Hindawi
Journal of Mathematics
Volume 2022, Article ID 1872695, 14 pages
https://doi.org/10.1155/2022/1872695

Research Article

On Square Sum Labeling of Two Families of Petersen Graphs

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Received 14 December 2021; Accepted 25 January 2022; Published 18 February 2022

Academic Editor: Gohar Ali

A labeling on a graph G with n vertices and m edges is called square sum if there exists a bijection $f: V(G) \longrightarrow \{0, 1, 2, 3, \ldots, n-1\}$ such that the function $f^*: E(G) \longrightarrow N$ defined by $f^*(st) = (f(s))^2 + (f(t))^2$, for all $st \in E(G)$, is injective. A graph G having a square sum labeling is called square sum. In this study, we have investigated the square sum labeling of generalized Petersen graph and double generalized Petersen graph.

1. Introduction

In [1], Germina et al. derived square sum labeling for basic graphs such as trees and cycles. Among various labeling methods that have evolved since 1960, one of the significant labeling methods is square sum. The vast history of the sum of squares of the integers motivated them to study and define this labeling in the particular graphs, and they named it square sum graphs. Many problems in real life and abstract thoughts, which are based on several conditions and with practical implications, can be studied by understanding modeling those problems by graphs, and this problem is one of the uses of graphs in handling and attempting to solve a problem of number theory. A graph which admits square sum labeling is called a square sum graph. In [1], Germina et al. proved that the complete graph K_n is square sum iff $n \leq 7$ and the other graphs which they proved to be square sum are trees, unicyclic graphs, mC_n, and cycles' chord; the graphs obtained by connected two copies of cycle C_n passes through the path P_k, a path union of k copies of C_n, and the path is P_2.

Let G be a graph and p and q be its vertices and edges, respectively. A labeling on the vertices G can also be called a function with domain $V(G)$. An injective function with domain $V(G)$ and range $\{0, 1, 2, \ldots, p-1\}$ is called square sum labeling if the induced function of f, namely, the function f^* with domain $E(G)$ and defined as $f^*(st) = (f(s))^2 + (f(t))^2$, for all $st \in E(G)$, is also injective. The square sum graphs are the graphs having a square sum labeling. We would also like to mention that not all graphs are square sum graphs. For instance, the complete graphs K_n, for $n \geq 6$, are not square sum graphs.

In number theory form, the statement that "a number n can be written as a sum of two squares integers" is equivalent to "every prime of the form $(4k+3)$ occurs an even number of times in the prime factorization of n." The edge labels in square sum labeling is of the form $n = a^2 + b^2$. So, a number n cannot be an edge label of a square sum graph if in its prime factorization a prime of the form $(4k+3)$ (if it exists) occurs an odd number of times. Seoud and Al-Harere [2] proved many necessary conditions of square sum graphs, and they showed that $2C_n$, P_{2n}, and C_{2n} are square sum graphs. In [3], Huilgol and Sriram prove that if G_1 and G_2 are square sum, then $G_1 \cup G_2 \cup G_3$ is also square sum, where G_3 is a set of isolated vertices, for more literature review (see [2–5]).

2. Main Results

In this section, we will present our results of this article.

Theorem 1. *The generalized Petersen graph $P(n,k)$ is a square sum graph, for all $n \geq 5$.*

Proof. Suppose $G \cong P(n,k)$ be the generalized Petersen graph, where $2 \leq k \leq \lfloor n/2 \rfloor$. We shall show that G is square sum graph by defining a square sum labeling on G. For this purpose, we shall consider two cases one is when n is even and other when n is odd.

So, suppose $f\colon V(G) \longrightarrow \{0,1,\ldots,|V(G)-1|\}$ be a function defined as follows. □

Case 1. When n is even.

The labeling of (y_i) for $2 \leq i \leq (n/2)-1$ is

$$f(y_i) = 4i - 2. \tag{1}$$

The labeling of $f(y_0)$ and $f(y_1)$ is fixed:

$$\begin{aligned} f(y_0) &= 1, \\ f(y_1) &= 2. \end{aligned} \tag{2}$$

The labeling of (y_i) for $(n/2)+1 \leq i \leq n-1$ is

$$\begin{aligned} f(y_i) &= 2n - 3 - 4\left(i - \frac{n}{2} - 1\right), \\ f(y_i) &= 2n - 1, \quad i = \frac{n}{2}. \end{aligned} \tag{3}$$

The labeling of $f(x_0)$ and $f(x_1)$ is fixed:

$$\begin{aligned} f(x_0) &= 0, \\ f(x_1) &= 3. \end{aligned} \tag{4}$$

The labeling of vertices x_i for $2 \leq i \leq (n/2)-1$ is

$$\begin{aligned} f(x_i) &= 4i - 1, \\ f(x_i) &= 2n - 2, \quad i = \frac{n}{2}. \end{aligned} \tag{5}$$

The labeling of x_i for $(n/2)+1 \leq i \leq n-1$ is

$$f(x_i) = 2n - 4 - 4\left(i - \frac{n}{2} - 1\right). \tag{6}$$

The labeling of x_i is

$$\begin{aligned} f(x_i) &= 2n + \frac{4n}{2} - 4i \\ &= 2n + 2n - 4i \\ &= 4n - 4i. \end{aligned} \tag{7}$$

This labeling is explained in Figure 1.

Now, we shall discuss the weights of the edges induced by the above labeling of vertices of G. The weight of the edge $uv \in E(G)$ will be denoted as $f^*(uv) = (f(u))^2 + (f(v))^2$. So, the induced labeling of the edge $x_i y_i$ for $0 \leq i \leq n-1$ is given as follows.

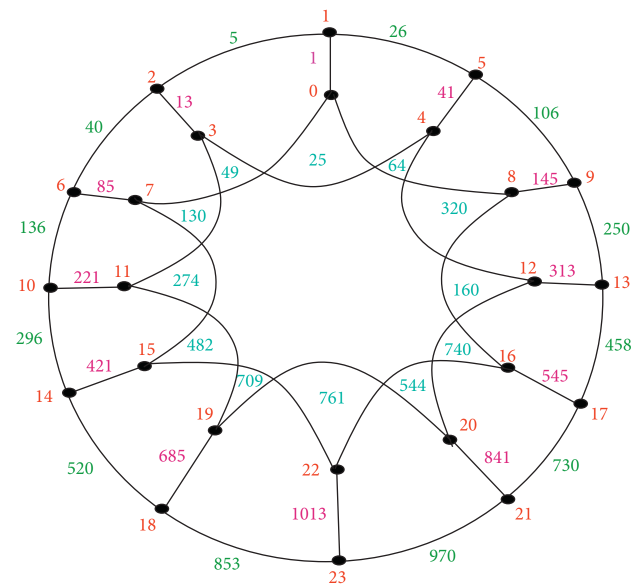

Figure 1: Square sum labeling of Petersen graph $P[12,2]$.

The labeling of $x_i y_i$, for $2 \leq i \leq (n/2)-1$, is

$$\begin{aligned} f^*(x_i y_i) &= f(x_i)^2 + f(y_i)^2; \quad 2 \leq i \leq \frac{n}{2} - 1, \\ &= (4i-1)^2 + (4i-2)^2 \\ &= 32i^2 - 24i + 5. \end{aligned} \tag{8}$$

The labeling of $x_0 y_1$ and $x_1 y_1$ is fixed:

$$\begin{aligned} f(x_0 y_1) &= 1, \\ f(x_1 y_1) &= 13. \end{aligned} \tag{9}$$

The square sum labeling of edge $x_i y_i$, for $i = (n/2)$, is

$$\begin{aligned} f^*(x_i y_i) &= f(x_i)^2 + f(y_i)^2 \\ &= (2n-1)^2 + (2n-2)^2 \\ &= \left(4n^2 - 4n + 1\right) + \left(4n^2 - 8n + 4\right) \\ &= 8n^2 - 12n + 5. \end{aligned} \tag{10}$$

The square sum labeling of $x_i y_i$, for $(n/2)+1 \leq i \leq n-1$, is

$$\begin{aligned} f^*(x_i y_i) &= f(x_i)^2 + f(y_i)^2 \\ &= \left((2n-4) - 4\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &\quad + \left((2n-3-4)\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 32n^2 - 64ni + 32i^2 + 8n - 8i + 1. \end{aligned} \tag{11}$$

The square sum labeling of $y_i y_{i+1}$, for $2 \leq i \leq (n/2)-2$, is

$$\begin{aligned} f^*(y_i y_{i+1}) &= f(y_i)^2 + f(y_{i+1})^2 \\ &= (4i-2)^2 + (4(i+1)-2)^2 \\ &= \left(16i^2 - 16i + 4\right) + \left(16i^2 + 16i + 4\right) \\ &= 32i^2 + 8. \end{aligned} \tag{12}$$

The following labels are fixed:

$$\begin{aligned} f^*(y_0 y_1) &= 5, \\ f^*(y_1 y_2) &= 40, \\ f^*(y_{n-1} y_0) &= 26. \end{aligned} \tag{13}$$

The square sum labeling of the edges $y_i y_{i+1}$, for $i = (n/2) - 1$, is

$$\begin{aligned} f^*(y_i y_{i+1}) &= f(y_i)^2 + f(y_{i+1})^2 \\ &= (4i-2)^2 + (2n-1)^2 \\ &= 16i^2 - 16i + 4 + 4n^2 - 4n + 1. \end{aligned} \tag{14}$$

The labeling of the edges $y_i y_{i+1}$, for $i = (n/2)$, is

$$\begin{aligned} f^*(y_i y_{i+1}) &= f(y_i)^2 + f(y_{i+1})^2 \\ &= (2n-1)^2 + (2n-3)^2 \\ &= 8n^2 - 16n + 10. \end{aligned} \tag{15}$$

The square sum labeling of the edges $y_i y_{i+1}$, for $(n/2) + 1 \le i \le n-2$, is

$$\begin{aligned} f^*(y_i y_{i+1}) &= f(y_i)^2 + f(y_{i+1})^2 \\ &= \left((2n-3) - 4\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &\quad + \left((2n-3) - 4\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 32n^2 - 64ni - 16n + 32i^2 + 16i + 10. \end{aligned} \tag{16}$$

The labeling of edges of $x_i x_{i+1}$, for $1 \le i \le (n/2) - 1$, is

$$\begin{aligned} f^*(x_i x_{i+k}) &= f(x_i)^2 + f(x_{i+k})^2 \\ &= (4i-1)^2 + (4(i+k)-1)^2 \\ &= 32i^2 - 16i^2 + 32ik + 16k^2 - 8k + 2. \end{aligned} \tag{17}$$

The labeling of the edges $x_i x_{i+1}$, for $1 \le i \le (n/2) -1$ and $i + k = (n/2)$, is

$$\begin{aligned} f^*(x_i x_{i+k}) &= f(x_i)^2 + f(x_{i+k})^2 \\ &= (4i-1)^2 + (2n-1)^2 \\ &= 16i^2 - 8i + 4n^2 - 4n + 2. \end{aligned} \tag{18}$$

The labeling of $x_i x_{i+1}$, for if $1 \le i \le (n/2) - 1$ and $i + k > (n/2)$, is

$$\begin{aligned} f^*(x_i x_{i+k}) &= f(x_i)^2 + f(x_{i+k})^2 \\ &= (4i-1)^2 + \left(f^2_{x_{i+k}}\right) \\ &= (4i-1)^2 + \left(2n - 4 - 4\left(i + k - \frac{n}{2} - 1\right)\right)^2 \\ &= \left(16i^2 - 8i + 1\right) + (4n - 4k - 4i) \\ &= 16i^2 - 12i + 4n - 4k + 1. \end{aligned} \tag{19}$$

The labeling of $x_i x_{i+k}$, for $i = (n/2)$, is

$$\begin{aligned} f^*(x_i x_{i+k}) &= f(x_i)^2 + f(x_{i+k})^2 \\ &= (2n-2)^2 + \left(2n - 4 - 4\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 20n^2 + 16i^2 + 16n^2 - 32ni + 8n - 4. \end{aligned} \tag{20}$$

The labeling of $x_i x_{i+k}$, for $(n/2) + 1 \le i \le n-1$ and $i + k \le n - 1$, is

$$\begin{aligned} f^*(x_i x_{i+k}) &= f(x_i)^2 + f(x_{i+k})^2 \\ &= \left(2n - 4 - 4\left(i - \frac{n}{2} - 1\right)\right)^2 + \left(2n - 4 - 4\left(i + k - \frac{n}{2} - 1\right)\right)^2 \\ &= \left(16n^2 - 32ni + 16i^2 + 16i^2\right) + \left(16n^2 - 32ni - 32nk + 32ik + 16i^2 + 16k^2\right) \\ &= 32n^2 - 64ni + 32i^2 + 32nk + 16i^2 + 16k^2. \end{aligned} \tag{21}$$

The labeling of $x_i x_{i+k}$, for $(n/2) + 1 \le i \le n-1$ and $i + k = n = 0 \bmod n$, is

$$f^*(x_i x_{i+k}) = f(x_i)^2 + f(x_{i+k})^2$$
$$= \left(2n - 4 - 4\left(i - \frac{n}{2} - 1\right)\right)^2 + 0$$
$$= 16n^2 + 16i^2 - 32ni - 32nk + 32ik + 16k^2. \quad (22)$$

The labeling of $x_i x_{i+k}$, for $(n/2) + 1 \le i \le n - 1$ and $i + k > n$ and (say $i + k = j$ and mod n), $i \le j \le (n/2) - 2$, is

$$f^*(x_i x_{i+k}) = f(x_i)^2 + f(x_{i+k})^2$$
$$= \left(2n - 4 - 4\left(i - \frac{n}{2} - 1\right)\right)^2 + (4j - 1)^2$$
$$= 16n^2 + 16i^2 - 32ni - 32nk + 32ik + 16k^2 + 16j^2 - 8j + 1. \quad (23)$$

From the computations of all the induced weights of the edges, we can easily see that they are distinct, so the graph is square sum for n even.

Case 2. When n is odd.

Suppose $G \cong P(n, k)$ be the generalized Petersen graph, where $2 \le k \le \lfloor n/2 \rfloor$ and n is odd. We shall show that G is square sum graph by defining a square sum labeling on G. So, let $\phi: V(G) \longrightarrow \{0, 1, \ldots, |V(G) - 1|\}$ be a function defined as follows.

The labeling of $(y_{n-1}), (y_{n-2}) \ldots (y_{n-6})$ is given below:

$$f(y_{n-1}) = 2, f(y_{n-2}) = 3, f(y_{n-3}) = 5,$$
$$f(y_{n-4}) = 8, f(y_{n-5}) = 10, f(y_{n-6}) = 12. \quad (24)$$

These are the fixed numbers of the above labeling.

The labeling of vertices of y_0 is

$$f(y_0) = 2n - 1. \quad (25)$$

The labeling of vertices y_i for $1 \le i \le (n/2)$ is

$$f(y_i) = 2n - 1 - 2i. \quad (26)$$

The labeling of vertices $x_{n-1}, x_{n-2} \ldots, x_{n-6}$ is given below:

$$f(x_{n-1}) = 1,$$
$$f(x_{n-2}) = 4,$$
$$f(x_{n-3}) = 6,$$
$$f(x_{n-4}) = 7,$$
$$f(x_{n-5}) = 9,$$
$$f(x_{n-6}) = 11. \quad (27)$$

These are the fixed numbers of the above labeling:

$$f(x_0) = 0. \quad (28)$$

The labeling x_i for $1 \le i \le n - 7$ is

$$f(x_i) = 2n - 2i. \quad (29)$$

This labeling is explained in Figure 2.

From all these computations of the weights of the edges, we can easily see that they are all distinct, that is, the induced labeling function f^* is injective just the same way as in Case 1. This shows that f is a square sum labeling, and therefore, $P(n, k)$ is a square sum graph.

2.1. Square Sum Labeling of Double Generalized Petersen Graph

The concept of double generalized Petersen graph was introduced by Zhou and Feng in 2012 (see [6]), where the automorphism group of these graph was characterized. Double generalized Petersen graph $DP(n, k)$ is defined as the graph with vertex and edge set, as in Figure 3, particularly, for $k = 1$.

In the next theorem, we have proved that the double generalized Petersen graph is a square sum graph for a particular case when $k = 1$.

Theorem 2. *Suppose $DP(n, 1)$ be the double generalized Petersen graph; then, there exists square sum labeling of $DP(n, 1)$, for all $n \ge 5$.*

Proof. Suppose $G \cong DP(n, 1)$ be the double generalized Petersen graph, where $0 \le i \le n - 1$. We want to show that G is a square sum graph by defining a square sum labeling on G. So, suppose $f: V(G) \longrightarrow \{0, 1, \ldots, |V(G) - 1|\}$ be a function defined as follows.

The labeling of a_i vertices, for $0 \le i \le n - 1$, is

$$f(a_i) = 4i. \quad (30)$$

The labeling of b_i, for $0 \le i \le n - 1$, is

$$f(b_i) = 4i + 1. \quad (31)$$

The labeling of c_i, for $0 \le i \le n - 1$, is

$$f(c_i) = 4i + 2. \quad (32)$$

The labeling of d_i, for $0 \le i \le n - 1$, is

$$f(d_i) = 4i = 3. \quad (33)$$

This above labeling is explained in Figure 4.

Now, we shall discuss the weights of the edges induced by the above labeling of vertices of G. The weight of the edge $uv \in E(G)$ will be denoted as $f^*(uv) = (f(u))^2 + (f(v))^2$. So, the induced labeling of the edge $a_i b_i$, for $0 \le i \le n - 1$, is given as

$$f^*(a_i b_i) = (f(a_i))^2 + (f(b_i))^2$$
$$= (4i)^2 + (4i + 1)^2$$
$$= 32i^2 + 8i + 1. \quad (34)$$

The labeling of $c_i d_i$, for $0 \le i \le n - 1$, is

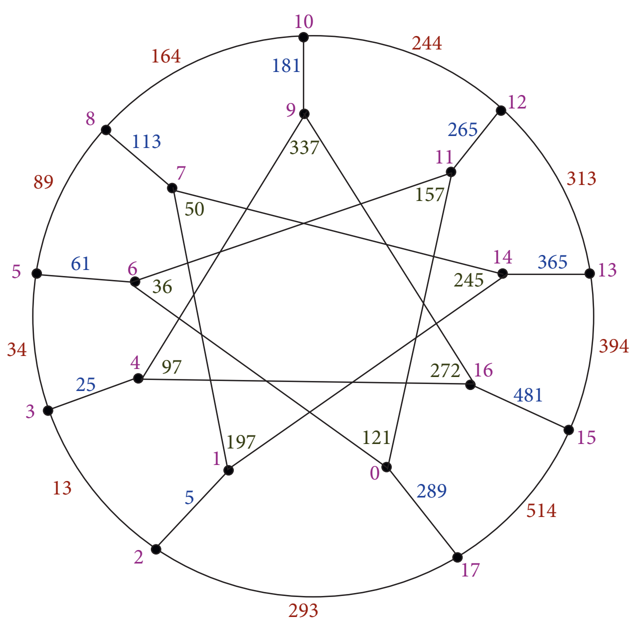

Figure 2: Square sum labeling of Petersen graph P [9,3].

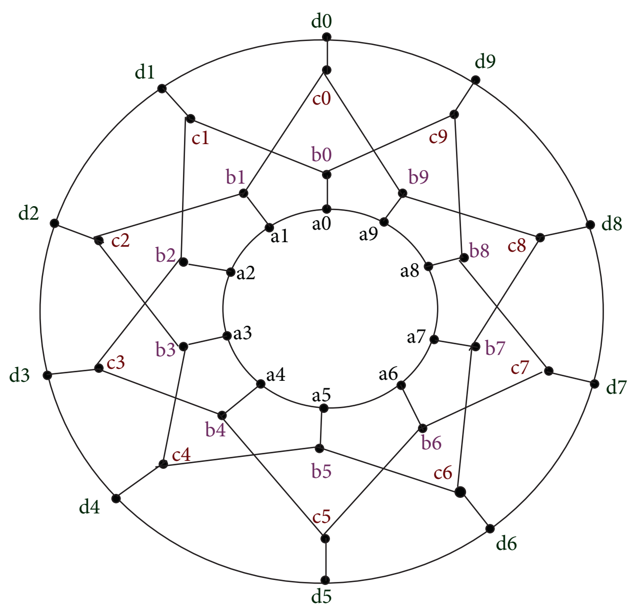

Figure 3: Generalized double Petersen graph $DP(10,1)$.

$$\begin{aligned} f^*(c_i d_i) &= f(c_i)^2 + f(d_i)^2 \\ &= (4i+2)^2 + (4i+3)^2 \\ &= 32i^2 + 40i + 13. \end{aligned} \tag{35}$$

The labeling of $b_i c_{i+1}$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(b_i c_{i+1}) &= f(b_i)^2 + f(c_{i+1})^2 \\ &= (4i+1)^2 + (4(i+1)+2)^2 \\ &= 16i^2 + 8i + 16(i+1)^2 + 16(i+1) + 5. \end{aligned} \tag{36}$$

The labeling of $a_i a_{i+1}$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(a_i a_{i+1}) &= f(a_i)^2 + f(a_{i+1})^2 \\ &= (4i)^2 + (4(i+1))^2 \\ &= 16i^2 + (4i+4)^2 \\ &= 16i^2 + 16i^2 + 32i + 16 \\ &= 32i^2 + 32i + 16. \end{aligned} \tag{37}$$

The labeling of $d_i d_{i+1}$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(d_i d_{i+1}) &= f(d_i)^2 + f(d_{i+1})^2 \\ &= (4i+3)^2 + (4(i+1)+3)^2 \\ &= 16i^2 + 24i + 9 + 16i^2 + 56i + 49 \\ &= 32i^2 + 80i + 58. \end{aligned} \tag{38}$$

The labeling of the edges $b_i c_i$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(b_i c_i) &= f(b_i)^2 + f(c_i)^2 \\ &= (4i+1)^2 + (4i+2)^2 \\ &= i = n-1, \quad i = 0 \\ &= (4(n-1)+1) + (4(0)+2)^2 \\ &= (4n-4+1)^2 + (4(0)+2)^2 \\ &= 16n^2 - 24n + 13. \end{aligned} \tag{39}$$

The labeling of the edges $b_{i+1} c_i$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(b_{i+1} c_i) &= f(b_{i+1})^2 + f(c_i)^2 \\ &= (4i+1)^2 + (4i+2)^2 \\ &= (4(i+1)+1)^2 + (4i+2)^2 \\ &= ((4i+4)+1)^2 + (4i+2)^2 \\ &= (4i+4+1)^2 + (4i+2)^2 \\ &= (4i+5)^2 + (4i+2)^2 \\ &= 16i^2 + 40i + 25 + 16i^2 + 16i + 4 \\ &= 32i^2 + 56i + 29. \end{aligned} \tag{40}$$

The labeling of the edges $a_{n-1} a_0$, for $0 \le i \le n-2$, is

$$\begin{aligned} f^*(a_{n-1} a_0) &= f(a_{n-1})^2 + f(a_0)^2 \\ &= (4i)^2 + (4i)^2 \\ &= (4(n-1))^2 + (4(0))^2 \\ &= (4n-4)^2 \\ &= 16n^2 + 32n + 16. \end{aligned} \tag{41}$$

The labeling of the edges $d_{n-1} d_0$, for $0 \le i \le n-2$, is

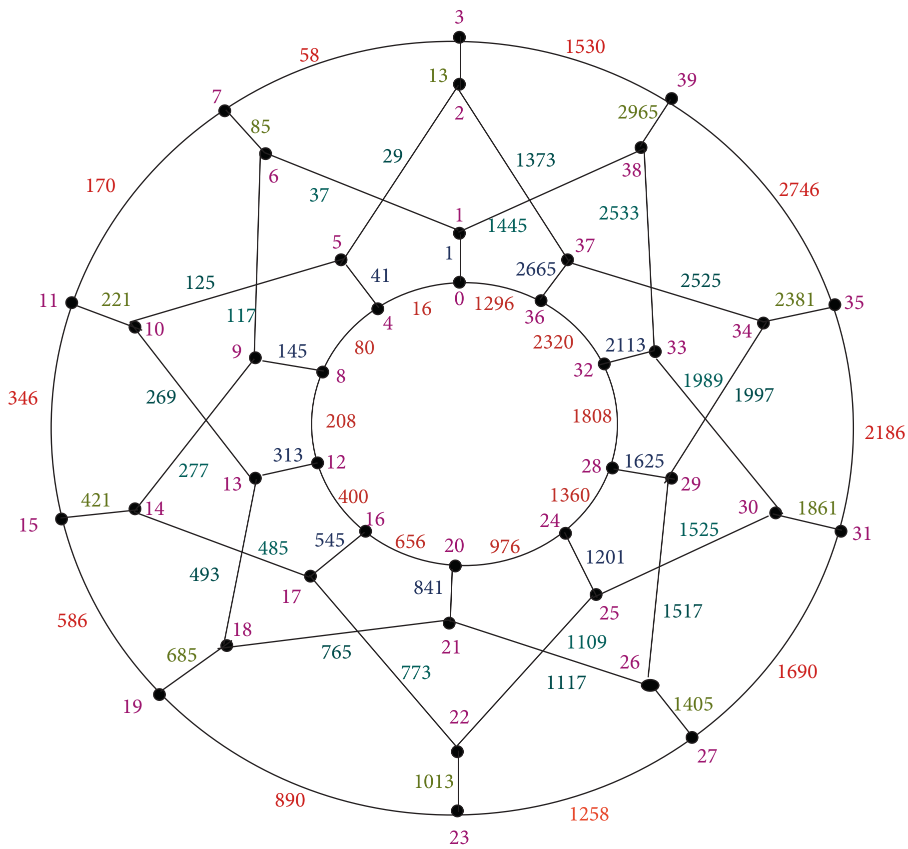

FIGURE 4: Square sum labeling of double generalized Petersen graph $DP[10, 1]$.

$$\begin{aligned} f^*(d_{n-1}d_0) &= f(d_{n-1})^2 + f(d_0)^2 \\ &= (4i+3)^2 + (4i+3)^2 \\ &= (4(n-1)+3)^2 + (4(0)+3)^2 \\ &= (4n-4+3)^2 + (4(0)+3)^2 \\ &= (4n-1)^2 + (3)^2 \\ &= 16n^2 - 8n + 10. \end{aligned} \tag{42}$$

From all these computations of the weights of the edges, we can easily see that they are all distinct, that is, the induced labeling function f^* is injective. This shows that f is a square sum labeling, and therefore, $DP(n, 1)$ is a square sum graph.

In the next theorem, we prove that "the double generalized Petersen graph $DP[n, 2]$ is a square sum graph for a particular case when $k = 2$." □

Theorem 3. *Double generalized Petersen graph $DP[n, 2]$ is square sum for $n \geq 5$.*

Proof. Let $G \cong DP(n, 2)$ be the double generalized Petersen graph. For this purpose, we shall consider two cases one is when n is even and other n is odd. □

Case 3. When n is even.

In this case, we define a labeling $f\colon V(G) \longrightarrow \{0, 1, 2, 3 \ldots |V(G)| - 1\}$, which will show that this labeling is a square sum labeling. Then, we discuss about the weights of the edges induced by the following labeling.

The labeling of the outer edges:

$$\begin{aligned} f(a_0) &= 0, \\ f(a_{n/2}) &= 1, \\ f(a_{n-1}) &= 2, \\ f\left(a_{(n/2)+1}\right) &= 2. \end{aligned} \tag{43}$$

The labeling of the vertices a_i, for $1 \leq i \leq (n/2)$, is

$$\begin{aligned} f(a_i) &= 2(i-1)+3, \\ f\left(a_{(n/2)-1}\right) &= n-1. \end{aligned} \tag{44}$$

The labeling of the vertices a_i, for $(n/2)+2 \leq i \leq n-1$, is

$$f(a_i) = n - 2 - 2\left(i - \frac{n}{2} - 1\right). \tag{45}$$

The labeling of inner vertices of b_i's is

$$f(b_0) = n,$$
$$f(b_{n/2}) = n + 1. \tag{46}$$

The labeling of the vertices b_i, for $1 \le i \le (n/2)$, is

$$f(b_i) = n + 2(i-1) + 3,$$
$$f\left(b_{(n/2)+1}\right) = 2n - 2,$$
$$f\left(b_{(n/2)-1}\right) = 2n - 2, \tag{47}$$
$$f\left(b_{(n/2)+2}\right) = 2n - 4.$$

The labeling of the vertices b_i, for $(n/2) + 2 \le i \le n - 1$, is

$$f(b_i) = 2n - 2 - 2\left(i - \frac{n}{2} - 1\right),$$
$$f\left(b_{(n/2)-2}\right) = 2n - 3. \tag{48}$$

The labeling of inner vertices of $c_{i's}$ is

$$f(c_0) = 2n,$$
$$f(c_{n/2}) = 2n + 1. \tag{49}$$

The labeling of c_i, for $1 \le i \le (n/2)$, is

$$f(c_i) = 2n + 2(i-1) + 3,$$
$$f(c_{n-1}) = 2n + 2,$$
$$f(c_{n-2}) = 2n + 4,$$
$$f\left(c_{(n/2)-2}\right) = 3n - 3, \tag{50}$$
$$f\left(c_{(n/2)-1}\right) = 3n - 1,$$
$$f\left(c_{(n/2)+1}\right) = 3n - 2,$$
$$f\left(c_{(n/2)+2}\right) = 3n - 4.$$

The labeling of c_i, for $(n/2) + 2 \le i \le n - 1$, is

$$f(c_i) = 3n - 2 - 2\left(i - \frac{n}{2} - 1\right). \tag{51}$$

The labeling of the inner vertices of $d_{i's}$ is

$$f(d_0) = 3n,$$
$$f\left(d_{(n/2)-1}\right) = 4n - 1, \tag{52}$$
$$f(d_{n-1}) = 3n + 2,$$
$$f(d_1) = 3n + 3.$$

The labeling of the vertices d_i, for $1 \le i \le (n/2)$, is

$$f(d_i) = 3n + 2(i-1) + 3,$$
$$f\left(d_{(n/2)+1}\right) = 4n - 2, \tag{53}$$
$$f(d_{n/2}) = 3n + 1.$$

The labeling of d_i, for $(n/2) + 2 \le i \le n - 1$, is

$$f(d_i) = 4n - 2 - 2\left(i - \frac{n}{2} - 1\right). \tag{54}$$

This labeling is explained in Figure 5.

The labeling of outer edges of a_i and inner vertices of b_i, for $1 \le i \le n - 1, i \ne 0, i \ne (n/2)$, is

$$\begin{aligned} f^*(a_i b_i) &= f(a_i)^2 + f(b_i)^2 \\ &= (2(i-1)+3)^2 + (n + 2(i-1) + 3)^2 \\ &= 8i^2 + 8i + n^2 + 4in + 2n + 2. \end{aligned} \tag{55}$$

The labeling of inner vertices of c_i and inner edges of b_i, for $1 \le i \le n - 1, i \ne 0, i = (n/2)$, is

$$\begin{aligned} f^*(c_i d_i) &= f(c_i)^2 + f(d_i)^2 \\ &= (2n + 2(i-1) + 3)^2 + (3n + 2(i-1) + 3)^2 \\ &= 13n^2 + 20in + 10n + 8i^2 + 8i + 2, \\ f^*\left(c_{(n/2)+1} d_{(n/2)+1}\right) &= \left(f\left(c_{(n/2)+1}\right)\right)^2 + \left(f\left(d_{(n/2)+1}\right)\right)^2 \\ &= (3n-2)^2 + (4n-2)^2 \\ &= 25n^2 - 28n + 8. \end{aligned} \tag{56}$$

The labeling of the edges $a_i b_i$, for $(n/2) + 2 \le i \le n - 1$, is

$$\begin{aligned} f^*(a_i b_i) &= (f(a_i))^2 + (f(b_i))^2 \\ &= \left(n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 + \left(2n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 13n^2 - 20in + 8i^2. \end{aligned} \tag{57}$$

The labeling of the edges $c_i d_i$, for $(n/2) + 2 \le i \le n - 1$, is

$$\begin{aligned} f^*(c_i d_i) &= (f(c_i))^2 + (f(d_i))^2 \\ &= \left(3n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 + \left(4n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 41n^2 - 36in + 8i^2. \end{aligned} \tag{58}$$

The labeling of the edges $b_i c_{i+2}$, for $1 \le i \le (n/2)$, is

$$\begin{aligned} f^*(b_i c_{i+2}) &= (f(b_i))^2 + (f(c_{i+2}))^2 \frac{n}{2} + 1 \le i \le n - 3 \\ &= (n + 2(i-1) + 3)^2 + (2n + 2(i + 2 - 1) + 3)^2 \\ &= 5n^2 + 8i^2 + 12in + 22n + 24i + 26. \end{aligned} \tag{59}$$

The labeling of $c_{(n/2)+1} b_{i+2}$ is given below:

$$\begin{aligned} f^*\left(c_{(n/2)+1} b_{i+2}\right) &= \left(f\left(c_{(n/2)}\right)\right)^2 + (f(b_{i+2}))^2 \\ &= (3n-2)^2 + \left(2n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= (3n-2)^2 + \left(2n - 2 - 2\left(i + 2 - \frac{n}{2} - 1\right)\right)^2 \\ &= 18n^2 + 4i^2 - 36n + 16i - 12in + 20. \end{aligned} \tag{60}$$

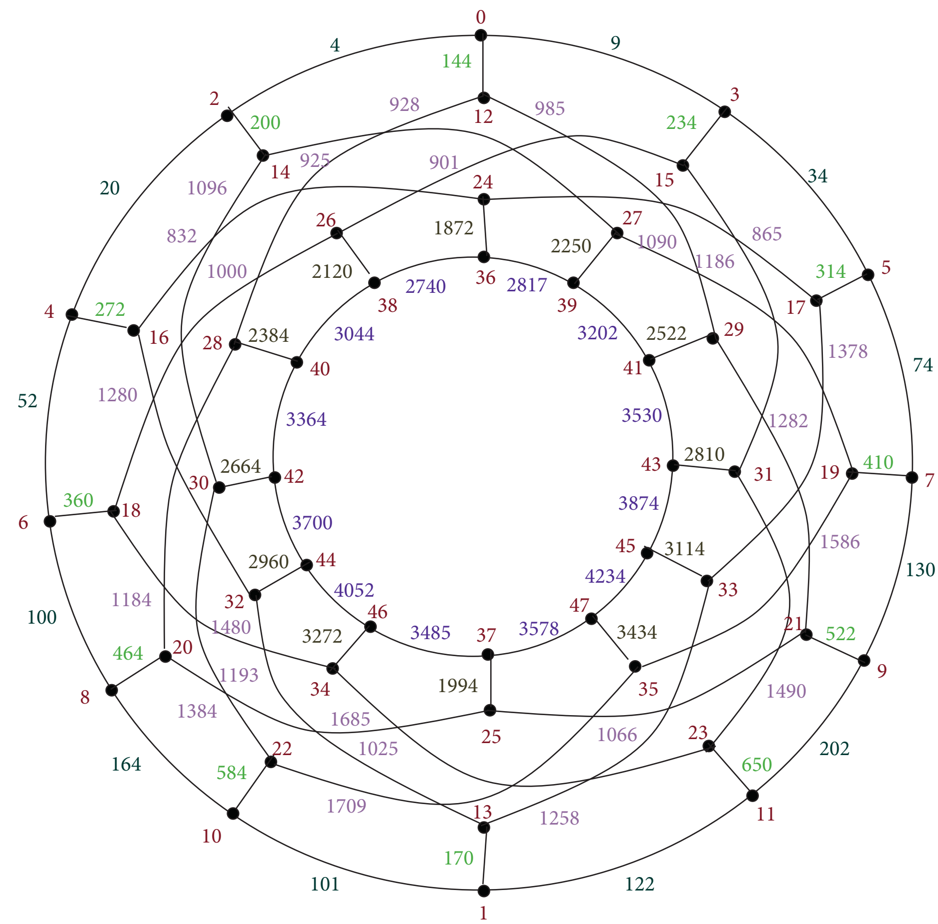

Figure 5: Square sum labeling of $DP[12, 2]$.

The labeling of the edges $b_{i+2}c_i$, for $(n/2) + 1 \leq i \leq n - 3$, is

$$\begin{aligned} f^*(b_{i+2}c_i) &= (f(b_{i+2}))^2 + (f(c_i))^2 \\ &= \left(2n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &\quad + \left(3n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= (3n - 2i + 4)^2 + (4n - 2i)^2 \\ &= 25n^2 - 28in - 24n + 8i^2 + 16i + 16. \end{aligned} \tag{61}$$

The labeling of the edges a_0b_0 is given below:

$$\begin{aligned} f^*(a_0b_0) &= (f(a_0))^2 + (f(b_0))^2 \\ &= (0)^2 + (n)^2 \\ &= n^2. \end{aligned} \tag{62}$$

The labeling of the edges $a_{n/2}b_{n/2}$ is given below:

$$\begin{aligned} f^*(a_{n/2}b_{n/2}) &= (f(a_{n/2}))^2 + \left(f\left(\frac{n}{2}\right)\right)^2 \\ &= (1)^2 + (n+1)^2 \\ &= n^2 + 2n + 2. \end{aligned} \tag{63}$$

The labeling of the edges c_0d_0 is given below:

$$\begin{aligned} f^*(c_0d_0) &= (f(c_0))^2 + (f(d_0))^2 \\ &= (2n)^2 + (3n)^2 \\ &= 4n^2 + 9n^2 \\ &= 13n^2. \end{aligned} \tag{64}$$

The labeling of edges $c_{n/2}d_{n/2}$ is given below:

$$\begin{aligned} f^*(c_{n/2}d_{n/2}) &= (f(c_{n/2}))^2 + (f(d_{n/2}))^2 \\ &= (2n+1)^2 + (3n+1)^2 \\ &= 13n^2 + 10n + 2. \end{aligned} \tag{65}$$

The labeling of vertices, for $b_{(n/2)-2}c_{n/2}$, is given below:

$$\begin{aligned} f^*\left(b_{(n/2)-2}c_{n/2}\right) &= \left(f\left(b_{(n/2)-2}\right)\right)^2 + \left(f\left(c_{n/2}\right)\right)^2 \\ &= (2n-3)^2 + (2n+1)^2 \\ &= 8n^2 - 8n + 10. \end{aligned} \tag{66}$$

The labeling of edges $b_{(n/2)-2}c_{n/2}$ is given below:

$$\begin{aligned} f^*\left(b_{(n/2)-1}c_{(n/2)+1}\right) &= \left(f\left(b_{(n/2)-1}\right)\right)^2 + \left(f\left(c_{(n/2)+1}\right)\right)^2 \\ &= (2n-1)^2 + (3n-2)^2 \\ &= 13n^2 - 16n + 5. \end{aligned} \tag{67}$$

The labeling of edges and vertices, for $b_{n/2}c_{(n/2)+2}$, is given below:

$$\begin{aligned} f^*\left(b_{n/2}c_{(n/2)+2}\right) &= \left(f\left(b_{n/2}\right)\right)^2 + \left(f\left(c_{(n/2)+2}\right)\right)^2 \\ &= (n+1)^2 + (3n-4)^2 \\ &= 10n^2 - 22n + 17. \end{aligned} \tag{68}$$

The labeling of vertices b_ic_{i+2}, for $(n/2)+1 \le i \le n-3$, is

$$\begin{aligned} f^*\left(b_ic_{i+2}\right) &= \left(f\left(b_i\right)\right)^2 + \left(f\left(c_{i+2}\right)\right)^2 \\ &= (2n-2)^2 + \left(3n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 \\ &= 4n^2 - 8n + 4 + \left(3n - 2 - 2\left(i + 2 - \frac{n}{2} - 1\right)\right)^2 \\ &= 4n^2 - 8n + 4 + (4n - 2i - 4)^2 \\ &= 20n^2 - 40n - 16in + 4i^2 + 16i + 20. \end{aligned} \tag{69}$$

The labeling of vertices $b_{n-2}c_0$ is given below:

$$\begin{aligned} f^*\left(b_{n-2}c_0\right) &= \left(f\left(b_{n-2}\right)\right)^2 + \left(f\left(c_o\right)\right)^2 \\ &= (n+4)^2 + (2n)^2 \\ &= 5n^2 + 8n + 16. \end{aligned} \tag{70}$$

The labeling of the edges $b_{n-1}c_1$ is given below:

$$\begin{aligned} f^*\left(b_{n-1}c_1\right) &= \left(f\left(b_{n-1}\right)\right)^2 + \left(f\left(c_1\right)\right)^2 \\ &= (n+2)^2 + (2n + 2(i-1) + 3)^2, \quad i = 1 \\ &= n^2 + 4n + 4 + (2n+3)^2 \\ &= 5n^2 + 16n + 13. \end{aligned} \tag{71}$$

The labeling of the edges $b_{i+2}c_i$, for $1 \le i \le (n/2) - 3$, is

$$\begin{aligned} f^*\left(b_{i+2}c_i\right) &= \left(f\left(b_{i+2}\right)\right)^2 + \left(f\left(c_i\right)\right)^2 \\ &= (n + 2(i-1) + 3)^2 + (2n + 2(i-1) + 3)^2 \\ &= (n + 2i + 5)^2 + (2n + i + 2)^2 \\ &= 5n^2 + 5i^2 + 8in + 18n + 24i + 29. \end{aligned} \tag{72}$$

The labeling of the edges c_0b_2 is given below:

$$\begin{aligned} f^*\left(c_0b_2\right) &= \left(f\left(c_o\right)\right)^2 + \left(f\left(b_2\right)\right)^2 \\ &= (2n)^2 + (n + 2(i-1) + 3)^2 \\ &= (2n)^2 + (n + 2(1) + 3)^2 \\ &= (2n)^2 + (n+5)^2 \\ &= 5n^2 + 10n + 25. \end{aligned} \tag{73}$$

The labeling of the vertices $c_{n-1}b_1$ is given below:

$$\begin{aligned} f^*\left(c_{n-1}b_1\right) &= \left(f\left(c_{n-1}\right)\right)^2 + \left(f\left(b_1\right)\right)^2 \\ &= (2n+2)^2 + (n + 2(i-1) + 3)^2 \\ &= 4n^2 + 8n + 4 + (n+3)^2 \\ &= 4n^2 + 8n + 4 + n^2 + 6n + 9 \\ &= 5n^2 + 14n + 13. \end{aligned} \tag{74}$$

The labeling of the vertices $c_{n-2}b_0$ is given below:

$$\begin{aligned} f^*\left(c_{n-2}b_0\right) &= \left(f\left(c_{n-2}\right)\right)^2 + \left(f\left(b_0\right)\right)^2 \\ &= (2n+4)^2 + (n)^2 \\ &= 4n^2 + 16n + 16 + n^2 \\ &= 5n^2 + 16n + 16. \end{aligned} \tag{75}$$

The labeling of the edges $b_{n/2}c_{(n/2)-2}$ is given below:

$$\begin{aligned} f^*\left(b_{n/2}c_{(n/2)-2}\right) &= \left(f\left(b_{n/2}\right)\right)^2 + \left(f\left(c_{(n/2)-2}\right)\right)^2 \\ &= (n+1)^2 + (3n-3)^2 \\ &= n^2 + 2n + 1 + 9n^2 - 18n + 9 \\ &= 10n^2 - 16n + 10. \end{aligned} \tag{76}$$

The labeling of the edges $b_{(n/2)+2}c_{n/2}$ is given below:

$$\begin{aligned} f^*\left(b_{(n/2)+2}c_{n/2}\right) &= \left(f\left(b_{(n/2)+2}\right)\right)^2 + \left(f\left(c_{n/2}\right)\right)^2 \\ &= (2n-4)^2 + (2n+1)^2 \\ &= 8n^2 - 12n + 17. \end{aligned} \tag{77}$$

The labeling of the edges $b_{n/2}c_{(n/2)+2}$ is given here:

$$\begin{aligned} f^*\left(b_{n/2}c_{(n/2)-2}\right) &= \left(f\left(b_{n/2}\right)\right)^2 + \left(f\left(c_{(n/2)-2}\right)\right)^2 \\ &= (n+1)^2 + (3n-3)^2 \\ &= n^2 + 2n + 1 + 9n^2 - 18n + 9 \\ &= 10n^2 - 16n + 10. \end{aligned} \tag{78}$$

The labeling of the edges $b_{(n/2)+1}c_{(n/2)-1}$ is given below:

$$\begin{aligned} f^*\left(b_{(n/2)+1}c_{(n/2)-1}\right) &= \left(f\left(b_{(n/2)+1}\right)\right)^2 + \left(f\left(c_{(n/2)-1}\right)\right)^2 \\ &= (2n-2)^2 + (3n-1)^2 \\ &= 4n^2 - 8n + 4 + 9n^2 - 6n + 1 \\ &= 13n^2 - 14n + 5. \end{aligned} \tag{79}$$

The labeling of the edges a_0a_1 is given below:

$$\begin{aligned} f^*(a_0a_1)^2 &= (f(a_0))^2 + (f(a_1))^2 \\ &= (0)^2 + (2(i-1)+3)^2, \quad i=1 \\ &= (0)^2 + (2(0)+3)^2 \\ &= (0)^2 + (3)^2 \\ &= 9. \end{aligned} \tag{80}$$

The labeling of the edges a_ia_{i+1}, for $1 \le i \le (n/2)-2$, is

$$\begin{aligned} f^*(a_ia_{i+1}) &= (f(a_i))^2 + (f(a_{i+1}))^2 \\ &= (2(i-1)+3)^2 + (2(i+1-1)+3)^2 \\ &= (2i+1)^2 + (2i+3)^2 \\ &= 8i^2 + 16i + 10. \end{aligned} \tag{81}$$

The labeling of the edges $a_{(n/2)-1}a_{n/2}$ is given below:

$$\begin{aligned} f^*\left(a_{(n/2)-1}a_{n/2}\right)^2 &= \left(f\left(a_{(n/2)-1}\right)\right)^2 + (f(a_{n/2}))^2 \\ &= (n-1)^2 + (1)^2 \\ &= n^2 - 2n + 2. \end{aligned} \tag{82}$$

The labeling of edges with vertices having labels $a_{n/2}a_{(n/2)+1}$ is

$$\begin{aligned} f^*\left(a_{n/2}a_{(n/2)+1}\right) &= f(a_{n/2})^2 + f\left(a_{(n/2)+1}\right)^2 \\ &= (1)^2 + (n-2)^2 \\ &= n^2 - 4n + 5. \end{aligned} \tag{83}$$

The labeling of the edges $a_{n-1}a_0$ is here:

$$\begin{aligned} f^*(a_{n-1}a_0) &= (f(a_{n-1}))^2 + (f(a_0))^2 \\ &= (2)^2 + (0)^2 \\ &= 4. \end{aligned} \tag{84}$$

The square sum labeling of the edges a_ia_{i+1}, for $(n/2)+2 \le i \le n-2$, is

$$\begin{aligned} f^*(a_ia_{i+1}) &= (f(a_i))^2 + f(a_{i+1})^2, \quad \frac{n}{2}+2 \le i \le n-2 \\ &= \left(n-2-2\left(i-\frac{n}{2}-1\right)\right)^2 \\ &\quad + \left(n-2-2(i+1)-\frac{n}{2}-1\right)^2 \\ &= (2n-2i)^2 + (2n-2i-2)^2 \\ &= 4n^2 - 8in + 4i^2 + 4n^2 - 8in - 8n + 8i + 4i^2 + 4 \\ &= 8n^2 - 16in + 8i^2 - 8n + 8i + 4. \end{aligned} \tag{85}$$

The labeling of d_0d_1 is the square sum:

$$\begin{aligned} f^*(d_0d_1) &= (f(d_0))^2 + (f(d_1))^2 \\ &= (3n)^2 + (3n+3)^2 \\ &= 9n^2 + 9n^2 + 18n + 9 \\ &= 18n^2 + 18n + 9. \end{aligned} \tag{86}$$

The labeling of the edges of d_0d_1 is the square sum:

$$\begin{aligned} f^*(d_id_{i+1}) &= (f(d_i))^2 + (f(d_{i+1}))^2 \\ &= (3n+2(i-1)+3)^2 + (3n+2(i+1-1)+3)^2 \\ &= (3n+2i+1)^2 + (3n+2i+3)^2 \\ &= 9n^2 + 12in + 18n + 4i^2 + 12i + 9 + 9n^2 + 12in + 4^2 + 4i + 6n + 1 \\ &= 18n^2 + 24in + 24n + 8i^2 + 16i + 10. \end{aligned} \tag{87}$$

The labeling of the edges $d_{(n/2)-1}d_{n/2}$ is given below:

$$\begin{aligned} f^*\left(d_{(n/2)-1}d_{n/2}\right) &= f\left(d_{(n/2)-1}\right)^2 + (f(d_{n/2}))^2 \\ &= (4n-1)^2 + (3n+1)^2 \\ &= 16n^2 - 8n + 1 + 9n^2 + 6n + 1 \\ &= 25n^2 - 2n + 2. \end{aligned} \tag{88}$$

The labeling of the edges $d_{n/2}d_{(n/2)+1}$ is the square sum:

$$\begin{aligned} f^*(d_{n/2}d_{n/2+1}) &= (f(d_{n/2}))^2 + \left(f\left(d_{(n/2)+1}\right)\right)^2 \\ &= (n+1)^2 + (4n-2)^2 \\ &= 9n^2 + 6n + 1 + 16n^2 - 16n + 5 \\ &= 25n^2 - 10n + 5. \end{aligned} \tag{89}$$

The labeling of the edges $d_{n-1}d_0$ is the square sum:

$$\begin{aligned} f^*(d_{n-1}d_0) &= (f(d_{n-1}))^2 + (f(d_0))^2 \\ &= (3n+2)^2 + (3n)^2 \\ &= 9n^2 + 6n + 6n + 4 + 9n^2 \\ &= 18n^2 + 12n + 4. \end{aligned} \tag{90}$$

The labeling of the edges $d_i d_{i+1}$ is the square sum:

$$\begin{aligned} f^*(d_i d_{i+1}) &= (f(d_i))^2 + (f(d_{i+1}))^2 \\ &= \left(4n - 2 - 2\left(i - \frac{n}{2} - 1\right)\right)^2 + \left(4n - 2 - 2\left(i + 1 - \frac{n}{2} - 1\right)\right)^2 \\ &= (5n - 2i)^2 + (5n - 2i - 2)^2 \\ &= 25n^2 - 20in + 4i^2 + 25n^2 - 20in - 20n + 4i^2 + 8i + 4 \\ &= 50n^2 - 40in + 8i^2 + 8i - 20n + 4. \end{aligned} \tag{91}$$

From all these computations of the weights of the edges, we can easily see that they are all distinct, that is, the induced labeling function f^* is injective. This shows that f is a square sum labeling, and therefore, $DP(n,2)$ is a square sum graph.

Here, we prove Case 2, when n is odd.

So, this graph shows square sum labeling. Let $G \cong DP(n,2)$ be the double generalized Petersen graph, where $1 \le i \le n-1$. We shall show that G is square sum graph by defining a square sum labeling on G.

Case 4. When n is odd.

For this case, we define a labeling $f\colon V(G) \longrightarrow 0, 1, 2, 3 \ldots |V(G)| - 1$, which will show that this labeling is a square sum labeling.

The labeling of a_0 is

$$f(a_0) = 1. \tag{92}$$

The labeling of b_0 is

$$f(b_0) = n + 2. \tag{93}$$

The labeling of c_0 is

$$f(c_0) = n + 1. \tag{94}$$

The labeling of d_0 is

$$f(d_0) = 0. \tag{95}$$

The labeling of a_i, for $1 \le i \le n-1$, is

$$f(a_i) = i + 1. \tag{96}$$

The labeling of b_i, for $1 \le i \le n-1$, is

$$f(b_i) = n + 2 + i. \tag{97}$$

The labeling of c_i, for $2 \le i \le n-1$, is

$$f(c_i) = 3n + i. \tag{98}$$

The labeling of d_i, for $1 \le i \le n-1$, is

$$f(d_i) = 2n + i + 1. \tag{99}$$

The labeling in this case is explained in Figure 6. The square sum labeling of the edge $a_0 b_0$ is

$$\begin{aligned} f^*(a_0 b_0) &= (f(a_0))^2 + (f(b_0))^2 \\ &= (1)^2 + (n+2)^2 \\ &= n^2 + 4n + 5. \end{aligned} \tag{100}$$

The labeling of edges with vertices having labels $a_i b_i$, for $1 \le i \le n-1$, is

$$\begin{aligned} f^*(a_i b_i) &= (f(a_i))^2 + (f(b_i))^2 \\ &= (i+1)^2 + (n+i+2)^2 \\ &= n^2 + 2i^2 + 6i + 4n + 2in + 5. \end{aligned} \tag{101}$$

The square sum labeling of the labels $c_0 d_0$ is

$$\begin{aligned} f^*(c_0 d_0) &= (f(c_0))^2 + (f(d_0))^2 \\ &= (n+1)^2 + (0)^2 \\ &= n^2 + 2n + 1. \end{aligned} \tag{102}$$

The square sum labeling of the labels $c_1 d_i$ is

$$\begin{aligned} f^*(c_1 d_i) &= (f(c_1))^2 + (f(d_i))^2 \\ &= (3n+1)^2 + (2n+i+1)^2 \\ &= 13n^2 + 10n + i^2 + 4in + 2i + 2. \end{aligned} \tag{103}$$

The square sum labeling of the labels $c_i d_i$, for $2 \le i \le n-1$, is

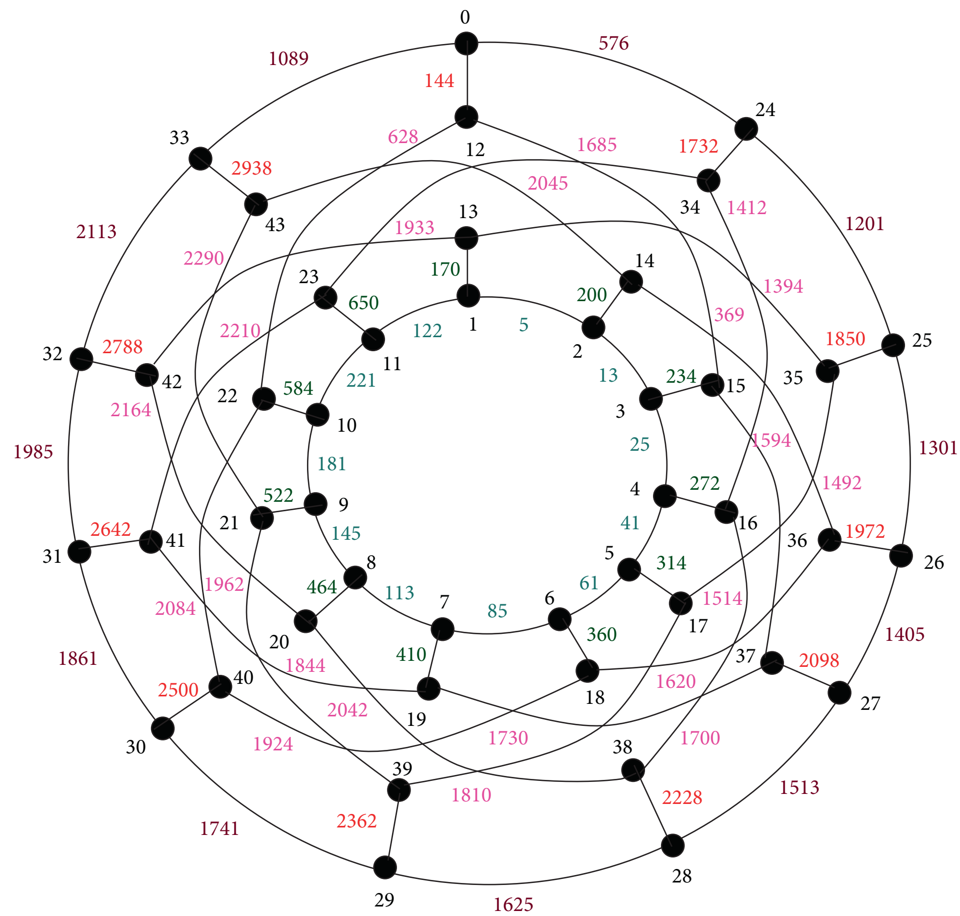

Figure 6: Square sum labeling of double generalized Petersen graph [11,2].

$$\begin{aligned} f^*(c_i d_i) &= (f(c_i))^2 + (f(d_i))^2 \\ &= (3n+i)^2 + (2n+i+1)^2 \\ &= 13n^2 + 10in + 2i^2 + 4n + 2i + 1. \end{aligned} \quad (104)$$

The square sum labeling of $d_i d_0$, for $i = n-1$, is

$$\begin{aligned} f^*(d_i d_0) &= (f(d_i))^2 + (f(d_0))^2 \\ &= (2n+i+1)^2 + (0)^2 \\ &= (2n+(n-1)+1)^2 + (0)^2 \\ &= (3n)^2 \\ &= 9. \end{aligned} \quad (105)$$

The square sum labeling of $a_i a_0$, for $i = n-1$, is

$$\begin{aligned} f^*(a_i a_0) &= (f(a_i))^2 + (f(a_0))^2 \\ &= (i+1)^2 + (1)^2 \\ &= ((n-1)+1)^2 + (1)^2 \\ &= n^2 + 1^2 \\ &= n^2 + 1. \end{aligned} \quad (106)$$

The labeling of the edges $a_i a_{i+1}$, for $0 \le i \le n-2$, is given below:

$$\begin{aligned} f^*(a_i a_{i+1}) &= (f(a_i))^2 + (f(a_{i+1}))^2 \\ &= (i+1)^2 + ((i+1)+1)^2 \\ &= (i+1)^2 + (i+2)^2 \\ &= 2i^2 + 6i + 5. \end{aligned} \quad (107)$$

The labeling of the edges $d_i d_{i+1}$, for $0 \le i \le n-2$, is given below:

$$\begin{aligned} f^*(d_i d_{i+1}) &= (f(d_i))^2 + (f(d_{i+1}))^2 \\ &= (2n+i+1)^2 + (2n+(i+1)+1)^2 \\ &= 8n^2 + 8in + 12n + 2i^2 + 6i + 5. \end{aligned} \quad (108)$$

The square sum labeling of the edges $c_0 b_i$, for $i = n-2$, is

$$\begin{aligned} f^*(c_0 b_i) &= (f(c_0))^2 + (f(b_i))^2 \\ &= (n+1)^2 + (n+i+2)^2 \\ &= (n+1)^2 + (n+2+(n-2))^2 \\ &= (n+1)^2 + (n+2+n-2)^2 \end{aligned}$$

$$= (n+1)^2 + (2n)^2$$
$$= 5n^2 + 2n + 1. \tag{109}$$

The square sum labeling of the edges b_0c_i, for $i = n-2$, is

$$f^*(b_0c_i) = (f(b_0))^2 + (f(c_i))^2$$
$$= (n+2)^2 + (3n+i)^2$$
$$= (n+2)^2 + (3n+(n-2))^2 \tag{110}$$
$$= n^2 + 4n + 4 + 16n^2 + 4 - 16n$$
$$= 17n^2 - 12n + 8.$$

The labeling of the edges b_ic_i is

$$f^* = (b_ic_i) = (f(b_i))^2 + (f(c_i))^2$$
$$= (n+2+i)^2 + (3n+i)^2$$
$$= (n+2+1)^2 + (3n+i)^2;\ (b_i) = n+2+i \Rightarrow i = 1$$
$$= (n+3)^2 + (3n+(n-1))^2;\ (c_i)$$
$$= 3n + i \Rightarrow i = n-1$$
$$= (n+3)^2 + (4n-1)^2$$
$$= 17n^2 - 2n + 10. \tag{111}$$

The square sum labeling of the edges c_0b_i, for $i = 2$, is

$$f^*(c_0b_i) = (f(c_0))^2 + (f(b_i))^2$$
$$= (n+1)^2 + (n+i+2)^2 \tag{112}$$
$$= 2n^2 + 2in + 6n + i^2 + 4i + 5.$$

The square sum labeling of the edges c_ib_{i+2}, for $1 \le i \le n-3$, is

$$f^*(c_ib_{i+2}) = (f(c_i))^2 + (f(b_{i+2}))^2$$
$$= (3n+i)^2 + (n+2+i)^2 \tag{113}$$
$$= 10n^2 + 8n + 8in + 2i^2 + 8i + 16.$$

The square sum labeling of the edges c_ib_i, for $0 \le i \le n-3$, is

$$f^*(c_ib_i) = (f(c_i))^2 + (f(b_i))^2$$
$$= (3n+i)^2 + (n+2+i)^2;\ (c_i) \Rightarrow i = i+2$$
$$= (3n+i+2)^2 + (n+2+i)^2 \tag{114}$$
$$= 10n^2 + 8in + 16n + 2i^2 + 8i + 8.$$

The square sum labeling of the edges c_ib_i, for $0 \le i \le n-3$, is

$$f^*(c_ib_i) = (f(c_i))^2 + (f(b_i))^2$$
$$= (3n+i)^2 + (n+2+i)^2;\ (c_i) \Rightarrow i = i+2$$
$$= (3n+i+2)^2 + (n+2+i)^2 \tag{115}$$
$$= 10n^2 + 8in + 16n + 2i^2 + 8i + 8.$$

The labeling of the edges c_1b_i is given below:

$$f^*(c_1b_i) = (f(c_1))^2 + (f(b_i))^2$$
$$= (3n+1)^2 + (n+2+i)^2;\ (c_i) \Rightarrow i = n-1$$
$$= (3n+1)^2 + (n+2+(n-1))^2 \tag{116}$$
$$= (3n+1)^2 + (2n+1)^2$$
$$= 13n^2 + 10n + 2.$$

From all these computations of the weights of the edges, we can easily see that they are all distinct, that is, the induced labeling function f^* is injective. This shows that f is a square sum labeling, and therefore, $DP(n, 2)$ is a square sum graph.

3. Conclusion

In this study, we have studied the square sum labeling of Generalized Petersen graph and double Generalized Petersen graph, and we prove that these graphs are square sum graphs.

Data Availability

All the data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Key Research and Development Program, under Grant 2018YFB0904205.

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