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Research Article

Minimal Doubly Resolving Sets of Some Classes of Convex Polytopes

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Source localization is one of the most challenging problems in complex networks. Monitoring and controlling complex networks is of great interest for understanding different types of systems, such as biological, technological, and complex physical systems. Modern research has made great developments in identifying sensors through which we can monitor or control complex systems. For this task, we choose a set of sensors with the smallest possible size so that the source may be identified. The problem of locating the source of an epidemic in a network is equivalent to the problem of finding the minimal doubly resolving sets (MDRSs) in a network. In this paper, we calculate the minimal doubly resolving sets (MDRSs) of some classes of convex polytopes in order to compute their double metric dimension (DMD).

1. Introduction and Preliminaries

Graph theory is the most natural and essential method for utilizing and studying these disciplines in fields of research where networks are the fundamental building blocks. For example, (1) in database designing, computer networking, clustering of web documents, mobile phone networks, image processing, and resource allocation; (2) cell biology structure, population genetics, bioinformatics, and cell-sample sequencing are only a few examples in biology; (3) minimal sum coloring, traveling salesman problem, optimization utilizing PERT (project evaluation review technique), game theory, and task and time table scheduling, are all terms used in operations research; and (4) some study blocks in chemistry include the three-dimensional sophisticated simulated structure of atoms, molecular bonds, chemoinformatics, and molecular descriptors.

For a simple, connected, and undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$, the distance $d(f, g)$ is the number of edges in the shortest path between two vertices $f, g \in V_\Gamma$. A vertex $e \in V_\Gamma$ is said to resolve two vertices f and g, if $d(e, f) \neq d(e, g)$. The representation of a vertex $g \in \Gamma$ with respect to the ordered subset $L_\Gamma = \{l_\iota | 1 \leq \iota \leq \rho\}$ is defined as a ρ vector $(d(g, l_\iota))_{\iota=1}^{\rho}$ (also known as vector of metric coordinates) and is denoted by $r(g|L_\Gamma)$. If any pair of distinct vertices of Γ has a unique vector of metric coordinates with respect to L_Γ, then L_Γ is called a resolving set of Γ. A resolving set or metric generator of minimum number of entries is called a basis for the graph Γ, and its count is known as the metric dimension (MD) of Γ, represented by $\dim(\Gamma)$.

The MD was first described in 1953 for general metric space [1]. Later, for the simple and undirected graph Γ, Slater [2] introduced the concept of a resolving set, in 1975. In 1976, Harary and Melter [3] individually introduced this graph theoretic parameter. A network intruder was the original purpose of the resolving sets, but Chartrand and Zhang [4] have published numerous uses of this technology in the method of positioning robot networks, chemical structures, and biological sciences. The findings of [5] discuss the applications of this invariant to chemistry, those of

[6] discuss applications to robot navigation in networks, and those of [7] discuss applications to pattern recognition and image processing. Several coin-weighing difficulties, such as those presented in [8, 9], as well as the full analysis of the game Mastermind, which is described in [10], have a strong link to the MD of Hamming graphs. Because of its wide applications in various branches of mathematics, such as discovery and verification in networks [11], methods of positioning robot networks [6], routing protocols geographically [12], optimization problems in combinatorics [13], and problems of sonar and coast guard LORAN [2], graph resolvability has become an important parameter in graph theory.

The MD of arbitrary graphs is a computationally challenging problem to solve. This has led to the discovery of useful boundaries for a number of graph classes. For instance, the bounds of MD for certain classes of Petersen graphs were evaluated by Shao et al. [14]. All the graphs with MD $n-1$, $n-2$, and 1 were classified by Chartrand et al. [5]. Tomescu et al. and Buczkowski et al. have researched the MD of specific distance-regular graph families, such as Jahangir graphs [15] and wheel graphs [16]. The MD for kayak paddle graphs and chorded cycles was determined by Ahmad et al. [17]. The MD of Mobius ladders was investigated by Ali et al. [18]. It has been shown that Mobius ladders are a family of cubic graphs with constant MD. The MD of regular bipartite graphs was calculated by Baca et al. (see [19]). Resolving sets were used to compute the MD for various classes of convex polytopes [20, 21]. Several years ago, Imran et al. [22] evaluated the MD of some plane graph families, and the MD of necklace graphs was computed using resolving sets in [23].

It is an intriguing task to determine the origins of propagation in complex networks. If a mysterious source of viral propagation is spreading throughout a network, the only information required to detect it is the infection time of a set of nodes known as sensing devices (or sensors). These devices may be able to track the shortest period of time they were contaminated. The only thing left to do is figure out how many devices will be required to ensure that the infection source is precisely located. The answer to this problem is a property called as double metric dimension [24, 25].

Identifying the infection source may be easy if one can watch the entire virus spreading process. However, the entire procedure may be too expensive due to the expense of data acquisition. Unless the initial timing of virus propagation is unclear, a doubly resolving sensor set may be able to reliably identify the sources of infection [26].

Detecting the virus source in a starlike network is more challenging than in a pathlike network [25]. For a star network with n vertices, the DMD is $n-1$, while for a path network it is 2. Furthermore, this indicates that the DMD is always dependent on the network's topology.

When we are interested in finding upper bounds on the MD of graphs, then DRSs are very helpful. DRSs were introduced by Caceres et al. [27] by proving their connection with the MD of the Cartesian product of the graph Γ and by showing that the least cardinality of a DRS is the upper limit of the MD of the graph under consideration. Let v_1, v_2, y_1, and y_2 be four distinct vertices of Γ; we say that v_1, v_2 doubly resolves y_1, y_2 or y_1, y_2 doubly resolves v_1, v_2, if $d(v_1, y_1) - d(v_1, y_2) \neq d(v_2, y_1) - d(v_2, y_2)$. A vertex set $D = \{v_\iota | 1 \leq \iota \leq \rho\} \subseteq V_\Gamma$ is termed as DRS of the graph Γ if any pair of distinct vertices of Γ are doubly resolved by some two vertices in $D \subseteq V_\Gamma$. A DRS having a minimum size is called the MDRS, and its size is called the DMD of Γ represented by $\psi(\Gamma)$. It is obvious that every DRS is a resolving set, which gives $\psi(\Gamma) \geq \dim(\Gamma)$ for all graphs Γ. The computational complexity of the DRSs and MD was investigated in [6, 28], respectively.

Thus, DRSs play a very important role while studying Cartesian products of graphs. The concept of finding upper bounds in the Cartesian product of graphs motivated us to work on DRSs of various classes of graphs. Ahmad et al. have created MDRSs for a number of Harary graph families [29]. The families of circulant graphs were found to have the same MD and MDRS [30]. The DMD and DRSs for the line graph of prism graphs and n-sunlet graphs have also been examined (for details, see [31]). The first explicit approximations of lower and upper bounds for the MDRSs problem were published by Chen et al. [32]. Ahmad et al. [33] have constructed the MD and MDRSs for the line graph of kayak paddle graphs. Lu et al. [34] developed a linear-time approach for the MDRS problem of all graphs, where each block represents a cycle or complete graph. In the case of prism and Hamming graphs and some convex polytopes, the MDRSs have been derived in [35–37], respectively. Liu et al. [38] considered the family of layer-sun graphs and their line graphs for the investigation of MDRSs. A recent study in [39] investigated the MDRSs for a variety of convex polytopes by Pan et al. The MD and MDRSs of jellyfish graphs and cocktail party graphs were evaluated by Liu et al. [40, 41]. Additionally, in [42], the line graphs of necklace graphs have been computed for the MDRS problem, as well as for the MD problem.

In the last few years, solving convex polytopes for the DMD has proven to be a tough problem. The DMD of double antiprism graphs A_n^* and convex polytopes U_n, described by Baca [43] and Imran et al. [21], respectively, are computed in this article. The MD of the convex polytopes A_n^* and U_n presented in the following theorems were calculated by Imran et al. [21, 44].

Theorem 1. *Let A_n^* be the graph of double antiprism, then $\dim(A_n^*) = 3$ for $n \geq 6$.*

Theorem 2. *Let U_n be the graph of convex polytopes, then $\dim(U_n) = 3$ for $n \geq 6$.*

In this paper, we computed the MDRSs for double antiprism graphs A_n^* and convex polytope U_n. In Section 2, we computed the MDRSs of double antiprism graphs A_n^* for $n \geq 6$. In Section 3, the MDRSs for the convex polytope U_n, where $n \geq 6$ have been conjectured. Section 4 concludes that the MDRSs for the double antiprism graphs A_n^* and convex polytopes U_n are constant.

2. Minimal Doubly Resolving Sets for the Double Antiprism Graphs A_n^*

Here, in this section, we computed the MDRSs for the double antiprism graphs A_n^*.

As demonstrated in Figure 1, the double antiprism graph A_n^* has 3-sided and n-sided faces.

We define the inner cycle vertices which are represented by $\{q_\mu\colon \forall\, 0 \le \mu \le n-1\}$, the middle cycle vertices which are represented by $\{r_\mu\colon \forall\, 0 \le \mu \le n-1\}$, and the outer cycle vertices which are represented by $\{s_\mu\colon \forall\, 0 \le \mu \le n-1\}$ as displayed in Figure 1.

Here, the DMD $\psi(A_n^*) \ge 3$, for $n \ge 3$ by applying Theorem 1. We are going to prove the DMD $\psi(A_n^*) = 3$, for $n \ge 3$ as well. To calculate the distances for the double antiprism graph A_n^*, let $S_\mu(r_0) = \{r \in V_{A_n^*}\colon d(r_0, r) = \mu\}$ be a vertex set in $V_{A_n^*}$ at a distance μ from r_0. Table 1 can be easily formulated for $S_\mu(r_0)$, and it will be used to calculate the distances between any two vertices in $V_{A_n^*}$.

The symmetry of A_n^*, for $n \ge 3$ shows that

$$\begin{aligned} d(q_\mu, q_\nu) &= d(r_\mu, r_\nu) = d(s_\mu, s_\nu) \\ &= d(r_0, r_{|\mu-\nu|}), \quad \text{if } 0 \le |\mu-\nu| \le n-1. \end{aligned} \tag{1}$$

When n is odd, we have

$$d(s_\mu, r_\nu) = \begin{cases} d(r_0, s_{|\mu-\nu|}) - 1, & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \nu > \mu, \\ d(r_0, s_{|\mu-\nu|}) + 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \nu > \mu, \\ d(r_0, s_{|\mu-\nu|}), & \text{if } 0 \le |\mu-\nu| \le n-1 \text{ for } \mu \ge \nu, \end{cases}$$

$$d(q_\mu, r_\nu) = \begin{cases} d(r_0, q_{|\mu-\nu|}) + 1, & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \nu > \mu, \\ d(r_0, q_{|\mu-\nu|}) - 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \nu > \mu, \\ d(r_0, q_{|\mu-\nu|}), & \text{if } 0 \le |\mu-\nu| \le n-1 \text{ for } \mu \ge \nu, \end{cases}$$

$$d(q_\mu, s_\nu) = \begin{cases} d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = 1 \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}), & \text{if } 2 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = \dfrac{n+1}{2} \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}) + 2, & \text{if } \dfrac{n+3}{2} \le |\mu-\nu| \le n-1 \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}) + 2, & \text{if } 0 \le |\mu-\nu| \le \dfrac{n-3}{2} \text{ for } \nu \ge \mu, \\ d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = \dfrac{n-1}{2} \text{ for } \nu > \mu, \\ d(r_0, r_{|\mu-\nu|}), & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-2 \text{ for } \nu > \mu, \\ d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = n-1 \text{ for } \nu > \mu. \end{cases} \tag{2}$$

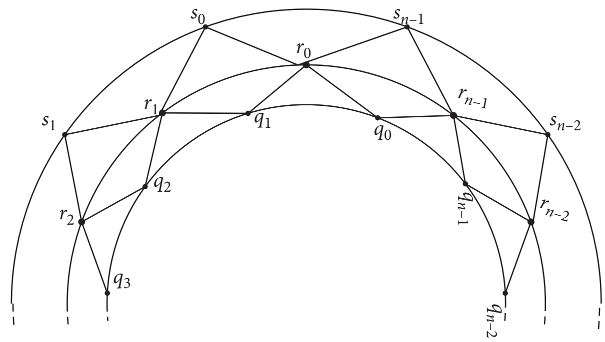

FIGURE 1: : Double antiprism graph A_n^*.

TABLE 1: $S_\mu(r_0)$ for A_n^*.

n	μ	$S_\mu(r_0)$
	1	$\{q_0, q_1, r_1, r_{n-1}, s_0, s_{n-1}\}$
	$2 \leq \mu \leq \lfloor n/2 \rfloor - 1$	$\{q_\mu, q_{n-\mu+1}, r_\mu, r_{n-\mu}, s_{\mu-1}, s_{n-\mu}\}$
Even	$n/2$	$\{q_{n/2}, q_{n+2/2}, r_{n/2}, s_{n-2/2}, s_{n/2}\}$
Odd	$n-1/2$	$\{q_{n-1/2}, q_{n+3/2}, r_{n-1/2}, r_{n+1/2}, s_{n-3/2}, s_{n+1/2}\}$
	$n+1/2$	$\{q_{n+1/2}, s_{n-1/2}\}$

When n is even, we have

$$
d(q_j, r_k) = \begin{cases} d(r_0, q_{|\mu-\nu|}) + 1, & \text{if } 1 \leq |\mu-\nu| \leq \dfrac{n-2}{2} \text{ for } \nu > \mu, \\ d(r_0, q_{|\mu-\nu|}), & \text{if } |\mu-\nu| = \dfrac{n}{2} \text{ for } \nu > \mu, \\ d(r_0, q_{|\mu-\nu|}) - 1, & \text{if } \dfrac{n+2}{2} \leq |\mu-\nu| \leq n-1 \text{ for } \nu > \mu, \\ d(r_0, q_{|\mu-\nu|}), & \text{if } 0 \leq |\mu-\nu| \leq n-1 \text{ for } \mu \geq \nu, \end{cases}
$$

$$
d(s_\mu, r_\nu) = \begin{cases} d(r_0, s_{|\mu-\nu|}) - 1, & \text{if } 1 \leq |\mu-\nu| \leq \dfrac{n-2}{2} \text{ for } \nu > \mu, \\ d(r_0, s_{|\mu-\nu|}), & \text{if } |\mu-\nu| = \dfrac{n}{2} \text{ for } \nu > \mu, \\ d(r_0, s_{|\mu-\nu|}) + 1, & \text{if } \dfrac{n+2}{2} \leq |\mu-\nu| \leq n-1 \text{ for } \nu > \mu, \\ d(r_0, s_{|\mu-\nu|}), & \text{if } 0 \leq |\mu-\nu| \leq n-1 \text{ for } \mu \geq \nu, \end{cases} \tag{3}
$$

$$
d(q_\mu, s_\nu) = \begin{cases} d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = 1 \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}), & \text{if } 2 \leq |\mu-\nu| \leq \dfrac{n}{2} \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}) + 2, & \text{if } \dfrac{n+2}{2} \leq |\mu-\nu| \leq n-1 \text{ for } \mu > \nu, \\ d(r_0, r_{|\mu-\nu|}) + 2, & \text{if } 0 \leq |\mu-\nu| \leq \dfrac{n-2}{2} \text{ for } \nu \geq \mu, \\ d(r_0, r_{|\mu-\nu|}), & \text{if } \dfrac{n}{2} \leq |\mu-\nu| \leq n-2 \text{ for } \nu > \mu, \\ d(r_0, r_{|\mu-\nu|}) + 1, & \text{if } |\mu-\nu| = n-1 \text{ for } \nu > \mu. \end{cases}
$$

As a result, if we know the distance $d(r_0, r)$ for each $r \in V_{A_n^*}$, we can rebuild the distances between any two vertices in $V_{A_n^*}$.

Lemma 1. *For $n \geq 3$, $\psi(A_n^*) = 3$, whenever n is even.*

Proof. In order to prove $\psi(A_n^*) = 3$, for even $n \geq 3$, it is sufficient to find a DRS of cardinality 3. Now, from Table 1, using the sets $S_\mu(r_0)$, Table 2 demonstrates the vectors of metric coordinates for each vertex of A_n^* in relation to the set $D = \{q_0, r_{n-2/2}, s_{n-1}\}$.

From Table 2, we can verify that if two vertices $a_1, a_2 \in S_\mu(r_0)$ for some $\mu = 1, 2, \ldots, 1/2$, then $r(a_1, D) - r(a_2, D) \neq 0$. Thus, if there are two vertices $a_1 \in S_\mu(r_0)$ and $a_2 \in S_\nu(r_0)$ for any $\mu, \nu = 1, 2, \ldots, 1/2$, $\mu \neq \nu$, then $r(a_1, D) - r(a_2, D) \neq \mu - \nu$. Therefore, the set $D = \{q_0, r_{n-2/2}, s_{n-1}\}$ is the MDRS. Thus, Lemma 1 holds. □

Lemma 2. *For $n \geq 3$, $\psi(A_n^*) = 3$, whenever n is odd.*

Proof. Here, the MDRS for A_3^* is $D = \{q_0, r_1, r_2\}$. In order to prove $\psi(A_n^*) = 3$, for odd $n \geq 5$, it is sufficient to find a DRS of cardinality 3. Now, from Table 1, using the sets $S_\mu(r_0)$, Table 3 demonstrates the vectors of metric coordinates for each vertex of A_n^* in relation to the set $D = \{q_0, r_1, r_{n+1/2}\}$.

From Table 3, we can verify that if two vertices $a_1, a_2 \in S_\mu(r_0)$ for some $\mu = 1, 2, \ldots, n + 1/2$, then $r(a_1, D) - r(a_2, D) \neq 0$. Thus, if there are two vertices $a_1 \in S_\mu(r_0)$ and $a_2 \in S_\nu(r_0)$ for any $\mu, \nu \in 1, 2, \ldots, n + 1/2$, $\mu \neq \nu$, then $r(a_1, D) - r(a_2, D) \neq \mu - \nu$. Therefore, the set $D = \{q_0, r_1, r_{n+1/2}\}$ is the MDRS. Thus, Lemma 2 holds. □

Using the entire technique, it is clearly shown $\psi(A_n^*) = 3$, for $n \geq 3$. Using Lemmas 1 and 2, the main theorem is stated as follows:

Theorem 3. *Let A_n^* be the double antiprism graph, then $\psi(A_n^*) = 3$ for $n \geq 3$.*

3. Minimal Doubly Resolving Sets for the Convex Polytope U_n

Here, in this section, we computed the MDRSs for convex polytope U_n.

As demonstrated in Figure 2, the convex polytope U_n has 4-sided, 5-sided, and n-sided faces.

We define, the inner cycle vertices are represented by $\{q_\mu : \forall\, 0 \leq \mu \leq n-1\}$, the interior cycle vertices are represented by $\{r_\mu : \forall\, 0 \leq \mu \leq n-1\}$, the set of exterior vertices are represented by $\{s_\mu : \forall\, 0 \leq \mu \leq n-1\} \cup \{t_\mu : \forall\, 0 \leq \mu \leq n-1\}$, and the outer cycle vertices are represented by $\{w_\mu : \forall\, 0 \leq \mu \leq n-1\}$ as displayed in Figure 2.

Here, $\psi(U_n) \geq 3$, for $n \geq 6$ by applying Theorem 2. We are going to prove $\psi(U_n) = 3$, for $n \geq 6$ as well. To calculate the distances for the convex polytope U_n, let $S_\mu(r_0) = \{r \in V_{U_n} : d(r_0, r) = \mu\}$ be a vertex set in V_{U_n} at a distance μ from r_0. Table 4 can be easily formulated for $S_\mu(r_0)$, and it will be used to calculate the distances between any two vertices in V_{U_n}.

The symmetry of U_n, where $n \geq 6$, shows that

$$
\begin{aligned}
d(q_\mu, s_\nu) &= d(r_0, s_{|\mu-\nu|}) + 1, \quad \text{if } 0 \leq |\mu-\nu| \leq n-1,\\
d(r_\mu, q_\nu) &= d(r_\mu, s_\nu) = d(t_\mu, w_\nu) = d(r_0, q_{|\mu-\nu|}), \quad \text{if } 0 \leq |\mu-\nu| \leq n-1,\\
d(r_\mu, r_\nu) &= d(q_\mu, q_\nu) = d(w_\mu, w_\nu) = d(r_0, r_{|\mu-\nu|}), \quad \text{if } 0 \leq |\mu-\nu| \leq n-1,\\
d(s_\mu, s_\nu) &= \begin{cases} d(r_0, s_{|\mu-\nu|}) - 1, & \text{if } |\mu-\nu| = 0,\\ d(r_0, s_{|\mu-\nu|}), & \text{if } |\mu-\nu| = 1,\\ d(r_0, s_{|\mu-\nu|}) + 1, & \text{if } 2 \leq |\mu-\nu| \leq n-2,\\ d(r_0, s_{|\mu-\nu|}), & \text{if } |\mu-\nu| = n-1, \end{cases}
\end{aligned}
\tag{4}
$$

When n is odd, we have

$$
d(q_\mu, t_\nu) = d(r_\mu, w_\nu) = \begin{cases} d(r_0, w_{|\mu-\nu|}) - 1, & \text{if } 1 \leq |\mu-\nu| \leq \dfrac{n-1}{2} \text{ for } \mu > \nu,\\ d(r_0, w_{|\mu-\nu|}) + 1, & \text{if } \dfrac{n+1}{2} \leq |\mu-\nu| \leq n-1 \text{ for } \mu > \nu,\\ d(r_0, w_{|\mu-\nu|}), & \text{if } 0 \leq |\mu-\nu| \leq n-1 \text{ for } \nu \geq \mu, \end{cases}
$$

Table 2: Vectors of metric coordinates for A_n^* for even $n \geq 3$.

μ	$S_\mu(r_0)$	$D = \{q_0, r_{n-2/2}, s_{n-1}\}$
0	r_0	$(1, (n-2/2), 1)$
	q_0	$(0, (n/2), 2)$
	q_1	$(1, (n-2/2), 2)$
1	r_1	$(2, (n-4/2), 2)$
	r_{n-1}	$(1, (n/2), 1)$
	s_0	$(2, (n-2/2), 1)$
	s_{n-1}	$(2, (n/2), 0)$
	q_μ	$(\mu, (n-2\mu/2), \mu+1)$
	$q_{n-\mu+1}$	$(\mu-1, (n-2\mu+4/2), \mu)$
$2 \leq \mu \leq (n-2/2)$	r_μ	$(\mu+1, (n-2\mu-2/2), \mu+1)$
	$r_{n-\mu}$	$(\mu, n-2\mu+2/2, \mu)$
	$s_{\mu-1}$	$(\mu+1, (n-2\mu/2), \mu)$
	$s_{n-\mu}$	$(\mu, (n-2\mu+4/2), \mu-1)$
	$q_{n/2}$	$((n/2), 1, (n+2/2))$
	$q_{n+2/2}$	$(n-2/2, 1, n/2)$
$n/2$	$r_{n/2}$	$(n/2, 1, n/2)$
	$s_{n-2/2}$	$(n+2/2, 1, n/2)$
	$s_{n/2}$	$(n/2, 2, n-2/2)$

Table 3: : Vectors of metric coordinates for A_n^* for odd $n \geq 5$.

μ	$S_\mu(f_0)$	$D = \{q_0, r_1, r_{n+1/2}\}$
0	r_0	$(1, 1, n-1/2)$
	q_0	$(0, 2, n-1/2)$
	q_1	$(1, 1, n+1/2)$
1	r_1	$(2, 0, n-1/2)$
	r_{n-1}	$(1, 2, n-3/2)$
	s_0	$(2, 1, n+1/2)$
	s_{n-1}	$(2, 2, n-1/2)$
	q_μ	$(\mu, \mu-1, (n-2\mu+3/2))$
	$q_{n-\mu+1}$	$(\mu-1, \mu+1, (n-2\mu+1/2))$
$2 \leq \mu \leq (n-3/2)$	r_μ	$(\mu+1, \mu-1, (n-2\mu+1/2))$
	$r_{n-\mu}$	$(\mu, \mu+2, (n-2\mu-1/2))$
	$s_{\mu-1}$	$(\mu+1, \mu-1, (n-2\mu+3/2))$
	$s_{n-\mu}$	$(\mu, \mu+1, (n-2\mu+1/2))$
	$q_{n-1/2}$	$(n-1/2, n-3/2, 2)$
	$q_{n+3/2}$	$(n-3/2, n+1/2, 1)$
	$r_{n-1/2}$	$(n+1/2, n-3/2, 1)$
$n-1/2$	$r_{n+1/2}$	$(n-1/2, n-1/2, 0)$
	$s_{n-3/2}$	$(n+1/2, n-3/2, 2)$
	$s_{n+1/2}$	$(n-1/2, n+1/2, 1)$
$n+1/2$	$q_{n+1/2}$	$(n-1/2, n-1/2, 1)$
	$s_{n-1/2}$	$(n+1/2, n-1/2, 1)$

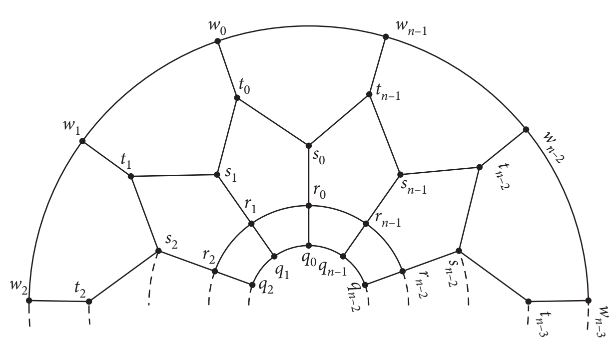

Figure 2: : Convex polytope U_n.

Table 4: $S_\mu(r_0)$ for U_n.

n	μ	$S_\mu(r_0)$
	1	$\{q_0, r_1, r_{n-1}, s_0\}$
	2	$\{q_1, q_{n-1}, r_2, r_{n-2}, t_0, t_{n-1}, s_1, s_{n-1}\}$
	$3 \le \mu \le \lfloor n/2 \rfloor - 1$	$\{q_{\mu-1}, q_{n-\mu+1}, r_\mu, r_{n-\mu}, s_{\mu-1}, s_{n-\mu+1}, t_{\mu-2}, t_{n-\mu+1}, w_{\mu-3}, w_{n-\mu+2}\}$
	$n/2$	$\{q_{n-2/2}, q_{n+2/2}, r_{n/2}, s_{n-2/2}, s_{n+2/2}, t_{n-4/2}, t_{n+2/2}, w_{n-6/2}, w_{n+4/2}\}$
Even	$n+2/2$	$\{q_{n/2}, s_{n/2}, t_{n-2/2}, t_{n/2}, w_{n-4/2}, w_{n+2/2}\}$
	$n+4/2$	$\{w_{n-2/2}, w_{n/2}\}$
	$n-1/2$	$\{q_{n-3/2}, q_{n+3/2}, r_{n-1/2}, r_{n+1/2}, s_{n-3/2}, s_{n+3/2}, t_{n-5/2}, t_{n+3/2}, w_{n-7/2}, w_{n+5/2}\}$
Odd	$n+1/2$	$\{q_{n-1/2}, q_{n+1/2}, s_{n-1/2}, s_{n+1/2}, t_{n-3/2}, t_{n+1/2}, w_{n-5/2}, w_{n+3/2}\}$
	$n+3/2$	$\{t_{n-1/2}, w_{n-3/2}, w_{n+1/2}\}$
	$n+5/2$	$\{w_{n-1/2}\}$

$$d(r_\mu, t_\nu) = \begin{cases} d(r_0, t_{|\mu-\nu|}) - 1, & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \mu > \nu, \\ d(r_0, t_{|\mu-\nu|}) + 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \mu > \nu, \\ d(r_0, t_{|\mu-\nu|}), & \text{if } 0 \le |\mu-\nu| \le n-1 \text{ for } \nu \ge \mu, \end{cases}$$

$$d(s_\mu, t_\nu) = \begin{cases} d(r_0, w_{|\mu-\nu|}) - 3, & \text{if } |\mu-\nu| = 1 \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) - 2, & \text{if } |\mu-\nu| = 2 \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) - 1, & \text{if } 3 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) + 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-2 \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}), & \text{if } |\mu-\nu| = n-1 \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) - 2, & \text{if } |\mu-\nu| = 0 \text{ for } \nu \ge \mu, \\ d(r_0, w_{|\mu-\nu|}) - 1, & \text{if } |\mu-\nu| = 1 \text{ for } \nu > \mu, \\ d(r_0, w_{|\mu-\nu|}), & \text{if } 2 \le |\mu-\nu| \le n-3 \text{ for } \nu > \mu, \\ d(r_0, w_{|\mu-\nu|}) - 1, & \text{if } |\mu-\nu| = n-2 \text{ for } \nu > \mu, \\ d(r_0, w_{|\mu-\nu|}) - 2, & \text{if } |\mu-\nu| = n-1 \text{ for } \nu > \mu, \end{cases}$$

$$d(q_\mu, w_\nu) = \begin{cases} d(r_0, w_{|\mu-\nu|}), & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) + 2, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \mu > \nu, \\ d(r_0, w_{|\mu-\nu|}) + 1, & \text{if } 0 \le |\mu-\nu| \le n-1 \text{ for } \nu \ge \mu, \end{cases}$$

$$d\left(t_\mu, t_\nu\right) = \begin{cases} d\left(r_0, t_{|\mu-\nu|}\right) - 2, & \text{if } |\mu-\nu| = 0, \\ d\left(r_0, t_{|\mu-\nu|}\right) - 1, & \text{if } |\mu-\nu| = 1, \\ d\left(r_0, t_{|\mu-\nu|}\right), & \text{if } 2 \le |\mu-\nu| \le \dfrac{n-1}{2}, \\ d\left(r_0, t_{|\mu-\nu|}\right) + 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-2, \\ d\left(r_0, t_{|\mu-\nu|}\right), & \text{if } |\mu-\nu| = n-1, \end{cases}$$

$$d\left(s_\mu, w_\nu\right) = \begin{cases} d\left(r_0, s_{|\mu-\nu|}\right), & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \mu > \nu, \\ d\left(r_0, s_{|\mu-\nu|}\right) + 1, & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \mu > \nu, \\ d\left(r_0, s_{|\mu-\nu|}\right) + 1, & \text{if } 0 \le |\mu-\nu| \le \dfrac{n-1}{2} \text{ for } \nu \ge \mu, \\ d\left(r_0, s_{|\mu-\nu|}\right), & \text{if } \dfrac{n+1}{2} \le |\mu-\nu| \le n-1 \text{ for } \nu > \mu, \end{cases} \tag{5}$$

When n is even, we have

$$d\left(q_\mu, t_\nu\right) = d\left(r_\mu, w_\nu\right) = \begin{cases} d\left(r_0, w_{|\mu-\nu|}\right), & \text{if } 0 \le |\mu-\nu| \le n-1 \text{ for } \nu \ge \mu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 1, & \text{if } 1 \le |\mu-\nu| \le \dfrac{n-2}{2} \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right), & \text{if } |\mu-\nu| = \dfrac{n}{2} \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right) + 1, & \text{if } \dfrac{n+2}{2} \le |\mu-\nu| \le n-1 \text{ for } \mu > \nu, \end{cases}$$

$$d\left(s_\mu, t_\nu\right) = \begin{cases} d\left(r_0, w_{|\mu-\nu|}\right) - 3, & \text{if } |\mu-\nu| = 1 \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 2, & \text{if } |\mu-\nu| = 2 \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 1, & \text{if } 3 \le |\mu-\nu| \le \dfrac{n-2}{2} \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right), & \text{if } |\mu-\nu| = \dfrac{n}{2} \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right) + 1, & \text{if } \dfrac{n+2}{2} \le |\mu-\nu| \le n-2 \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right), & \text{if } |\mu-\nu| = n-1 \text{ for } \mu > \nu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 2, & \text{if } |\mu-\nu| = 0 \text{ for } \nu \ge \mu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 1, & \text{if } |\mu-\nu| = 1 \text{ for } \nu > \mu, \\ d\left(r_0, w_{|\mu-\nu|}\right), & \text{if } 2 \le |\mu-\nu| \le n-3 \text{ for } \nu > \mu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 1, & \text{if } |\mu-\nu| = n-2 \text{ for } \nu > \mu, \\ d\left(r_0, w_{|\mu-\nu|}\right) - 2, & \text{if } |\mu-\nu| = n-1 \text{ for } \nu > \mu, \end{cases}$$

$$d(r_j,t_k)=\begin{cases} d(r_0,t_{|\mu-\nu|}), & \text{if } 0\le|\mu-\nu|\le n-1 \text{ for } \nu\ge\mu,\\ d(r_0,t_{|\mu-\nu|})-1, & \text{if } 1\le|\mu-\nu|\le\dfrac{n-2}{2} \text{ for } \mu>\nu,\\ d(r_0,t_{|\mu-\nu|}), & \text{if } |\mu-\nu|=\dfrac{n}{2} \text{ for } \mu>\nu,\\ d(r_0,t_{|\mu-\nu|})+1, & \text{if } \dfrac{n+2}{2}\le|\mu-\nu|\le n-1 \text{ for } \mu>\nu,\end{cases}$$

$$d(q_j,w_k)=\begin{cases} d(r_0,w_{|\mu-\nu|}), & \text{if } 1\le|\mu-\nu|\le\dfrac{n-2}{2} \text{ for } \mu>\nu,\\ d(r_0,w_{|\mu-\nu|})+1, & \text{if } |\mu-\nu|=\dfrac{n}{2} \text{ for } \mu>\nu,\\ d(r_0,w_{|\mu-\nu|})+2, & \text{if } \dfrac{n+2}{2}\le|\mu-\nu|\le n-1 \text{ for } \mu>\nu,\\ d(r_0,w_{|\mu-\nu|})+1, & \text{if } 0\le|\mu-\nu|\le n-1 \text{ for } \nu\ge\mu,\end{cases}$$

$$d(t_j,t_k)=\begin{cases} d(r_0,t_{|\mu-\nu|})-2, & \text{if } |\mu-\nu|=0,\\ d(r_0,t_{|\mu-\nu|})-1, & \text{if } |\mu-\nu|=1,\\ d(r_0,t_{|\mu-\nu|}), & \text{if } 2\le|\mu-\nu|\le\dfrac{n-2}{2},\\ d(r_0,t_{|\mu-\nu|})+1, & \text{if } \dfrac{n}{2}\le|\mu-\nu|\le n-2,\\ d(r_0,t_{|\mu-\nu|}), & \text{if } |\mu-\nu|=n-1,\end{cases}$$

$$d(s_j,w_k)=\begin{cases} d(r_0,s_{|\mu-\nu|}), & \text{if } 1\le|\mu-\nu|\le\dfrac{n}{2} \text{ for } \mu>\nu,\\ d(r_0,s_{|\mu-\nu|})+1, & \text{if } \dfrac{n+2}{2}\le|\mu-\nu|\le n-1 \text{ for } \mu>\nu,\\ d(r_0,s_{|\mu-\nu|})+1, & \text{if } 0\le|\mu-\nu|\le\dfrac{n-2}{2} \text{ for } \nu\ge\mu,\\ d(r_0,s_{|\mu-\nu|}), & \text{if } \dfrac{n}{2}\le|\mu-\nu|\le n-1 \text{ for } \nu>\mu.\end{cases} \tag{6}$$

As a result, if we know the distance $d(r_0, r)$ for each $r \in V_{U_n}$, we can rebuild the distances between any two vertices in V_{U_n}.

Lemma 3. *For $n \geq 6$, $\psi(U_n) = 3$, whenever n is even.*

Proof. In order to prove $\psi(U_n) = 3$, for even $n \geq 6$, it is sufficient to find a DRS of cardinality 3. Now, from Table 4, using the sets $S_\mu(r_0)$, Table 5 demonstrates the vectors of metric coordinates for each vertex of U_n in relation to the set $D = \{q_0, q_{n/2}, w_0\}$.

From Table 5, we can verify that if two vertices $b_1, b_2 \in S_\mu(r_0)$ for some $\mu = 1, 2, \ldots, n + 4/2$, then $r(b_1, D) - r(b_2, D) \neq 0$. Thus, if there are two vertices $b_2 \in S_\mu(r_0)$ and $b_2 \in S_\nu(r_0)$ for any $\mu = 1, 2, \ldots, n + 4/2$, $\mu \neq \nu$, then $r(b_1, D) - r(b_2, D) \neq \mu - \nu$. Therefore, the set $D = \{q_0, q_{n/2}, w_0\}$ is the MDRS. Thus, Lemma 3 holds. □

Lemma 4. *For $n \geq 6$, $\psi(U_n) = 3$, whenever n is odd.*

Proof. In order to prove $\psi(U_n) = 3$, for odd $n \geq 6$, it is sufficient to find a DRS of cardinality 3. Now, from Table 4, using the sets $S_\mu(r_0)$, Table 6 demonstrates the vectors of metric coordinates for each vertex of U_n in relation to the set $D = \{q_0, q_{n-1/2}, w_{n+7/2}\}$.

From Table 6, we can verify that if two vertices $b_1, b_2 \in S_\mu(r_0)$ for some $\mu = 1, 2, \ldots, n + 5/2$, then $r(b_1, D) - r(b_2, D) \neq 0$. Thus, if there are two vertices $b_2 \in S_\mu(r_0)$ and $b_2 \in S_\nu(r_0)$ for any $\mu = 1, 2, \ldots, n + 5/2$, $\mu \neq \nu$, then $r(b_1, D) - r(b_2, D) \neq \mu - \nu$. Therefore, the set $D = \{q_0, q_{n-1/2}, w_{n+7/2}\}$ is the MDRS. Thus, Lemma 4 holds. □

Using the entire technique, it is clearly shown $\psi(U_n) = 3$, for $n \geq 6$. Using Lemmas 3 and 4, the main theorem is stated as follows:

Table 5: Vectors of metric coordinates for U_n for even $n \geq 6$.

μ	$S_\mu(r_0)$	$D = \{q_0, q_{n/2}, w_0\}$
0	r_0	$(1, n+2/2, 3)$
1	q_0	$(0, n/2, 4)$
	r_1	$(2, n/2, 3)$
	r_{n-1}	$(2, n/2, 4)$
	s_0	$(2, n+4/2, 1)$
2	q_1	$(1, n-2/2, 4)$
	q_{n-1}	$(1, n-2/2, 5)$
	r_2	$(3, n-2/2, 4)$
	r_{n-2}	$(3, n-2/2, 5)$
	s_1	$(3, n+2/2, 2)$
	s_{n-1}	$(3, n+2/2, 3)$
	t_0	$(3, n+4/2, 1)$
	t_{n-1}	$(3, n+4/2, 2)$
$3 \leq \mu \leq n-2/2$	$q_{\mu-1}$	$(\mu-1, n-2\mu+2/2, \mu+2)$
	$q_{n-\mu+1}$	$(\mu-1, n-2\mu+2/2, \mu+3)$
	r_μ	$(\mu+1, n-2\mu+2/2, \mu+2)$
	$r_{n-\mu}$	$(\mu+1, n-2\mu+2/2, \mu+3)$
	$s_{\mu-1}$	$(\mu+1, n-2\mu+6/2, \mu)$
	$s_{n-\mu+1}$	$(\mu+1, n-2\mu+6/2, \mu+1)$
	$t_{\mu-2}$	$(\mu+1, n-2\mu+8/2, \mu-1)$
	$t_{n-\mu+1}$	$(\mu+1, n-2\mu+8/2, \mu)$
	$w_{\mu-3}$	$(\mu+1, n-2\mu+12/2, \mu-3)$
	$w_{n-\mu+2}$	$(\mu+1, n-2\mu+12/2, \mu-2)$
$n/2$	$q_{n-2/2}$	$(n-2/2, 1, n+4/2)$
	$q_{n+2/2}$	$(n-2/2, 1, n+6/2)$
	$r_{n/2}$	$(n+2/2, 1, n+4/2)$
	$s_{n-2/2}$	$(n+2/2, 3, n/2)$
	$s_{n+2/2}$	$(n+2/2, 3, n+2/2)$
	$t_{n-4/2}$	$(n+2/2, 4, n-2/2)$
	$t_{n+2/2}$	$(n+2/2, 4, n/2)$
	$w_{n-6/2}$	$(n+2/2, 6, n-6/2)$
	$w_{n+4/2}$	$(n+2/2, 6, n-4/2)$
$n+2/2$	$q_{n/2}$	$(n/2, 0, n+6/2)$
	$s_{n/2}$	$(n+4/2, 2, n+2/2)$
	$t_{n-2/2}$	$(n+4/2, 3, n/2)$
	$t_{n/2}$	$(n+4/2, 3, n+2/2)$
	$w_{n-4/2}$	$(n+4/2, 5, n-4/2)$
	$w_{n+2/2}$	$(n+4/2, 5, n-2/2)$
$n+4/2$	$w_{n-2/2}$	$(n+6/2, 4, n-2/2)$
	$w_{n/2}$	$(n+6/2, 4, n/2)$

Table 6: Vectors of metric coordinates for U_n for odd $n \geq 6$.

μ	$S_\mu(r_0)$	$D = \{q_0, q_{n-1/2}, w_{n+7/2}\}$
0	r_0	$(1, n+1/2, n-3/2)$
1	q_0	$(0, n-1/2, n-1/2)$
	r_1	$(2, n-1/2, n-1/2)$
	r_{n-1}	$(2, n+1/2, n-5/2)$
	s_0	$(2, n+3/2, n-5/2)$
2	q_1	$(1, n-3/2, n+1/2)$
	q_{n-1}	$(1, n-1/2, n-3/2)$
	r_2	$(3, n-3/2, n+1/2)$
	r_{n-2}	$(3, n-1/2, n-7/2)$
	s_1	$(3, n+1/2, n-3/2)$
	s_{n-1}	$(3, n+3/2, n-7/2)$
	t_0	$(3, n+5/2, n-5/2)$
	t_{n-1}	$(3, n+5/2, n-7/2)$

Table 6: Continued.

μ	$S_\mu(r_0)$	$D = \{q_0, q_{n-1/2}, w_{n+7/2}\}$
$3 \le \mu \le n-3/2$	$q_{\mu-1}$	$= \begin{cases} (\mu-1, n-2\mu+1/2, n+2\mu-3/2), & \text{if } \mu \le 5; \\ (\mu-1, n-2\mu+1/2, n-2\mu+17/2), & \text{if } \mu > 5 \end{cases}.$
	$q_{n-\mu+1}$	$= \begin{cases} (\mu-1, n-2\mu+3/2, n-2\mu+1/2), & \text{if } \mu \le n-7/2; \\ (\mu-1, n-2\mu+3/2, 2\mu-n-11/2), & \text{if } \mu > n-7/2 \end{cases}.$
	r_μ	$= \begin{cases} (\mu+1, n-2\mu+1/2, n+2\mu-3/2), & \text{if } \mu \le 4; \\ (\mu+1, n-2\mu+1/2, n-2\mu+9/2), & \text{if } \mu > 4 \end{cases}.$
	$r_{n-\mu}$	$= \begin{cases} (\mu+1, n-2\mu+3/2, n-2\mu-3/2), & \text{if } \mu < n-7/2; \\ (\mu+1, n-2\mu+3/2, 2\mu-n+13/2), & \text{if } \mu \ge n-7/2 \end{cases}.$
	$s_{\mu-1}$	$= \begin{cases} (\mu+1, n-2\mu+5/2, n+2\mu-7/2), & \text{if } j \le 5; \\ (j+1, n-2j+5/2, n-2\mu+13/2), & \text{if } j > 5 \end{cases}.$
	$s_{n-\mu+1}$	$= \begin{cases} (\mu+1, n-2\mu+7/2, n-2\mu-3/2), & \text{if } \mu \le n-7/2; \\ (\mu+1, n-2\mu+7/2, 2\mu-n+9/2), & \text{if } \mu > n-7/2. \end{cases}.$
	$t_{\mu-2}$	$= \begin{cases} (\mu+1, n-2\mu+7/2, n+2\mu-9/2), & \text{if } \mu \le 5; \\ (\mu+1, n-2\mu+7/2, n-2\mu+13/2), & \text{if } \mu > 5 \end{cases}.$
	$t_{n-\mu+1}$	$= \begin{cases} (\mu+1, n-2\mu+9/2, n-2\mu-3/2), & \text{if } \mu \ne n-3/2; \\ (n-1/2, 6, 2), & \text{if } \mu = n-3/2 \end{cases}.$
	$w_{\mu-3}$	$= \begin{cases} (\mu+1, n-2\mu+11/2, n+2\mu-13/2), & \text{if } \mu \le 6; \\ (\mu+1, n-2\mu+11/2, n-2\mu+5/2), & \text{if } \mu > 6 \end{cases}.$
	$w_{n-\mu+2}$	$(\mu+1, n-2\mu+13/2, n-2\mu-3/2)$
$n-1/2$	$q_{n-3/2}$	$(n-3/2, 1, 9)$
	$q_{n+3/2}$	$(n-3/2, 2, 6)$
	$r_{n-1/2}$	$(n+1/2, 1, 7)$
	$r_{n+1/2}$	$(n+1/2, 2, 6)$
	$s_{n-3/2}$	$(n+1/2, 3, 7)$
	$s_{n+3/2}$	$(n+3/2, 3, 5)$
	$t_{n-5/2}$	$(n+1/2, 4, 7)$
	$t_{n+3/2}$	$(n+1/2, 5, 3)$
	$w_{n-7/2}$	$(n+1/2, 6, 7)$
	$w_{n+5/2}$	$(n+1/2, 7, 1)$
$n+1/2$	$q_{n-1/2}$	$(n-1/2, 0, 8)$
	$q_{n+1/2}$	$(n-1/2, 1, 7)$
	$s_{n-1/2}$	$(n+3/2, 2, 6)$
	$s_{n+1/2}$	$(n+3/2, 3, 5)$
	$t_{n-3/2}$	$(n+3/2, 3, 6)$
	$t_{n+1/2}$	$(n+3/2, 4, 4)$
	$w_{n-5/2}$	$(n+3/2, 5, 6)$
	$w_{n+3/2}$	$(n+3/2, 6, 2)$
$n+3/2$	$t_{n-1/2}$	$(n+5/2, 3, 5)$
	$w_{n-3/2}$	$(n+5/2, 4, 5)$
	$w_{n+1/2}$	$(n+5/2, 5, 3)$
$n+5/2$	$w_{n-1/2}$	$(n+7/2, 4, 4)$

Theorem 4. *Let U_n be the convex polytope for $n \ge 6$. Then* $\psi(U_n) = 3$.

4. Conclusion

In this article, we calculate the MDRSs for the double antiprism graphs A_n^* and convex polytopes U_n. Our results show that the DMD of these plane graph families is limited and does not depend on the parity of n. In order to resolve all the vertices of these plane graph families, only three appropriately selected vertices are necessary.

Data Availability

No data were used in this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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