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Research Article

Uniformly Convergent Scheme for Singularly Perturbed Space Delay Parabolic Differential Equation with Discontinuous Convection Coefficient and Source Term

Mulunesh Amsalu Ayele, Awoke Andargie Tiruneh, and Getachew Adamu Derese

Department of Mathematics, College of Sciences, Bahir Dar University, Bahir Dar, Ethiopia

Correspondence should be addressed to Awoke Andargie Tiruneh; awoke248@yahoo.com

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A singularly perturbed delay parabolic problem of convection-diffusion type with a discontinuous convection coefficient and source term is examined. In the problem, strong interior layers and weak boundary layers are exhibited due to a large delay in the spatial variable and discontinuity of convection coefficient and source. The problem is discretized by a nonstandard finite difference scheme in the spatial variable and for the time derivative, we used the Crank–Nicolson scheme. To enhance the order of convergence of the spatial variable, the Richardson extrapolation technique is applied. The error analysis of the proposed scheme was carried out and proved that the scheme is uniformly convergent of second order in both spatial and temporal variables. Numerical experiments are performed to verify the theoretical estimates.

1. Introduction

In many real life situations, we encounter problems with having small parameters multiplying the highest order derivative terms, involving at least one shift term. We call these singularly perturbed delay differential equations. Such problems arise frequently in the mathematical modeling of various physical and biological phenomena. The delay terms in the models enable us to include some past behavior to get more practical models for the phenomena. For example, Stein's model is a well-known space dependent model which represents a commonly-used description of spontaneous neuronal activity [1]. Many other examples can be found in [2, 3]. Extensive numerical methods have been developed for singularly perturbed delay differential equations, such as [4–7] and references therein.

Among the recently conducted studies on time dependent large spatial delay differential equations, some to mention are [8–11] but still, all these are reaction-diffusion problems with smooth data.

Nonetheless, there are numerical methods for singularly perturbed ordinary differential equations with nonsmooth data (discontinuous source term and/or convection coefficient) using special piecewise uniform meshes; see [12–16] and references therein.

When we came to time dependent differential equations, in [17] the authors studied singularly perturbed parabolic delay differential equation with discontinuous coefficient and source term based on the upwind finite difference method on a specially generated mesh in the spatial direction with backward Euler method for the discretization of the time variable. Providing $\varepsilon-$uniform numerical method for singularly perturbed differential equations with discontinuous coefficients and source terms is not that easy. In this case, the situation is more complicated, especially when the delay is large.

Motivated by the above-given works, we have designed a numerical scheme for singularly perturbed time dependent delay differential equation with discontinuous convection coefficient and source therm, using a nonstandard finite difference scheme in the spatial variable and Crank–Nicolson method in a temporal variable. To increase the order of convergence of the spatial variable, the Richardson extrapolation technique is applied. It is proved that the proposed scheme is uniformly convergent of order $(h^2 + (\Delta t)^2)$.

The paper is organized as follows: in Section 2, we formulate the problem. In Section 3, the bounds on the solution and its derivatives is discussed. Section 4 shows the derivation of the numerical scheme. In Section 5, the convergence of the full discrete scheme is analyzed. In Section 6, we describe the Richardson extrapolation technique and its convergence analysis. Numerical results are presented in Section 7, and lastly, conclusions are given in Section 8.

2. Problem Formulation

Consider the following singularly perturbed delay parabolic problems with a discontinuous convection coefficient and source term:

$$\varepsilon\frac{\partial^2 u}{\partial x^2}+\alpha(x)\frac{\partial u}{\partial x}-\beta(x)u(x,t)-\gamma(x)u(x-1,t)-\frac{\partial u}{\partial t} = f(x,t),\ (x,t)\in\Omega, \tag{1}$$

subject to the following initial condition and interval boundary conditions

$$\begin{aligned}
&u(x,0)=\psi_0(x),\ (x,t)\in\Gamma_0=\{(x,t)\colon\ 0\le x\le 2\ \text{and}\ t=0\},\\
&u(x,t)=\psi_l(x,t),\ (x,t)\in\Gamma_l=\{(x,t)\colon\ (x,t)\in[-1,0]\times[0,T]\},\\
&u(x,t)=\psi_r(x,t),\ (x,t)\in\Gamma_r=\{(x,t)\colon\ x=2,\ \text{and}\ 0\le t\le 2\},
\end{aligned} \tag{2}$$

where $0<\varepsilon\ll 1$ is the perturbation parameter, $\Omega=\Omega_1\cup\Omega_2$, $\Omega_1=(0,1)\times(0,T]$, $\Omega_2=(1,2)\times(0,T]$, $\overline{\Omega}=[0,2]\times[0,T]$, the function $\beta(x)$ and $\gamma(x)$ are sufficiently smooth functions such that $\beta(x)>0, \gamma(x)<0$ and $\beta(x)+\gamma(x)\ge 0$ for all $x\in[0,2]$. Moreover, assume that

$$\begin{aligned}
&\alpha(x)=\begin{cases}\alpha_1(x), & \text{if}\quad 0\le x\le 1,\\ \alpha_2(x), & \text{if}\quad 1\le x\le 2,\end{cases}\\
&f(x,t)=\begin{cases} f_1(x,t), & \text{if}\quad (x,t)\in\overline{\Omega}_1,\\ f_2(x,t), & \text{if}\quad (x,t)\in\Omega_2^*,\end{cases}\\
&\alpha_1(x)<-\eta_1<-2\eta,\ \alpha_2(x)>\eta_2>2\eta>0,\\
&|[\alpha]|\le C,\ |[f]|\le C,
\end{aligned} \tag{3}$$

where $\eta=\min\{\eta_1,\eta_2\}$, $\overline{\Omega}_1=[0,1]\times[0,T]$ and $\Omega_2^*=(1,2]\times[0,T]$. The solution of (1) satisfies $[u]=u(1^+,t)-u(1^-,t)=0$ and $[u_x]=\partial/\partial x(1^+,t)-\partial/\partial x(1^-,t)=0$ at $x=1$, here $u(1^-,t)$ and $u(1^+,t)$ are left and right side limit of u at $x=1$.

The problem defined in (1) can be rewritten as follows:

$$\widetilde{\mathscr{L}}_\varepsilon u(x,t)=F(x,t), \tag{4}$$

where

$$\widetilde{\mathscr{L}}_\varepsilon u(x,t)\equiv\begin{cases}\varepsilon\dfrac{\partial^2 u}{\partial x^2}+\alpha(x)\dfrac{\partial u}{\partial x}-\beta(x)u(x,t)-\dfrac{\partial u}{\partial t}, & \text{for}\,(x,t)\in\Omega_1,\\[2ex] \varepsilon\dfrac{\partial^2 u}{\partial x^2}+\alpha(x)\dfrac{\partial u}{\partial x}-\beta(x)u(x)-\gamma(x,t)u(x-1,t)-\dfrac{\partial u}{\partial t}, & \text{for}\,(x,t)\in\Omega_2,\end{cases} \tag{5a}$$

$$F(x,t)=\begin{cases} f(x,t)-\gamma(x,t)\psi(x-1,t), & \text{for}\,(x,t)\in\Omega_1,\\ f(x,t), & \text{for}\,(x,t)\in\Omega_2,\end{cases} \tag{5b}$$

with initial and boundary condition

$$
\begin{cases}
u(x,0) = \psi_0(x), & (x,t) \in \Gamma_0 = \{(x,t); 0 \le x \le 2 \text{ and } t = 0\}, \\
u(x,t) = \psi_l(x,t), & (x,t) \in \Gamma_l = \{(x,t); (x,t) \in [-1,0] \times [0,T]\}, \\
u(1^-,t) = u(1^+,t), & \dfrac{\partial}{\partial x}(1^-,t) = \dfrac{\partial}{\partial x}(1^+,t) \\
u(2,t) = \psi_r(x,t), & (x,t) \in \Gamma_r = \{(x,t); x = 2 \text{ and } 0 \le t \le T\},
\end{cases}
\tag{6}
$$

The solution $u(x,t)$ of problem (1) exhibits a strong interior layer and weak boundary layer in the neighborhood of the point $x = 1$ and $x = 2$, respectively [17].

3. Bounds on the Solution and Its Derivatives

In this section, the analytical aspects of the solution of problem (1) and its derivatives are studied. The differential operator $\widetilde{\mathscr{L}}_\varepsilon$ satisfies the following minimum principle.

Lemma 1 (Minimum principle). *Suppose $\vartheta(x,t) \in C^{(0,0)}(\overline{\Omega}) \cap C^{(1,0)}(\Omega) \cap C^{(2,1)}(\Omega_1 \cup \Omega_2)$ and assume that $\vartheta(0,t) \ge 0, \vartheta(x,0) \ge 0, \vartheta(2,t) \ge 0$, and $\widetilde{\mathscr{L}}_\varepsilon \vartheta(x,t) \le 0$, $\forall (x,t) \in \Omega$ and $[\vartheta_x](1,t) = \vartheta_x(1^+,t) - \vartheta_x(1^-,t) \le 0$. Then, $\vartheta(x,t) \ge 0$, $\forall (x,t) \in \overline{\Omega}$.*

Proof. Define a test function

$$
s(x,t) = \begin{cases} \dfrac{3}{2} + \dfrac{x}{2}, & (x,t) \in [0,1] \times [0,2], \\ 3 - x, & (x,t) \in [1,2] \times [0,2]. \end{cases}
\tag{7}
$$

Note that, $s(x,t) > 0, \forall (x,t) \in \overline{\Omega}, \widetilde{\mathscr{L}}_\varepsilon s(x,t) > 0, \forall (x, t) \in \Omega_1 \cup \Omega_2$ and $[s_x](1,t) < 0$. Let

$$
\mu = \max\left\{ \frac{-\vartheta(x,t)}{\varphi(x,t)} : (x,t) \in \overline{\Omega} \right\}.
\tag{8}
$$

Then, there exists (x^*,t^*) such that $\vartheta(x^*,t^*) + \mu s(x^*,t^*) = 0$ and $\vartheta(x,t) + \mu s(x,t) \ge 0$, $\forall (x,t) \in \overline{\Omega}$. Therefore, the function attains minimum at (x^*,t^*). Suppose the theorem does not hold true, then $\mu > 0$.

Case 1: $(x^*,t^*) \in \Omega_1$

$$
0 > \widetilde{\mathscr{L}}_\varepsilon(\vartheta + \mu s)(x^*,t^*) = \varepsilon \frac{\partial^2(\vartheta + \mu s)}{\partial x^2}(x^*,t^*) + \alpha_1(x^*) \frac{\partial(\vartheta + \mu s)}{\partial x}(x^*,t^*) - \beta(x^*)(\vartheta + \mu s)(x^*,t^*) - \frac{\partial(\vartheta + \mu s)}{\partial t}(x^*,t^*) \ge 0.
\tag{9}
$$

Case 2: $(x^*,t^*) = (1,t^*)$

$$
0 \le [(\vartheta + \mu s)_x](1,t^*) = [\vartheta_x](1,t^*) + \mu[s_x](1,t^*) < 0.
\tag{10}
$$

Case 3: $(x^*,t^*) \in \Omega_2$

$$
\begin{aligned}
0 > \widetilde{\mathscr{L}}_\varepsilon(\vartheta + \mu s)(x^*,t^*) &= \varepsilon \frac{\partial^2(\vartheta + \mu s)}{\partial x^2}(x^*,t^*) + \alpha_2(x^*) \frac{\partial(\vartheta + \mu s)}{\partial x}(x^*,t^*) \\
&\quad - \beta(x^*)(\vartheta + \mu s)(x^*,t^*) - \gamma(x^*)(\vartheta + \mu s)(x^* - 1,t^*) - \frac{\partial(\vartheta + \mu s)}{\partial t}(x^*,t^*) \\
&= \varepsilon \frac{\partial^2(\vartheta + \mu s)}{\partial x^2}(x^*,t^*) + \alpha_2(x^*) \frac{\partial(\vartheta + \mu s)}{\partial x}(x^*,t^*) - (\beta(x^*) + \gamma(x^*))(\vartheta + \mu s)(x^*,t^*) \\
&\quad - \gamma(x^*)((\vartheta + \mu s)(x^* - 1,t^*) - (\vartheta + \mu s)(x^* - 1,t^*)) - \frac{\partial(\vartheta + \mu s)}{\partial t}(x^*,t^*) \ge 0.
\end{aligned}
\tag{11}
$$

In all the cases we reached at a contradiction. Hence, the required result follows. □

Lemma 2. (Stability result). If $u(x,t)$ satisfies problem (1), then the bound

$$\left\|u\left(x,t_{j+1}\right)\right\|_{\overline{\Omega}} \leq C\max\left\{\|u\|_{\Gamma_l},\|u\|_{\Gamma_0},\left\|\widetilde{\mathscr{L}}_{\varepsilon}u\right\|_{\Omega},\|u\|_{\Gamma_r}\right\}. \tag{12}$$

Proof. Defining the following barrier functions

$$\Phi(x,t)^{\pm} = C\max\left\{\|u\|_{\Gamma_l},\|u\|_{\Gamma_0},\left\|\widetilde{\mathscr{L}}_{\varepsilon}u\right\|_{\Omega},\|u\|_{\Gamma_r}\right\}s(x,t) \pm u(x,t), \tag{13}$$

and using the minimum principle in Lemma 1 we can obtain the required estimate. □

4. Description of the Numerical Scheme

To obtain the totally discrete scheme, we discretized the temporal variable and space variable separately, then formulate the fully discrete scheme.

4.1. Temporal Semidiscretization. On the time domain $[0,T]$, we use uniform mesh given by: $\Omega_t^M = \left\{t_j: t_j = j\Delta t, \Delta t = T/M, \text{for } j = 0,1,\ldots,M\right\}$, where M is the number of mesh intervals in $[0,T]$ and Δt is the time step. Then, on Ω_t^M the continuous problem (1), is discretized by using the Crank–Nicolson method, defined by the following equation:

$$\varepsilon\frac{\partial^2}{\partial x^2}u^{j+1/2} + \alpha(x)\frac{\partial}{\partial x}u^{j+1/2} - \beta(x)u\left(x,t_{j+1/2}\right) - \gamma(x)u\left(x-1,t_{j+1/2}\right) - \frac{u^{j+1}-u^j}{\Delta t} = f\left(x,t_{j+1/2}\right), \tag{14}$$

where

$$\begin{aligned} u\left(x,t_{j+1/2}\right) &= \frac{u\left(x,t_{j+1}\right)+u\left(x,t_j\right)}{2},\\ f\left(x,t_{j+1/2}\right) &= \frac{f\left(x,t_{j+1}\right)+f\left(x,t_j\right)}{2}. \end{aligned} \tag{15}$$

After some rearrangement, the temporal discretized form of (1) is given by the following equation:

$$\mathscr{L}_{\varepsilon}^{M}u\left(x,t_{j+1}\right) = Q\left(x,t_j\right), \tag{16}$$

where

$$\mathscr{L}_{\varepsilon}^{M}u\left(x,t_{j+1}\right) = \begin{cases} \varepsilon\dfrac{\partial^2}{\partial x^2}u^{j+1} + \alpha(x)\dfrac{\partial}{\partial x}u^{j+1} - \left(\beta(x)+\dfrac{2}{\Delta t}\right)u^{j+1}, & \text{for } x \in (0,1],\\ \varepsilon\dfrac{\partial^2}{\partial x^2}u^{j+1} + \alpha(x)\dfrac{\partial}{\partial x}u^{j+1} - \left(\beta(x)+\dfrac{2}{\Delta t}\right)u^{j+1} - \gamma(x,t)u\left(x-1,t_{j+1}\right), & \text{for } x \in (0,2), \end{cases} \tag{17}$$

where

$$Q\left(x,t_j\right) = \begin{cases} -\varepsilon\dfrac{\partial^2}{\partial x^2}u^{j} - \alpha(x)\dfrac{\partial}{\partial x}u^{j} + \left(\beta(x)-\dfrac{2}{\Delta t}\right)u^{j} + 2\gamma(x)u\left(x-1,t_{j+1/2}\right) + 2f\left(x,t_{j+1/2}\right), & x \in (0,1],\\ -\varepsilon\dfrac{\partial^2}{\partial x^2}u^{j} - \alpha(x)\dfrac{\partial}{\partial x}u^{j} + \left(\beta(x)-\dfrac{2}{\Delta t}\right)u^{j} + \gamma(x,t)u\left(x-1,t_j\right) + 2f\left(x,t_{j+1/2}\right), & x \in (1,2). \end{cases} \tag{18}$$

and $u^{j+1}(x) = u(x,t_{j+1})$ is the semi-discrete approximation to the exact solution $u(x,t)$ of (1) at the $(j+1)^{\text{th}}$ time level. The local truncation error of the semi-discrete method (16) is given by the following equation:

$$\widetilde{e}_{j+1} = u\left(x,t_{j+1}\right) - \widetilde{u}^{j+1}(x), \tag{19}$$

where $\widetilde{u}^j(x)$ is the solution evaluated after one step of the semi-discrete scheme (16) taking the exact value $u(x,t_j)$ instead of u^j as the initial data.

Lemma 3. *Let $\vartheta(x,t_{j+1})$ be a smooth function such that $\vartheta(0,t_{j+1}) \geq 0, \vartheta(2,t_{j+1}) \geq 0$, $\mathscr{L}_{\varepsilon}^{M}\vartheta(x,t_{j+1}) \leq 0$, and $[\vartheta_x](1^+,t_{j+1}) = \vartheta_x(1^+, t_{j+1}) - \vartheta_x(1^-,t_{j+1}) = 0, \forall x \in (0,2)$. Then, $\vartheta(x,t_{j+1}) \geq 0, \forall x \in [0,2]$.*

Proof. Apply the same procedure as Lemma 1. □

Lemma 4. *If $u(x,t_{j+1})$ satisfies problem (16), then the bound*

$$\left\|u\left(x,t_{j+1}\right)\right\| \leq \max\left\{\left|u\left(0,t_{j+1}\right)\right|, \frac{\|Q\|}{\eta}, \left|u\left(2,t_{j+1}\right|\right\}, \quad \text{for all } x \in [0,2]. \tag{20}$$

Proof. Consider a barrier function

$$\Phi^{\pm}\left(x,t_{j+1}\right) = \max\left\{|u|\left(0,t_{j+1}\right), \frac{\|Q\|}{\eta}, \left|u\left(2,t_{j+1}\right|\right\} \pm u\left(x,t_{j+1}\right), \tag{21}$$

Clearly, $\Phi^{\pm}(x,t_{j+1}) \geq 0$ and $\Phi^{\pm}(2,t_{j+1}) \geq 0$. Also, for $\mathscr{L}_{\varepsilon}^{M}\Phi^{\pm}(0,t_{j+1})$ we have two cases.

Case 1: $x \in [0,1]$

$$\mathscr{L}_{\varepsilon}^{M}\Phi^{\pm}\left(x,t_{j+1}\right) = -\widetilde{\beta}_1(x)\max\left\{\left|u\left(x,t_{j+1}\right)\right|, \frac{\|Q\|}{\eta}, \left|u\left(2,t_{j+1}\right|\right\} \pm \mathscr{L}_{\varepsilon}^{M}u\left(x,t_{j+1}\right) \leq 0, \text{ since } \widetilde{\beta}_1(x) = \left(\beta(x) + \frac{2}{\Delta t}\right) > 0, \text{and using equation (16)} \tag{22}$$

Case 2: $x \in (0,2]$

$$\mathscr{L}_{\varepsilon}^{M}\Phi^{\pm}\left(x,t_{j+1}\right) = -\widetilde{\beta}_2(x)\max\left\{\left|u\left(0,t_{j+1}\right)\right|, \frac{\|Q\|}{\eta}, \left|u\left(2,t_{j+1}\right|\right\} \pm \mathscr{L}_{\varepsilon}^{M}u\left(x,t_{j+1}\right) \leq 0, \text{since}\,(\beta(x)+\gamma) > 0, \widetilde{\beta}_2(x) = \left(\beta(x) + \frac{2}{\Delta t}\right) + \gamma > 0 \text{ and using equation (16).} \tag{23}$$

Therefore, from Lemma 3 we get $\Phi^{\pm}(x,t_{j+1}) \geq 0$ for all $x \in [0,2]$. □

Lemma 5. *Suppose that* $|\partial^{\nu}/\partial t^{\nu}u(x,t)| \leq C$, $(x,t) \in \overline{\Omega}$, $\nu = 0,1,2,3$, *the local truncation error associated to scheme (16) satisfies:*

$$\left\|\widetilde{e}_{j+1}\right\|_{\infty} \leq C_1(\Delta t)^3, \quad j = 1,2,\ldots,M. \tag{24}$$

Proof. Using Taylor's series expansion, expand $u(x,t_{j+1})$ and $u(x,t_j)$ centered at $t_{j+1/2}$, we get the following equation:

$$\frac{u\left(x,t_{j+1}\right) - u\left(x,t_j\right)}{\Delta t} = u_t\left(x,t_{j+1/2}\right) + O\left((\Delta t)^2\right). \tag{25}$$

Then, substitute (25) into (1), we obtain the following equation:

$$\frac{u\left(x,t_{j+1}\right) - u\left(x,t_j\right)}{\Delta t} = \varepsilon\frac{d}{dx^2}u\left(x,t_{j+1/2}\right) + \alpha(x)\frac{d}{dx}u\left(x,t_{j+1/2}\right) - \beta(x)u\left(x,t_{j+1/2}\right) - \gamma(x)u\left(x-1,t_{j+1/2}\right) - f\left(x,t_{j+1/2}\right) + O\left((\Delta t)^2\right), \tag{26}$$

where

$$f\left(x,t_{j+1/2}\right) = \frac{f\left(x,t_{j+1}\right) + f\left(x,t_j\right)}{2} + O\left((\Delta t)^2\right),$$
$$u\left(x,t_{j+1/2}\right) = \frac{u\left(x,t_{j+1}\right) + u\left(x,t_j\right)}{2} + O\left((\Delta t)^2\right). \tag{27}$$

From (26), the local truncation error $\|\widetilde{e}_{j+1}\|$ is the solution of the following BVP:

$$\begin{cases} \mathscr{L}_\varepsilon^M \widetilde{e}_{j+1} = O\left((\Delta t)^3\right) \\ \widetilde{e}_{j+1}(0) = 0, \widetilde{e}_{j+1}(2) = 0. \end{cases} \tag{28}$$

Next, using the maximum principle for the operator $\mathscr{L}_\varepsilon^M$ proves the result, for further detail one can refer to [18]. □

Theorem 1. (Global error estimate). Under the hypothesis of Lemma 5, the global error estimate $E_{j+1} = u(x, t_{j+1}) - u^{j+1}(x) = \sum_{k=1}^{j} \widetilde{e}_k$, associated with the Crank–Nicolson scheme in the time direction at $j + 1^{th}$ time level is given by the following equation:

$$\left\|E_{j+1}\right\|_\infty \leq C(\Delta t)^2, \quad \text{for}\, j = 1, 2, \ldots, M. \tag{29}$$

Proof. Using the local error estimate given by Lemma 5, we obtain the following global error estimates at $(j+1)^{th}$ time level:

$$\begin{aligned} \left\|E_{j+1}\right\| &= \left\|\sum_{\xi=1}^{j} \widetilde{e}_\xi\right\| \leq \left\|\widetilde{e}_1\right\| + \left\|\widetilde{e}_2\right\| + \cdots + \left\|\widetilde{e}_j\right\| \\ &\leq C_0 j (\Delta t)^3, \text{by Lemma 5} \\ &\leq C_0 (j\Delta t)(\Delta t)^2, \\ &\leq C_0 T (\Delta t)^2, \text{since}\, j\Delta t \leq T \\ &\leq C(\Delta t)^2, C_0 T = C. \end{aligned} \tag{30}$$

□

Lemma 6. *The solution of the semidiscretized problem (16) satisfies the following equation:*

$$\left|\frac{d^p u\left(x, t_{j+1}\right)}{dx^p}\right| \leq C \begin{cases} 1 + \varepsilon^{-p} \exp\left(\dfrac{-\eta(1-x)}{\varepsilon}\right), x \in (0,1), \\ 1 + \varepsilon^{-p} \exp\left(\dfrac{-\eta(x-1)}{\varepsilon}\right) + \varepsilon^{-p+1} \exp\left(\dfrac{-\eta(2-x)}{\varepsilon}\right), x \in (1,2), \end{cases} \tag{31}$$

for $p = 0, 1, 2, 3, 4.$

Proof See [14, 15]. □

4.2. Spatial Discretization. On the spatial domain [0, 2], is discretized uniformly as follows:

$$\overline{\Omega}_x^N = \left\{0 = x_0, x_1, \ldots, x_N = 2\right\}, \tag{32}$$

where $x_i = x_0 + ih, h = 2/N$ for $i = 1, 2, .., N-1$, and N is the number of mesh intervals in the spatial variable. Thus, the discretized domain is defined as $\overline{\Omega}^{N,M} = \overline{\Omega}_x^N \times \Omega_t^M$. In the spatial discretization, the problem (16) is further discretized using the nonstandard finite difference method. The main idea of the nonstandard finite difference method is to replace the denominator of the finite difference approximation of the derivatives by positive functions.

4.3. Nonstandard Finite Difference. For the construction of an exact finite difference scheme, consider the following constant coefficient homogeneous differential equation, corresponding to (16):

$$\varepsilon \frac{d}{dx^2} u^{j+1} + \widehat{\alpha} \frac{d}{dx} u^{j+1} - \widehat{\beta} u^{j+1} = 0, \tag{33}$$

$$\varepsilon \frac{d}{dx^2} u^{j+1} + \widehat{\alpha} \frac{d}{dx} u^{j+1} = 0, \tag{34}$$

where $|\alpha(x)| > \widehat{\alpha} > 0$, $(\beta(x) + (2/\Delta t)) > \widehat{\beta}$. Now, the homogeneous differential equation (33) possesses two linearly independent solutions, given as follows:

$$\exp(\lambda_1) \text{ and } \exp(\lambda_2), \lambda_{1,2} = \frac{-\widehat{\alpha} \pm \sqrt{(\widehat{\alpha})^2 + 4\varepsilon\widehat{\beta}}}{2\varepsilon}. \tag{35}$$

The target is to find a difference equation which has the same general solution as the differential equations in (16) at the mesh point x_i given by $U_{i,j+1} = a_1 \exp(\lambda_1) + a_2(\lambda_2)$. Using the theory of difference equations in [19], by taking there consecutive points we can get the following set of equations:

$$\begin{aligned} U_{i-1,j+1} - u\left(x_{i-1}, t_{j+1}\right) &= 0, \\ U_{i,j+1} - u\left(x_i, t_{j+1}\right) &= 0, \\ U_{i+1,j+1} - u\left(x_{i+1}, t_{j+1}\right) &= 0, \end{aligned} \tag{36}$$

or,

$$\begin{vmatrix} U_{i-1,j+1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ U_{i,j+1} & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1,j+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0. \tag{37}$$

Simplifying and substituting the values of λ_1 and λ_2, we obtain the exact difference scheme(in the sense that it has the same general solution as the corresponding differential equation) for (33) as follows:

$$\exp\left(-\frac{\widehat{\alpha}h}{2\varepsilon}\right)U_{i-1,j+1} - 2\cosh\left(\frac{h\sqrt{(\widehat{\alpha})^2 + 4\varepsilon\widehat{\beta}}}{2\varepsilon}\right)U_{i,j+1} + \exp\left(\frac{\widehat{\alpha}h}{2\varepsilon}\right)U_{i+1,j+1} = 0. \tag{38}$$

The main goal is finding a suitable denominator function for the second order derivative, the extraction of the denominator function from equation (38) is not straightforward. As a result, assume layer behaviors of the solution of problem (1) and that of the problem in the case when $\beta(x) = 0$ are similar, so that in equation (38) use the approximation $(h\sqrt{(\widehat{\alpha})^2 + 4\varepsilon\widehat{\beta}}/2\varepsilon) \approx (h\widehat{\alpha}/2\varepsilon)$ and following steps in [20], we obtain the following equation:

$$\begin{aligned}
&\varepsilon\frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{(h\varepsilon/-\widehat{\alpha})(\exp(-h\widehat{\alpha}/\varepsilon) - 1)} + \widehat{\alpha}\frac{U_{i,j+1} - U_{i-1,j+1}}{h} = 0, \quad \text{for } i = 1, 2, \ldots, \frac{N}{2}\\
&\varepsilon\frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{(h\varepsilon/\widehat{\alpha})(\exp(h\widehat{\alpha}/\varepsilon) - 1)} + \widehat{\alpha}\frac{U_{i+1,j+1} - U_{i,j+1}}{h} = 0, \quad \text{for } i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N - 1.
\end{aligned} \tag{39}$$

Therefore, the denominator function for the second order derivative approximation becomes

$$\sigma^2(\varepsilon, h) = \begin{cases} \dfrac{h\varepsilon}{-\widehat{\alpha}}\left(\exp\left(\dfrac{-h\widehat{\alpha}}{\varepsilon}\right) - 1\right), \text{for } 0 \le x \le 1,\\[2ex] \dfrac{h\varepsilon}{\widehat{\alpha}}\left(\exp\left(\dfrac{h\widehat{\alpha}}{\varepsilon}\right) - 1\right), \text{for } 1 < x \le 2. \end{cases} \tag{40}$$

This denominator function can be modified to variable coefficient problem as follows:

$$\sigma_i^2(\varepsilon, h) = \begin{cases} \dfrac{h\varepsilon}{-\alpha(x_i)}\left(\exp\left(\dfrac{-h\alpha(x_i)}{\varepsilon}\right) - 1\right), & \text{for } i = 1, 2, \ldots, \dfrac{N}{2},\\[2ex] \dfrac{h\varepsilon}{\alpha(x_i)}\left(\exp\left(\dfrac{h\alpha(x_i)}{\varepsilon}\right) - 1\right), & \text{for } i = \dfrac{N}{2} + 1, \dfrac{N}{2} + 2, \ldots, N. \end{cases} \tag{41}$$

Remark 1. For the discretization of the first derivative of the spatial variable we use backward and forward finite differences depending on the convection coefficient term,

$$D_x^- U_i^{j+1} = \frac{U_i^{j+1} - U_{i-1}^j}{h}, D_x^+ U_i^{j+1} = \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h}. \tag{42}$$

4.4. Fully Discrete Scheme. Using the denominator function σ_i^2 in (41) into the discretized form of the scheme in (16), we obtain the following difference scheme:

$$\begin{cases} \mathscr{L}^{M,N}U_{i,j+1} = Q_{i,j+1}, i = 1, 2, \ldots, N - 1,\\ j = 0, 1, 2, \ldots, M - 1,\\ \text{with the conditions}\\ \begin{cases} U_{i,0} = \psi_0(x_i), \text{ for } i = 0, 1, 2, \ldots, N - 1,\\ U_{i,j+1} = \psi_l(x_i, t_{j+1}), \text{ for } i = -\dfrac{N}{2}, -\left(\dfrac{N}{2} - 1\right), \ldots, 0,\\ U_{N,j+1} = \psi_r(t_{j+1}), \text{ for } j = 0, 1, 2, \ldots, M - 1, \end{cases} \end{cases} \tag{43}$$

where

$$\mathscr{L}_{\varepsilon}^{N,M}U_{i,j+1}=\begin{cases}\varepsilon\dfrac{U_{i-1,j+1}-2U_{i,j+1}+U_{i+1,j+1}}{\sigma_i^2(\varepsilon,h)}+\alpha_1(x_i)\dfrac{U_{i,j+1}-U_{i-1,j+1}}{h}-\beta_1(x_i)U_{i,j+1}, & \text{for } i=1,2,\ldots,\dfrac{N}{2},\ j=0,1,\ldots,M-1,\\[2ex] \varepsilon\dfrac{U_{i-1,j+1}-2U_{i,j+1}+U_{i+1,j+1}}{\sigma_i^2(\varepsilon,h)}+\alpha_2(x_i)\dfrac{U_{i+1,j+1}-U_{i,j+1}}{h}-\beta_1(x_i)U_{i,j+1}-\gamma(x_i)U_{i-N/2,j+1}, & \text{for } i=\dfrac{N}{2}+1,\dfrac{N}{2}+2,\ldots,N-1,\ j=0,1,\ldots,M-1.\end{cases}$$

$$Q_{i,j}=\begin{cases}-\varepsilon\dfrac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{\sigma_i^2(\varepsilon,h)}-\alpha_1(x_i)\dfrac{U_{i,j}-U_{i-1,j}}{h}+\beta_2(x_i)U_{i,j}+f(x_i,t_{j+1})+f(x_i,t_j)+\gamma(x_i)\left(U_{i-N/2,j+1}+U_{i-N/2,j}\right), & \text{for } i=1,2,\ldots,\dfrac{N}{2},\ j=0,1,\ldots,M-1,\\[2ex] -\varepsilon\dfrac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{\sigma_i^2(\varepsilon,h)}-\alpha_2(x_i)\dfrac{U_{i+1,j}-U_{i,j}}{h}+\beta_2(x_i)U_{i,j}+f(x_i,t_{j+1})+f(x_i,t_j)+\gamma(x_i)U_{i-N/2,j}, & \text{for } i=\dfrac{N}{2}+1,\dfrac{N}{2}+2,\ldots,N-1,\ j=0,1,\ldots,M-1.\end{cases}\tag{44}$$

where $\beta_1(x_i)=(\beta(x_i)+2/\Delta t), \beta_2(x_i)=(\beta(x_i)-2/\Delta t)$.

5. Uniform Convergence of the Fully Discrete Scheme

Lemma 7 (Discrete Minimum principle). *Let $\vartheta_{i,j+1}$ be a mesh function such that $\vartheta_{i,j+1}\geq 0, \forall i=0,1,\ldots,N, \mathscr{L}_{\varepsilon}^{N,M}\vartheta_{i,j+1}\leq 0$ for all $i\in\{0,1,\ldots,N\}/\{N/2\}$ and $D_x^{+}\vartheta_{N/2,j+1}-D_x^{-}\vartheta_{N/2,j+1}=0$. Then, $\vartheta_{i,j+1}\geq 0$ for all $i=0,1,\ldots,N$.*

Proof. Define a test function

$$s(x_i,t_{j+1})=\begin{cases}\dfrac{3}{2}+\dfrac{x_i}{2}, & (x_i,t_{j+1})\in\Omega_1^{N,M},\\[2ex] 3-x_i, & (x_i,t_{j+1})\in\Omega_2^{N,M},\end{cases}\tag{45}$$

where $\Omega_1^{N,M}=([0,1]\cap\overline{\Omega}_x^N)\times\Omega_t^M, \Omega_2^{N,M}=([1,2]\cap\overline{\Omega}_x^N)\times\Omega_t^M$. Note that, $s(x_i,t_{j+1})>0, \forall(x_i,t_{j+1})\in\overline{\Omega}^{N,M}$, $\mathscr{L}_{\varepsilon}s(x_i,t_{j+1})>0, \forall(x_i,t_{j+1})\in(\Omega\cap\overline{\Omega}_x^N)\times\Omega_t^M$ and $[s_x](N/2,t_{j+1})<0$. Let

$$\mu=\max\left\{\frac{-\vartheta(x_i,t_{j+1})}{s(x_i,t_{j+1})}: (x_i,t_{j+1})\in\overline{\Omega}^{N,M}\right\}.\tag{46}$$

Then, there exists $(x_w,t_{j+1})\in\overline{\Omega}^{N,M}$ such that $\vartheta(x_w,t_{j+1})+\mu s(x_w,t_{j+1})=0$ and $\vartheta(x_i,t_{j+1})+\mu s(x_i,t_{j+1})\geq 0, \forall(x_i,t_{j+1})\in\overline{\Omega}^{N,M}$. Therefore, the function attains a minimum at (x_w,t_{j+1}). Suppose the theorem does not hold true, then $\mu>0$.

Case 1: $w\in 1,2,\ldots N/2-1$

$$\begin{aligned}0>\mathscr{L}_{\varepsilon}^{N,M}(\vartheta+\mu s)_{w,j+1}&=\varepsilon\frac{(\vartheta+\mu s)_{w-1,j+1}-2(\vartheta+\mu s)_{w,j+1}+(\vartheta+\mu s)_{w+1,j+1}}{\sigma_w^2(\varepsilon,h)}\\&\quad+\alpha_1(x_w)\frac{(\vartheta+\mu s)_{w,j+1}-(\vartheta+\mu s)_{w-1,j+1}}{h}-\beta_1(x_w)(\vartheta+\mu s)_{w,j+1}\\&=\varepsilon\frac{\left((\vartheta+\mu s)_{w-1,j+1}-(\vartheta+\mu s)_{w,j+1}\right)}{\sigma_w^2(\varepsilon,h)}+\frac{\left((\vartheta+\mu s)_{w+1,j+1}-(\vartheta+\mu s)_{w,j+1}\right)}{\sigma_w^2(\varepsilon,h)}\\&\quad+\alpha_1(x_w)\frac{(\vartheta+\mu s)_{w,j+1}-(\vartheta+\mu s)_{w-1,j+1}}{h}-\beta_1(x_w)(\vartheta+\mu s)_{w,j+1}>0, \left(\text{since } \alpha_1(x_w)<0\right).\end{aligned}\tag{47}$$

Case 2: $(x_w,t_{j+1})=(x_{N/2},t_{j+1})$

$$0\leq[(\vartheta+\mu s)_x]\left(x_{N/2},t_{j+1}\right)=[\vartheta_x]\left(x_{N/2},t_{j+1}\right)+\mu[s_x]\left(x_{N/2},t_{j+1}\right)<0.\tag{48}$$

Case 3: $w \in N/2+1, N/2+2, \ldots, N-1$

$$
\begin{aligned}
0 > \mathscr{L}_{\varepsilon}^{N,M}(\vartheta+\mu s)_{w,j+1} &= \varepsilon \frac{(\vartheta+\mu s)_{w-1,j+1} - 2(\vartheta+\mu s)_{w,j+1} + (\vartheta+\mu s)_{w+1,j+1}}{\sigma_w^2(\varepsilon,h)} \\
&\quad + \alpha_2(x_w)\frac{(\vartheta+\mu s)_{w,j+1} - (\vartheta+\mu s)_{w-1,j+1}}{h} - \beta_1(x_w)(\vartheta+\mu s)_{w,j+1} - \gamma(x_w)(\vartheta+\mu s)_{w-N/2,j+1} \\
&= \varepsilon \frac{\left((\vartheta+\mu s)_{w-1,j+1} - (\vartheta+\mu s)_{w,j+1}\right)}{\sigma_w^2(\varepsilon,h)} + \frac{\left((\vartheta+\mu s)_{w+1,j+1} - (\vartheta+\mu s)_{w,j+1}\right)}{\sigma_w^2} \\
&\quad + \alpha_2(x_w)\frac{(\vartheta+\mu s)_{w+1,j+1} - (\vartheta+\mu s)_{w,j+1}}{h} - (\beta_1(x_w)+\gamma(x_w))(\vartheta+\mu s)_{w,j+1} \\
&\quad - \gamma(x_w)\left((\vartheta+\mu s)_{w-N/2,j+1} - (\vartheta+\mu s)_{w,j+1}\right) > 0, \\
&\quad (\text{since } \alpha_2(x_w) > 0,\ (\beta_1(x_w)+\gamma(x_w)) > 0 \text{ and } \gamma(x_w) < 0),
\end{aligned}
\tag{49}
$$

%

in all cases, the result contradicts with our assumption. Hence, the required result follows.

An immediate consequence of the discrete minimum principle is the following uniform stability property of the operator $\mathscr{L}_{\varepsilon}^{N,M}$. □

Lemma 8. *The solution $U_{i,j+1}$ of the discrete scheme in (25) satisfies the bound*

$$
\left\|U_{i,j+1}\right\| \le C \max\left\{\left\|\psi_l\left(0,t_{j+1}\right)\right\|, \left\|\psi_0\left(0,t_{j+1}\right)\right\|, \left\|\mathscr{L}_{\varepsilon}^{N,M}U_{i,j+1}\right\|, \left\|\psi_r\left(2,t_{j+1}\right)\right\|\right\}, \quad \forall i = 0,1,\ldots N-1, \quad j = 1,2,\ldots,M-1.
\tag{50}
$$

Proof. Define barrier function

$$
\Phi_{i,j+1}^{\pm} = C \max\left\{\left\|\psi_l\left(0,t_{j+1}\right)\right\|, \left\|\mathscr{L}_{\varepsilon}^{N,M}U_{i,j+1}\right\| s\left(x_i,t_{j+1}\right), \left\|\psi_r\left(1,t_{j+1}\right)\right\|\right\} \pm \vartheta_{i,j+1},
\tag{51}
$$

then following the procedure applied for the proof of Lemma 7, we obtain $\Phi_{0,j+1}^{\pm} \ge 0, \Phi_{N,j+1}^{\pm} \ge 0$ and $\mathscr{L}_{\varepsilon}^{N,M}\Phi_{i,j+1}^{\pm} \ge 0$. The required result follows from Lemma 7. □

5.1. Error Estimates

Theorem 2. *Let $\alpha(x)$ and $Q(x,t_{j+1})$ be sufficiently smooth functions so that $U_{j+1}(x) \in C^4[0,2]$. Then, the truncation error of the discrete scheme satisfies the bound.*

$$
\mathscr{L}_{\varepsilon}^{N,M}\left(U_{j+1}(x_i) - U_{i,j+1}\right) \le
\begin{cases}
Ch\left(1 + \max\limits_{x_i \in (0,1)\cap\overline{\Omega}_x^N} \dfrac{\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^3}\right), \\
Ch\left(1 + \max\limits_{x_i \in (1,2)\cap\overline{\Omega}_x^N} \dfrac{\exp(-\eta(x_i-1)/\varepsilon)}{\varepsilon^3} + \max\limits_{x_i \in (1,2)\cap\overline{\Omega}_x^N} \dfrac{\exp(-\eta(2-x_i)/\varepsilon)}{\varepsilon^5}\right).
\end{cases}
\tag{52}
$$

Proof. The truncation error in the spatial discretization is

$$\begin{aligned}
\left|\mathscr{L}_\varepsilon^{N,M}\left(U_{j+1}(x_i)-U_{i,j+1}\right)\right| &= \left|\mathscr{L}_\varepsilon^{N,M}U_{j+1}(x_i)-\mathscr{L}_\varepsilon^{N,M}U_{i,j+1}\right| \\
&\le C\left|\varepsilon\left(\frac{d^2}{dx^2}-\frac{D^+D^-h^2}{\sigma_i^2(\varepsilon,h)}\right)U_{j+1}(x_i)+\alpha_i\left(\frac{d}{dx}-D_x^-\right)U_{j+1}(x_i)\right| \\
&\le C\varepsilon\left|\left(\frac{d^2}{dx^2}-D_x^+D_x^-\right)U_{j+1}(x_i)\right|+C\varepsilon\left|\left(\frac{h^2}{\sigma_i^2(\varepsilon,h)}-1\right)D_x^+D_x^-U_{j+1}(x_i)\right|+Ch\left|\frac{d^2}{dx^2}U_{j+1}(x_i)\right| \\
&\le C\varepsilon h^2\left|\frac{d^4}{dx^4}U_{j+1}(x_i)\right|+Ch\left|\frac{d^2}{dx^2}U_{j+1}(x_i)\right|,
\end{aligned} \tag{53}$$

depending on the behavior of $\sigma_i^2(\varepsilon,h)$, the estimate $\varepsilon|h^2/\sigma(\varepsilon,h)-1|\le Ch$. Then apply Lemma 6 in to the truncation error bound in (53).

Case 1: When $i\in 1,2,\ldots N/2-1$

$$\begin{aligned}
\left|\mathscr{L}_\varepsilon^{N,M}\left(U_{j+1}(x_i)-U_{i,j+1}\right)\right| &\le C\varepsilon h^2\left|1+\varepsilon^{-4}\exp\left(\frac{-\eta(1-x_i)}{\varepsilon}\right)\right|+Ch\left|1+\varepsilon^{-2}\exp\left(\frac{-\eta(1-x_i)}{\varepsilon}\right)\right| \\
&\le Ch^2\left|\varepsilon+\varepsilon^{-3}\exp\left(\frac{-\eta(1-x_i)}{\varepsilon}\right)\right|+Ch^2\left|1+\varepsilon^{-2}\exp\left(\frac{-\eta(1-x_i)}{\varepsilon}\right)\right| \\
&\le Ch\left|1+\max\frac{\exp(-\eta(1-x_i)/\varepsilon)}{\varepsilon^3}\right|.
\end{aligned} \tag{54}$$

Case 2: When $i\in N/2+1,N/2+2,\ldots,N-1$, applying the same procedure like case 1, we will get the following equation:

$$\left|\mathscr{L}_\varepsilon^{N,M}\left(U_{j+1}(x_i)-U_{i,j+1}\right)\right|\le Ch\left|1+\max_{x_i\in\Omega_2^N}\frac{\exp(-\eta(x_i-1)/\varepsilon)}{\varepsilon^3}+\max_{x_i\in\Omega_2^N}\frac{\exp(-\eta(2-x_i)/\varepsilon)}{\varepsilon^5}\right|. \tag{55}$$

□

Lemma 9. *For a fixed mesh, as* $\varepsilon\longrightarrow 0$, *it holds*

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\max_{1\le i\le N-1}\frac{\exp(-\eta x_i/\varepsilon)}{\varepsilon^p}=0, \\
&\lim_{\varepsilon\longrightarrow 0}\max_{1\le i\le N-1}\frac{\exp(-\eta(2-x_i)/\varepsilon)}{\varepsilon^p}=0,\quad \forall p\in Z^+,
\end{aligned} \tag{56}$$

where $x_i=ih,h=2/N,\forall i=1,2,\ldots,N-1$.

Proof. For the proof see [21]. □

Theorem 3. *Let* u *and* $U_{i,j+1}$ *are solutions of problem (1) and problem (25), respectively. Then,*

$$\sup_{0<\varepsilon\ll 1}\left\|u\left(x_i,t_{j+1}\right)-U_{i,j+1}\right\|\le C\left(N^{-1}+\Delta t^2\right). \tag{57}$$

Proof. Using Lemma 9 in Theorem 1 and also apply discrete minimum principle Lemma 8, we obtain the following equation:

$$\sup_{0<\varepsilon\ll 1}\left\|u\left(x_i,t_j\right)-U_{i,j+1}\right\|\le\sup_{0<\varepsilon\ll 1}\left(\left\|u\left(x_i,t_{j+1}\right)-U_{j+1}(x_i)\right\|+\left\|U_{j+1}(x_i)-U_{i,j+1}\right\|\right)\le C\left(N^{-1}+\Delta t^2\right). \tag{58}$$

□

6. Richardson Extrapolation Technique

To increase the accuracy of the numerical solution of the proposed scheme, we used the Richardson extrapolation technique in a spatial variable. Let $U_{i,j+1}^{2N}$ be the solution of the discrete problem (25) on the mesh $\Omega^{2N,M}$. From Theorem 2, we have the following equation:

$$U_{j+1}(x_i) - U_{i,j+1} \leq CN^{-1} + O\left(N^{-2}\right) \leq CN^{-1} + R_N, \quad (59)$$

where R_N is the remainder term in the spatial direction. Then, (59) also holds for any $h/2 \neq 0$, which is

$$U_{j+1}(x_i) - U_{i,j+1}^{2N} \leq \left(CN^{-1}/2\right) + R_{2N^{-1}}. \quad (60)$$

Next, to eliminate the term $O(N^{-1})$ subtract twice of (60) from (59), we get the following equation:

$$U_{j+1}(x_i) - U_{i,j+1} - 2\left(U_{j+1}(x_i) - U_{i,j+1}^{2N}\right) \leq R_N - 2R_{2N}, \quad (61)$$

after simplifying, we obtain the following equation:

$$U_{j+1}(x_i) - \left(2U_{i,j+1}^{2N} - U_{i,j+1}\right) \leq O\left(N^{-2}\right). \quad (62)$$

Therefore, we used the following extrapolation formula:

$$\overline{U}_{i,j+1} = 2U_{i,j+1}^{2N} - U_{i,j+1}. \quad (63)$$

The approximate solution $\overline{U}_{i,j+1}$ is more accurate than either $U_{i,j+1}^{N}$ or $U_{i,j+1}^{2N}$. The truncation error of the spatial discretization in the approximation of (63) becomes

$$\max_{\forall x_i \in (0,2)} \left|U_{j+1}(x_i) - \overline{U}_{i,j+1}\right| \leq CN^{-2}. \quad (64)$$

For more detail see [22].

Theorem 4. *Let $u(x_i, t_{j+1})$ and $\overline{U}_{i,j+1}$ be the solution of problems in (1) and (63) respectively, then the proposed scheme satisfies the following error estimate*

$$\sup_{0<\varepsilon\ll 1} \max_{0\leq x_i, t_{j+1} \leq 2} \left|u\left(x_i, t_{j+1}\right) - \overline{U}_{i,j+1}\right| \leq C\left(N^{-2} + (\Delta t)^2\right). \quad (65)$$

Proof. Combining the error in the temporal and spatial discretization gives the required bound. □

7. Numerical Illustration

In this section, we test the performance of the proposed scheme through numerical experiments. The exact solution of the problem is not known, so to compute the maximum point-wise errors, we use the double mesh principle given by the formula,

(i) Before extrapolation

$$E_\varepsilon^{N,M} = \max_{0\leq i,j\leq N,M} \left|U^{N,M}\left(x_i, t_j\right) - U^{2N,2M}\left(x_{2i}, t_{2j}\right)\right|. \quad (66)$$

(ii) After extrapolation

$$E_\varepsilon^{N,M} = \max_{0\leq i,j\leq N,M} \left|\overline{U}^{N,M}\left(x_i, t_j\right) - \overline{U}^{2N,2M}\left(x_{2i}, t_{2j}\right)\right|, \quad (67)$$

where $U^{N,M}(x_i, t_j)$ and $\overline{U}(x_i, t_j)$ denote the numerical solutions obtained by N mesh intervals in the spatial direction and M mesh intervals in the time direction, such that $M = T/\Delta t$. The corresponding rate of convergence by the following formula is

$$\mathrm{ROC}_\varepsilon^{N,M} = \log 2\left(\frac{E_\varepsilon^{N,M}}{E_\varepsilon^{2N,2M}}\right). \quad (68)$$

Also, the ε–uniform maximum point-wise error $E^{N,M}$ is computed as follows:

$$E^{N,M} = \max_{\varepsilon} E_\varepsilon^{N,M}, \quad (69)$$

and the corresponding ε–uniform rate of convergence $\mathrm{ROC}^{N,M}$ is given by the following equation:

$$ROC^{N,M} = \ln\left(\frac{E^{N,M}}{E^{2N,2M}}\right). \quad (70)$$

Example 1. Consider the following singularly perturbed convection diffusion problem [17]:

$$\begin{gathered}\varepsilon\frac{\partial^2 u}{\partial x^2} + \alpha(x)\frac{\partial u}{\partial x} - 5u(x,t) + 2u(x-1,t) - \frac{\partial u}{\partial t} = f(x,t),\ (x,t) \in (0,2)\times(0,2],\\ u(x,0) = 0, x \in [0,2], u(x,t) = 0,\ (x,t) \in [-1,0]\times[0,2], u(2,0) = 0, t \in [0,2],\end{gathered} \quad (71)$$

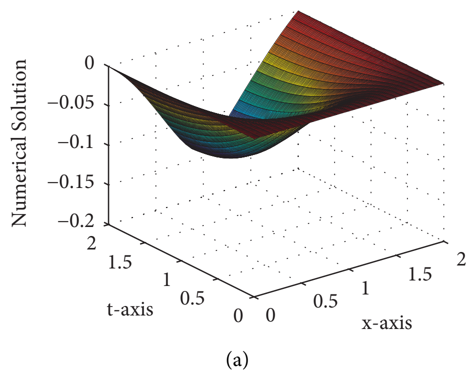


FIGURE 1: Surface plot of numerical solution of Example 1 at $N = 32$, $M = 128$. (a) $\varepsilon = 10^0$, (b) $\varepsilon = 10^{-4}$.

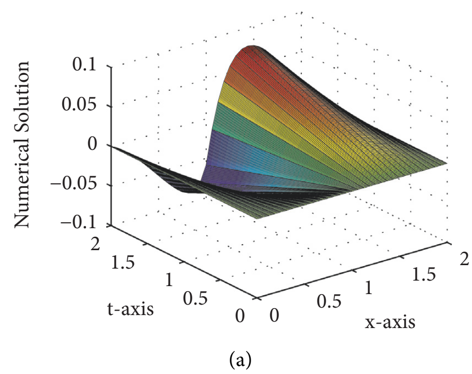


FIGURE 2: Surface plot of numerical solution of Example 2 at $N = 32$, $M = 128$. (a) $\varepsilon = 10^0$, (b) $\varepsilon = 10^{-4}$.

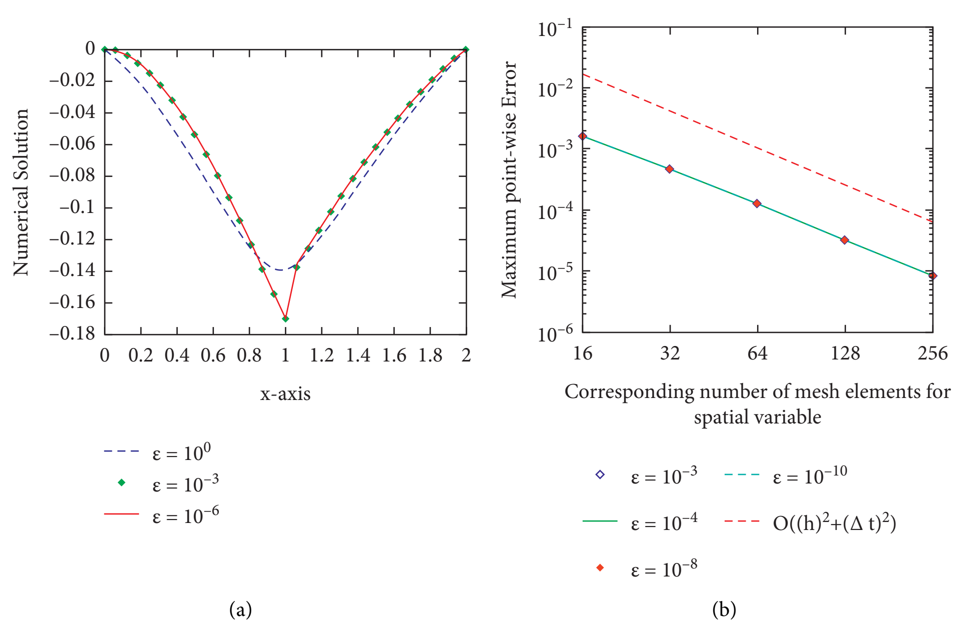
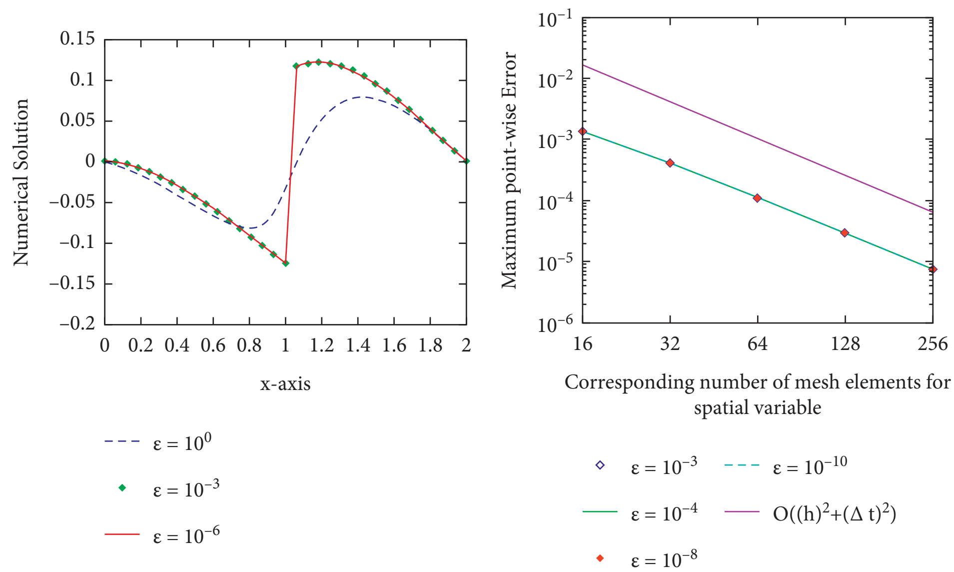


FIGURE 3: One dimensional plot and log-log plot for Example 1 (a) Numerical solution at $t = 2$, when $N = 32$, (b) Log-log plot.



(a) (b)

FIGURE 4: One dimensional plot and log-log plot for Example 2 (a) Numerical solution at $t=2$, when $N=32$, (b) Log-log plot.

TABLE 1: Computed maximum point-wise errors, ε-uniform errors $(E^{N,M})$, and the ε–uniform rate of convergence $\mathrm{ROC}^{N,M}$ for Example 1 when $M=N$, before extrapolation.

$\varepsilon\downarrow$	$N \longrightarrow 32$	64	128	256	512
10^{-3}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-4}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-5}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-6}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-7}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-8}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-9}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
10^{-10}	$2.3872e-03$	$1.1497e-03$	$5.6604e-04$	$2.8110e-04$	$1.4002e-04$
$E^{N,M}$	**2.3872e − 03**	**1.1497e − 03**	**5.6604e − 04**	**2.8110e − 04**	**1.4002e − 04**
$\mathrm{ROC}^{N,M}$	**1.0541**	**1.0223**	**1.0098**	**1.0055**	

where

$$\alpha(x) = \begin{cases} -\left(4+x^2\right), & x \in [0,1], \\ \left(8-x^2\right), & x \in (1,2], \end{cases}$$
$$f(x,t) = \begin{cases} 4xt^2e^{-t}, & (x,t) \in [0,1]\times[0,2], \\ 4(2-x)t^2e^{-t}, & (x,t) \in (1,2]\times[0,2]. \end{cases} \quad (72)$$

Example 2. Consider the following singularly perturbed convection diffusion problem:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \alpha(x)\frac{\partial u}{\partial x} - (x+3)u(x,t) + u(x-1,t) - \frac{\partial u}{\partial t}\varsigma$$
$$u(x,0) = 0, \quad x \in [0,2],$$
$$u(x,t) = 0, \quad (x,t) \in [-1,0]\times[0,2], \quad (73)$$
$$u(2,0) = 0, \quad t \in [0,2],$$

where

Table 2: Computed maximum point-wise errors, ε-uniform errors $(E^{N,M})$, and the ε–uniform rate of convergence $(\mathrm{ROC}^{N,M})$ for Example 1, after extrapolation.

$\varepsilon\downarrow$	$N/M \longrightarrow$ 16/64	32/128	64/256	128/512	256/1024
10^{-3}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1246e-06$
10^{-4}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-5}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-6}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-7}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-8}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-9}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
10^{-10}	$1.6250e-03$	$4.6468e-04$	$1.2399e-04$	$3.2008e-05$	$8.1304e-06$
$\mathbf{E^{N,M}}$	**1.6250e − 03**	**4.6468e − 04**	**1.2399e − 04**	**3.2008e − 05**	**8.1304e − 06**
$\mathbf{ROC^{N,M}}$	**1.8062**	**1.9060**	**1.9537**	**1.9770**	

Table 3: Computed maximum point-wise errors, ε-uniform errors $(E^{N,M})$ and the ε–uniform rate of convergence $(\mathrm{ROC}^{N,M})$ for Example 2 when $M = N$, before extrapolation.

$\varepsilon\downarrow$	$N \longrightarrow$ 32	64	128	256	512
10^{-3}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5616e-04$
10^{-4}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-5}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-6}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-7}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-8}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-9}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
10^{-10}	$3.6703e-03$	$1.9480e-03$	$1.0030e-03$	$5.0885e-04$	$2.5627e-04$
$\mathbf{E^{N,M}}$	**3.6703e − 03**	**1.9480e − 03**	**1.0030e − 03**	**5.0885e − 04**	**2.5627e − 04**
$\mathbf{ROC^{N,M}}$	**0.9139**	**0.9577**	**0.9790**	**0.9896**	

Table 4: Computed maximum point-wise errors, ε-uniform errors $(E^{N,M})$, and the ε–uniform rate of convergence $(\mathrm{ROC}^{N,M})$ for Example 2, after extrapolation.

$\varepsilon\downarrow$	$N/M \longrightarrow$ 16/64	32/128	64/256	128/512	256/1024
10^{-3}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4504e-06$
10^{-4}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-5}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-6}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-7}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-8}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-9}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
10^{-10}	$1.3438e-03$	$4.0562e-04$	$1.1130e-04$	$2.9142e-05$	$7.4556e-06$
$\mathbf{E^{N,M}}$	**1.3438e − 03**	**4.0562e − 03**	**1.1130e − 03**	**2.9142e − 04**	**7.4556e − 04**
$\mathbf{ROC^{N,M}}$	**1.7281**	**1.8658**	**1.9332**	**1.9677**	

$$\alpha(x) = \begin{cases} e^{x}(x-4), & x \in [0,1], \\ e^{x} + x, & x \in (1,2], \end{cases}$$
$$f(x,t) = \begin{cases} xt, & (x,t) \in [0,1] \times [0,2], \\ \cos\left(\dfrac{\pi x}{2}\right)t, & (x,t) \in (1,2] \times [0,2]. \end{cases} \tag{74}$$

8. Conclusion

Singularly perturbed time dependent convection-diffusion problem with discontinuous convection coefficient and source term is treated via nonstandard finite difference method for the spatial derivative and Crank–Nicolson method for the time derivative. In addition to this, to enhance the order of convergence of the spatial variable, the Richardson extrapolation technique is used. Due to the presence of large delay and discontinuity in coefficient and source term, the problem exhibits strong interior layers at $x = 1$ and weak boundary layer at $x = 2$. Figures 1(a), 1(b) and 2–4 of the surface plot for the numerical solution of the problem in Examples 1 and 2, respectively, demonstrate the existence of a strong boundary layer at $x = 1$. Figures 3(a) and 3(b) The maximum point-wise error of Examples 1 and 2 is plotted in Figures 3(b) and 4(b), respectively, in the log-log plot. From these figures, it is clear to observe that the maximum error is ε–uniform convergent of second order in both space and time variable. The error analysis of the proposed scheme is carried out and the scheme is ε-uniform convergent of order $O(h^2 + (\Delta t)^2)$. This result clearly shown in Tables 1–4 of the analogous computed errors and uniform convergence rates for the given examples, before and after the extrapolation technique. The results confirm that the theoretical error estimates are in agreement with the numerical results.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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