4} + \frac{z^4}{\overline{\zeta}\left(\overline{\zeta}^4 - r^{8k}z^4\right)} + \frac{\overline{\zeta}^3 z^4}{r^{8k}\overline{\zeta}^4 z^4 - 1} + \frac{1}{\overline{\zeta}\left(\overline{\zeta}^4 z^4 - r^{8k}\right)}\right)\right]\right\} d\xi d\eta, \tag{48}$$

$f \in L_1(R_4;\mathbb{C})$.

□

4. The Operator $\mathbf{T}_{\mathbf{R}_4}$

The Cauchy–Pompeiu's operator in R_4 includes an infinite series; therefore, we have to prove its convergence; this will be done through the following two lemmas.

Lemma 1. *For every z, $\zeta \in R_4$, with $z \neq \zeta$, we have the following:*

$$\begin{aligned} &|\zeta - z| < |\zeta - z_k|,\\ &z_k = \pm\frac{1}{r^{2k}z}, \pm\frac{i}{r^{2k}z}, \pm\frac{z}{r^{2k}}, \pm\frac{iz}{r^{2k}},\\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < |\zeta - z_k|,\\ &z_k = \pm r^{2k}z, \pm ir^{2k}z, \pm\frac{r^{2k}}{z}, \pm\frac{ir^{2k}}{z}. \end{aligned} \tag{49}$$

where $k \in \mathbb{N}$.

Proof. Let $= \xi + i\eta$ and $z = x + iy$; then,

$$\begin{aligned} \left|\zeta - \frac{1}{\overline{z}}\right|^2 - |\zeta - z|^2 &= \left(\zeta - \frac{1}{\overline{z}}\right)\left(\overline{\zeta} - \frac{1}{z}\right) - (\zeta - z)(\overline{\zeta} - \overline{z})\\ &= |\zeta|^2 - \frac{\overline{\zeta}}{\overline{z}} - \frac{\zeta}{z} + \frac{1}{|z|^2} - |\zeta|^2 + z\overline{\zeta} + \overline{z}\zeta - |z|^2\\ &= \frac{\overline{\zeta}\left(|z|^2 - 1\right)}{\overline{z}} + \frac{\zeta\left(|z|^2 - 1\right)}{z} - \frac{\left(|z|^2 - 1\right)\left(|z|^2 + 1\right)}{|z|^2}\\ &= \frac{\left(|z|^2 - 1\right)}{|z|^2}\left(z\overline{\zeta} + \overline{z}\zeta - |z|^2 - 1\right) = \left(\frac{1}{x^2 + y^2} - 1\right)\left(x^2 + y^2 - 2\xi x - 2\eta y + 1\right)\\ &\geq \left(\frac{1}{x^2 + y^2} - 1\right)\left(1 - \xi^2 - \eta^2\right) > 0 \Rightarrow |\zeta - z| < \left|\zeta - \frac{1}{\overline{z}}\right|, \end{aligned} \tag{50}$$

where $(x - \xi)^2 \geq 0 \Rightarrow x^2 - 2\xi x \geq -\xi^2$, $(y - \eta)^2 \geq 0 \Rightarrow y^2 - 2y\eta \geq -\eta^2$,

$$\begin{aligned} &|z|^2 = x^2 + y^2 < 1 \Rightarrow \frac{1}{x^2 + y^2} - 1 > 0,\\ &|\zeta|^2 = \xi^2 + \eta^2 < 1 \Rightarrow 1 - \xi^2 - \eta^2 > 0. \end{aligned} \tag{51}$$

Since $|\zeta - 1/\overline{z}| < |\zeta - 1/z| \le |\zeta - 1/r^{2k}z|$, for every $k \in \mathbb{N}$, we have

$$|\zeta - z| < \left|\zeta - \frac{1}{z}\right| \le \left|\zeta - \frac{1}{r^k z}\right|. \tag{52}$$

By the symmetry properties, we have

$$\begin{aligned} &\left|\zeta - \frac{1}{z}\right| < \left|\zeta - \frac{i}{z}\right| \le \left|\zeta - \frac{i}{r^{2k}z}\right|, \\ &\left|\zeta - \frac{1}{z}\right| < \left|\zeta + \frac{i}{z}\right| \le \left|\zeta + \frac{i}{r^{2k}z}\right|, \\ &\left|\zeta - \frac{1}{z}\right| < \left|\zeta + \frac{1}{z}\right| \le \left|\zeta + \frac{1}{r^{2k}z}\right|. \end{aligned} \tag{53}$$

This leads to

$$\begin{aligned} &|\zeta - z| < \left|\zeta - z_k\right|, \\ &z_k = \pm \frac{1}{r^{2k}z}, \pm \frac{i}{r^{2k}z}, \end{aligned} \tag{54}$$

for every $k \in \mathbb{N}$.

On the contrary, we have

$$|\zeta - z| \le \left|\zeta - \frac{z}{r^{2k}}\right|. \tag{55}$$

By the symmetry properties, we have

$$\begin{aligned} &\left|\zeta - \frac{z}{r^{2k}}\right| < \left|\zeta + \frac{z}{r^{2k}}\right|, \\ &\left|\zeta - \frac{z}{r^{2k}}\right| < \left|\zeta - \frac{iz}{r^{2k}}\right|, \\ &\left|\zeta - \frac{z}{r^{2k}}\right| < \left|\zeta + \frac{iz}{r^{2k}}\right|. \end{aligned} \tag{56}$$

This leads to

$$\begin{aligned} &|\zeta - z| < \left|\zeta - z_k\right|, \\ &z_k = \pm \frac{z}{r^{2k}}, \pm \frac{iz}{r^{2k}}, \end{aligned} \tag{57}$$

for every $k \in \mathbb{N}$.

We have

$$\begin{aligned} &\left|r^2 z\right| = r^2|z| < r^2, \\ &\left|\frac{r^2}{\overline{z}}\right| = \frac{r^2}{|z|} > r^2 \Rightarrow \left|r^2 z\right| < r^2 < \left|\frac{r^2}{\overline{z}}\right| < r < |\zeta|. \end{aligned} \tag{58}$$

Therefore,

$$\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - r^2 z\right|. \tag{59}$$

Since

$$\left|\zeta - r^2 z\right| \le \left|\zeta - r^{2k} z\right|, \quad k \in \mathbb{N}, \tag{60}$$

we can obtain

$$\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - r^{2k} z\right|, \quad k \in \mathbb{N}. \tag{61}$$

By symmetry properties, we have

$$\begin{aligned} &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - \frac{r^2}{z}\right| \le \left|\zeta - \frac{r^{2k}}{z}\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - \frac{r^2}{z}\right| < \left|\zeta + \frac{r^2}{z}\right| \le \left|\zeta + \frac{r^{2k}}{z}\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - \frac{r^2}{z}\right| < \left|\zeta - \frac{ir^2}{z}\right| \le \left|\zeta - \frac{ir^{2k}}{z}\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - \frac{r^2}{z}\right| < \left|\zeta + \frac{ir^2}{z}\right| \le \left|\zeta + \frac{ir^{2k}}{z}\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - r^2 z\right| < \left|\zeta + r^2 z\right| \le \left|\zeta + r^{2k} z\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - r^2 z\right| < \left|\zeta - ir^2 z\right| \le \left|\zeta - ir^{2k} z\right|, \\ &\left|\zeta - \frac{r^2}{\overline{z}}\right| < \left|\zeta - r^2 z\right| < \left|\zeta + ir^2 z\right| \le \left|\zeta + ir^{2k} z\right|, \end{aligned} \tag{62}$$

for every $k \in \mathbb{N}$.

This means

$$\left|\zeta-\frac{r^2}{\overline{z}}\right|<\left|\zeta-z_k\right|,$$
$$z_k=\pm r^{2k}z,\ \pm ir^{2k}z,\ \pm\frac{r^{2k}}{z},\ \pm\frac{ir^{2k}}{z}. \tag{63}$$

□

Lemma 2. *For every $z,\quad \zeta\in R_4$, with $z\neq\zeta$, $0<r<1$, the series is as follows:*

$$A=\sum_{k=1}^{\infty}r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4-z^4}+\frac{z^4}{\zeta\left(\zeta^4-r^{8k}z^4\right)}+\frac{\zeta^3z^4}{r^{8k}\zeta^4z^4-1}+\frac{1}{\zeta\left(z^4\zeta^4-r^{8k}\right)}\right). \tag{64}$$

Is it uniformly converged.

Proof. First, we have

$$\left|\frac{\zeta^3}{r^{8k}\zeta^4-z^4}\right|<\left|\frac{1}{r^{8k}\zeta^4-z^4}\right|=\left|\frac{1}{4z^3r^{2k}}\right|.\left|\frac{1}{\zeta-z/r^{2k}}-\frac{1}{\zeta+z/r^{2k}}+\frac{i}{\zeta-iz/r^{2k}}-\frac{i}{\zeta+iz/r^{2k}}\right|$$
$$<\frac{1}{4r^{2k+3}}\left[\frac{1}{\left|\zeta-z/r^{2k}\right|}+\frac{1}{\left|\zeta+z/r^{2k}\right|}+\frac{1}{\left|\zeta-iz/r^{2k}\right|}+\frac{1}{\left|\zeta+iz/r^{2k}\right|}\right],\quad |\zeta|<1,\ r<|z|. \tag{65}$$

From Lemma 1, we have

$$\frac{1}{\left|\zeta-z_k\right|}<\frac{1}{|\zeta-z|},$$
$$z_k=\pm\frac{z}{r^{2k}},\ \pm\frac{iz}{r^{2k}},$$
$$\Rightarrow\left|\frac{\zeta^3}{r^{8k}\zeta^4-z^4}\right|<\frac{1}{r^{2k+3}}.\frac{1}{|\zeta-z|}. \tag{66}$$

Second, we have

$$\left|\frac{z^4}{\zeta\left(\zeta^4-r^{8k}z^4\right)}\right|=\frac{|z|}{4r^{6n}|\zeta|}\left|\frac{1}{\zeta-r^{2k}z}-\frac{1}{\zeta+r^{2k}z}+\frac{i}{\zeta-ir^{2k}z}-\frac{i}{\zeta+ir^{2k}z}\right|$$
$$<\frac{1}{4r^{6k+1}}\left[\frac{1}{\left|\zeta-r^{2k}z\right|}+\frac{1}{\left|\zeta+r^{2k}z\right|}+\frac{1}{\left|\zeta-ir^{2k}z\right|}+\frac{1}{\left|\zeta+ir^{2k}z\right|}\right]. \tag{67}$$

From Lemma 1, we have

$$\frac{1}{\left|\zeta-z_k\right|}<\frac{1}{\left|\zeta-r^2/\overline{z}\right|},$$
$$z_k=\pm r^{2k}z,\ \pm ir^{2k}z$$
$$\Rightarrow\left|\frac{z^4}{\zeta\left(\zeta^4-r^{8k}z^4\right)}\right|<\frac{1}{r^{6k+1}}.\frac{1}{\left|\zeta-r^2/\overline{z}\right|}. \tag{68}$$

Third, we have

$$\left|\frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1}\right| < \left|\frac{z^4}{r^{8k}\zeta^4 z^4 - 1}\right| = \left|\frac{1}{4z^3 r^{2k}}\right| \cdot \left|\frac{1}{\zeta - 1/r^{2k}z} - \frac{1}{\zeta + 1/r^{2k}z} + \frac{i}{\zeta - i/r^{2k}z} - \frac{i}{\zeta + i/r^{2k}z}\right|. \tag{69}$$

From Lemma 1, we have

$$\begin{aligned} &\frac{1}{|\zeta - z_k|} < \frac{1}{|\zeta - z|}, \\ &z_k = \pm \frac{1}{r^{2k}z} \pm \frac{i}{r^{2k}z}, \\ &\Rightarrow \left|\frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1}\right| < \frac{1}{r^{2k+3}} \cdot \frac{1}{|\zeta - z|}. \end{aligned} \tag{70}$$

Fourth, we have

$$\begin{aligned} \left|\frac{1}{\zeta\left(\zeta^4 z^4 - r^{8k}\right)}\right| &= \frac{1}{|\zeta z r^6|} \cdot \left|\frac{1}{\zeta - r^{2k}/z} - \frac{1}{\zeta + r^{2k}/z} + \frac{i}{\zeta - ir^{2k}/z} - \frac{i}{\zeta + ir^{2k}/z}\right| \\ &< \frac{1}{4r^{6n+2}} \left[\frac{1}{|\zeta - r^{2k}/z|} + \frac{1}{|\zeta + r^{2k}/z|} + \frac{1}{|\zeta - ir^{2k}/z|} + \frac{1}{|\zeta + ir^{2k}/z|}\right]. \end{aligned} \tag{71}$$

From Lemma 1, we have

$$\begin{aligned} &\frac{1}{|\zeta - z_k|} < \frac{1}{|\zeta - r^2/\overline{z}|}, \\ &z_k = \pm \frac{r^{2k}}{z}, \pm \frac{ir^{2k}}{z}, \\ &\Rightarrow \left|\frac{1}{\zeta\left(\zeta^4 z^4 - r^{8k}\right)}\right| < \frac{1}{r^{6k+2}} \cdot \frac{1}{|\zeta - r^2/\overline{z}|}. \end{aligned} \tag{72}$$

When using (66), (68), (70), and (72), we can obtain

$$\begin{aligned} |A| &\leq \sum_{k=1}^{\infty} r^{8k} \left|\frac{\zeta^3}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)} + \frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1} + \frac{1}{\zeta\left(z^4\zeta^4 - r^{8k}\right)}\right| \\ &< \sum_{k=1}^{\infty} r^{8k} \left[\left|\frac{\zeta^3}{r^{8k}\zeta^4 - z^4}\right| + \left|\frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)}\right| + \left|\frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1}\right| + \left|\frac{1}{\zeta\left(z^4\zeta^4 - r^{8k}\right)}\right|\right] \\ &< \sum_{k=1}^{\infty} r^{8k} \left(\frac{1}{r^{2k+3}|\zeta - z|} + \frac{1}{r^{6k+1}|\zeta - r^2/\overline{z}|} + \frac{1}{r^{2k+3}|\zeta - z|} + \frac{1}{r^{6k+2}|\zeta - r^2/\overline{z}|}\right) \\ &\leq \frac{2}{|\zeta - z|} \sum_{k=1}^{\infty} r^{6k-3} + \frac{2}{|\zeta - r^2/\overline{z}|} \sum_{k=1}^{\infty} r^{2k-2} < +\infty, \quad \left|r^6\right| < \left|r^2\right| < 1. \end{aligned} \tag{73}$$

And, the series A is uniformly converged. □

5. The Boundedness of Cauchy–Pompeiu's Operator in R_4

The boundedness of the operator T is proved by Tutschke and Mshimba in [8] by applying the Schmitz inequality. In this section, we will use the same technique to prove the boundedness of the modulated Cauchy–Pompeiu's operator in R_4.

Theorem 3. *(Schmitz's inequality [8]). For any regular domain $D \subset \mathbb{C}$, we have*

$$\int_D \frac{d\xi d\eta}{|\zeta - z|^\alpha} \leq \frac{2\pi}{2-\alpha}\left(\frac{S_D}{\pi}\right)^{1-\alpha/2}, \tag{74}$$

where $\zeta = \xi + i\eta$, $z \in D$, α is a real number consist $0 < \alpha < 2$, and S_D is the area of the domain D.

Theorem 4. *Let $D \subset \mathbb{C}$ be a bounded domain; then, for $f \in L_p(\overline{D};\mathbb{C})$, $p > 2$, T is a completely linear operator from $L_p(\overline{D};\mathbb{C})$ into $C^\alpha(\mathbb{C})$ with $\alpha = p - 2/p$.*

Theorem 5. *The Cauchy–Pompeiu's operator in the octant ring R_4 is*

$$\begin{aligned}
&T_{R_4}: C^P(R_4;\mathbb{C}) \longrightarrow C^\alpha(\overline{R_4};\mathbb{C}),\\
&T_{R_4} f(z) = -\frac{4}{\pi},\\
&\int_{R_4}\Bigg\{f(\zeta)\Bigg[\frac{\zeta^3}{\zeta^4 - z^4} + \frac{\zeta^3 z^4}{z^4\zeta^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)} + \frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1} + \frac{1}{\zeta\left(z^4\zeta^4 - r^{8k}\right)}\right)\Bigg]\\
&-\overline{f(\zeta)}\Bigg[\frac{\overline{\zeta}^3}{\overline{\zeta}^4 - z^4} + \frac{\overline{\zeta}^3 z^4}{\overline{\zeta}^4 z^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4 - z^4} + \frac{z^4}{\overline{\zeta}\left(\overline{\zeta}^4 - r^{8k}z^4\right)} + \frac{\overline{\zeta}^3 z^4}{r^{8k}\overline{\zeta}^4 z^4 - 1} + \frac{1}{\overline{\zeta}\left(\overline{\zeta}^4 z^4 - r^{8k}\right)}\right)\Bigg]\Bigg\} d\xi d\eta,
\end{aligned} \tag{75}$$

which is bounded, where $p > 2$ and $\alpha = p - 2/p$.

Proof. First, we have

$$\begin{aligned}
&\frac{4\zeta^3}{\zeta^4 - z^4} = \frac{1}{\zeta - z} + \frac{1}{\zeta + z} + \frac{1}{\zeta - iz} + \frac{1}{\zeta + iz} \Rightarrow \left|\frac{4\zeta^3}{\zeta^4 - z^4}\right| < \frac{4}{|\zeta - z|},\\
&\frac{4\zeta^3 z^4}{\zeta^4 z^4 - 1} = \frac{1}{\zeta - 1/z} + \frac{1}{\zeta + 1/z} + \frac{1}{\zeta - i/z} + \frac{1}{\zeta + i/z} \Rightarrow \left|\frac{4\zeta^3 z^4}{\zeta^4 z^4 - 1}\right| < \frac{4}{|\zeta - z|}.
\end{aligned} \tag{76}$$

From Lemmas 1 and 2, we can write

$$\left|T_{R_4}f(z)\right| \le \frac{4}{\pi}\int_{R_4}\left\{|f(\zeta)|.\left|\frac{\zeta^3}{\zeta^4 - z^4} + \frac{\zeta^3 z^4}{z^4\zeta^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\zeta^3}{r^{8k}\zeta^4 - z^4} + \frac{z^4}{\zeta\left(\zeta^4 - r^{8k}z^4\right)} + \frac{\zeta^3 z^4}{r^{8k}\zeta^4 z^4 - 1} + \frac{1}{\zeta\left(z^4\zeta^4 - r^{8k}\right)}\right)\right|\right.$$

$$+ |f(\zeta).\left|\frac{\overline{\zeta}^3}{\overline{\zeta}^4 - z^4} + \frac{\overline{\zeta}^3 z^4}{\overline{\zeta}^4 z^4 - 1} + \sum_{k=1}^{\infty} r^{8k}\left(\frac{\overline{\zeta}^3}{r^{8k}\overline{\zeta}^4 - z^4} + \frac{z^4}{\overline{\zeta}\left(\overline{\zeta}^4 - r^{8k}z^4\right)} + \frac{\overline{\zeta}^3 z^4}{r^{8k}\overline{\zeta}^4 z^4 - 1} + \frac{1}{\overline{\zeta}\left(\overline{\zeta}^4 z^4 - r^{8k}\right)}\right)\right|\}d\xi d\eta, \tag{77}$$

$f \in L_p(R_4;\mathbb{C}),\ p > 2$

$$< \frac{8}{\pi}\int_{R_4}\left\{f_{C^p}\left[\frac{8}{|\zeta - z|} + \sum_{k=1}^{\infty}\left(\frac{2r^{6k-3}}{|\zeta - z|} + \frac{2r^{2k-2}}{|\zeta - r^2/\overline{z}|}\right)\right]\right\}d\xi d\eta,$$

where $f_{C^p} = \max_{\zeta\in R_4}|f(\zeta)|$, with $f \in C^p(R_4;\mathbb{C}) \subset L_p(R_4;\mathbb{C})$,

$$\begin{aligned}
\left|T_{R_4}f(z)\right| &< \frac{8f_{C^p}}{\pi}\int_{R_4}\left\{\frac{8}{|\zeta - z|} + 2\sum_{k=1}^{\infty}\left(\frac{r^{2k-3}}{|\zeta - z|} + \frac{r^{2k-3}}{|\zeta - r^2/\overline{z}|}\right)\right\}d\xi d\eta \\
&\le \frac{16f_{C^p}}{\pi}\int_{R_4}\left\{\frac{4}{|\zeta - z|} + \frac{1}{1 - r^2}\left(\frac{r^{-1}}{|\zeta - z|} + \frac{r^{-1}}{|\zeta - r^2/\overline{z}|}\right)\right\}d\xi d\eta \\
&\le \frac{16f_{C^p}}{\pi}\int_{R_4}\left\{\left(4 + \frac{1}{r\left(1 - r^2\right)}\right)\frac{1}{|\zeta - z|} + \frac{1}{r\left(1 - r^2\right)}.\frac{1}{|\zeta - r^2/\overline{z}|}\right\}d\xi d\eta \\
&\Rightarrow \left|T_{R_4}f(z)\right| < \frac{16}{\pi}f_{C^p}(I_1 + I_2),
\end{aligned} \tag{78}$$

where

$$I_1 = \left(4 + \frac{1}{r\left(1 - r^2\right)}\right)\int_{R_4}\frac{d\xi d\eta}{|\zeta - z|}. \tag{79}$$

By Schmitz's inequality, we have

$$I_1 \le \left(4 + \frac{1}{r\left(1 - r^2\right)}\right).\frac{2\pi}{2 - 1}\left(\frac{\pi/8\left(1 - r^2\right)}{\pi}\right)^{1-1/2} = \left(4 + \frac{1}{r\left(1 - r^2\right)}\right).\frac{\pi\sqrt{1 - r^2}}{\sqrt{2}}. \tag{80}$$

On the contrary, we have

$$\begin{aligned}
I_2 &= \frac{1}{r\left(1 - r^2\right)}\int_{R_4}\frac{d\xi d\eta}{\left|\zeta - r^2/\overline{z}\right|} \le \frac{1}{r\left(1 - r^2\right)}\int_{R_4}\frac{d\xi d\eta}{\left||\zeta| - r^2/|z|\right|} \\
&\le \int_r^1\int_0^{\pi/4}\frac{t\,dt\,d\theta}{\left|t - r^2/|z|\right|} = \frac{\pi}{4}\int_r^1\frac{t\,dt}{t - r^2/|z|},
\end{aligned} \tag{81}$$

where $r < |z| < 1 \Rightarrow 1 < 1/|z| < 1/r \Rightarrow r^2 < r^2/|z| < r \Rightarrow -r < -r^2/|z| < -r^2$, $r < t < 1$.

Therefore, we have

$$
\begin{aligned}
&0 < t - \frac{r^2}{|z|} < 1 - r^2 \Rightarrow \left| t - \frac{r^2}{|z|} \right| = t - \frac{r^2}{|z|} \\
&I_2 \leq \frac{\pi}{4} \int_r^1 \left(1 + \frac{r^2/|z|}{t - r^2/|z|} \right) \mathrm{d}t = \frac{\pi}{4} \left[t + \frac{r^2}{|z|} \ln \left| t - \frac{r^2}{|z|} \right| \right]_r^1 \\
&I_2 \leq \frac{\pi}{4} \left[1 - r + \frac{r^2}{r} \ln \left(\frac{|z| - r^2}{r|z| - r^2} \right) \right] < \frac{\pi}{4} \left[1 - r + r \left(\frac{|z| - r^2}{r(|z| - r)} \right) \right] \\
&\leq \frac{\pi}{4} [1 - r + |z| + r] < \pi/2 \\
&\Rightarrow I_1 + I_2 < \left(4 + \frac{1}{r(1 - r^2)} \right) . \frac{\pi \sqrt{1 - r^2}}{\sqrt{2}} + \frac{\pi}{2} = \lambda \\
&\Rightarrow \left| T_{R_4} f(z) \right| < \frac{16\lambda}{\pi} f_{C^p}.
\end{aligned}
\tag{82}
$$

And, the operator T_{R_4} is bounded. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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