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Research Article

Novel Analysis of Fractional-Order Fifth-Order Korteweg–de Vries Equations

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In this paper, the ρ-homotopy perturbation transformation method was applied to analysis of fifth-order nonlinear fractional Korteweg–de Vries (KdV) equations. This technique is the mixture form of the ρ-Laplace transformation with the homotopy perturbation method. The purpose of this study is to demonstrate the validity and efficiency of this method. Furthermore, it is demonstrated that the fractional and integer-order solutions close in on the exact result. The suggested technique was effectively utilized and was accurate and simple to use for a number of related engineering and science models.

1. Introduction

A number of researchers have recently become interested in fractional calculus, which was first developed during Newton's period. Within the fractional calculus structure, many interesting and significant steps have been discovered within the last thirty decades. A fractional derivative was invented as a result of the complexity of a heterogeneous phenomenon. The fractional derivative operators, by incorporating diffusion methods, are capable of capturing the attitudes of multidimensional media [1–4]. The use of differential equations of any scale has proved useful in showing a number of problems more quickly and accurately. Increasingly, scholars turned to generalized calculus to convey their viewpoints while analyzing complex phenomena in the context of mathematical methods using software [5–10].

Nonlinear impacts occur in several implemented scientific fields, such as fluid, mathematical biology, nonlinear image sensors, quantum field theory, kinetics, thermodynamics, and fluid dynamics. It is based on nonlinear partial differential equations of various degrees of complexity to model these processes. Partial differential equations are generally applied in the description of physical processes [11–15]. Most of the essential physical systems do not exhibit linear behavior. There is no way to determine the exact result of such nonlinear phenomena. Only techniques that are appropriate for solving nonlinear equations can be used to investigate this phenomenon [16–22].

In 1895, Korteweg and de Vries proposed a KdV equation to design Russell's soliton phenomenon, such as small and huge water waves. Solitons are steady solitary waves, which mean that these solitary waves are a particle. KdV equations are applied in different applied fields such as quantam mechanics, fluid dynamics, optics, and plasma physics. Fifth-order KdV form equations were utilized to analyze many nonlinear phenomena in particle physics [23–25]. It plays a vital role in the distribution of waves [26]. In their analysis, the KdV form equation has dispersive terms of the third and fifth-order relevant to the magnetoacoustic wave problem in cold plasma free collision plasma and dispersive terms appear near-critical angle propagation [27]. Plasma is a dynamic, quasineutral, and electrically conductive fluid. It consists of neutral particles, electrons, and ions. It consists of magnetic and electric areas due to the electrically conducting behavior of plasma. The mixture of particles and areas supports plasma waves of various forms. A magnetic lock is a less longitudinal ion dispersion. The magnetoacoustic wave behaves as an ion-acoustic wave in the low magnetic field range, while in the low-temperature capacity, it acts as an Alfven wave [28, 29].

The general model for the analysis of magnetic properties-acoustic waves in plasma and shallow water waves with surface tension is equated with the fifth order of KdV. Recent study reveals that the solutions to this equation for travelling waves do not vanish at infinity [30, 31]. Consider the well-known three types of the fifth-order KdV equations as follows [32, 33]:

$$D_\tau^\gamma \mathscr{V} + \mathscr{V}_\zeta + \mathscr{V}^2\mathscr{V}_{2\zeta} + \mathscr{V}_\zeta\mathscr{V}_{2\zeta} - 20\mathscr{V}^2\mathscr{V}_{3\zeta} + \mathscr{V}_{5\zeta} = 0, \quad 0<\beta\leq 1, \tag{1}$$

with initial condition $\mathscr{V}(\zeta,0) = 1/\zeta$,

$$D_\tau^\beta \mathscr{V} + \mathscr{V}\mathscr{V}_\zeta - \mathscr{V}\mathscr{V}_{3\zeta} + \mathscr{V}_{5\zeta} = 0, \quad 0<\beta\leq 1, \tag{2}$$

with initial condition $\mathscr{V}(\zeta,0) = e^\zeta$, and

$$D_\tau^\beta \mathscr{V} + \mathscr{V}\mathscr{V}_\zeta + \mathscr{V}_{3\zeta} - \mathscr{V}_{5\zeta} = 0, \quad 0<\beta\leq 1, \tag{3}$$

with initial condition $\mathscr{V}(\zeta,0) = 105/169\mathrm{sech}^4(\zeta - \phi/2\sqrt{13})$.

(1) and (2) are called fifth-order KdV equations and (3) is called the Kawahara equation. Analytic techniques for these mathematical model are particularly difficult to come across due to their severe nonlinearity. Several researchers have employed various analytical and computational strategies to the solution of linear and nonlinear KdV equations throughout the last decade, such as the multisymplectic method [34], variational iteration method [33], He's homotopy perturbation method [35], and Exp-function method [36].

Recently, Fahd and Abdeljawad [37] developed the Laplace transform of the generalized fractional Caputo derivatives. We established a novel methodology with ρ-Laplace transform for solving fractional differential equations with a generalized fractional Caputo derivative. The homotopy perturbation method is merged with the Laplace transform method to create a highly effective method for handling nonlinear terms which is known as the homotopy perturbation transformation technique. This technique can provide the result in quick convergent series. Ghorbani pioneered the use of He's polynomials in nonlinear terms [38–40]. Later on, many scholars utilized the homotopy perturbation transformation method for linear and nonlinear differential equations such as heat-like equations [41], Navier–Stokes equations [42], hyperbolic equation and Fisher's equation [43], and gas dynamic equation [44].

2. Basic Definitions

2.1. Definition. The fractional generalized integral of order β of a continuous function (CF) g: $[0,+\infty] \longrightarrow R$ is defined as [37]

$$\left(I^{\beta,\rho} g\right)(\zeta) = \frac{1}{\Gamma(\beta)} \int_0^\zeta \left(\frac{\zeta^\rho - s^\rho}{\rho}\right)^{\beta-1} \frac{g(s)ds}{s^{1-\rho}}, \quad \rho>0,\ \zeta>0,\ 0<\beta<1. \tag{4}$$

2.2. Definition. The order β fractional generalized derivative of a CF g: $[0,+\infty] \longrightarrow R$ is given as [37]

$$\begin{aligned}\left(D^{\beta,\rho} g\right)(\zeta) &= \left(I^{1-\beta,\rho} g\right)(\zeta) \\ &= \frac{1}{\Gamma(1-\beta)} \left(\frac{d}{d\zeta}\right) \int_0^\zeta \left(\frac{\zeta^\rho - s^\rho}{\rho}\right)^{-\beta-1} \frac{g(s)ds}{s^{1-\rho}}, \\ &\rho>0,\ \zeta>0 \text{ and } 0<\beta<1.\end{aligned} \tag{5}$$

2.3. Definition. The Caputo derivative of fractional-order β of a CF g: $[0,+\infty] \longrightarrow R$ is defined as [37]

$$\left(D^{\beta,\rho} g\right)(\zeta) = \frac{1}{\Gamma(1-\beta)} \left(\frac{d}{d\zeta}\right) \int_0^\zeta \left(\frac{\zeta^\rho - s^\rho}{\rho}\right)^{-\beta-1} \beta^n \frac{g(s)\mathrm{ds}}{s^{1-\rho}}, \tag{6}$$

where $\rho>0$, $\zeta>0$, $\beta = \zeta^{1-\beta} d/d\zeta$, and $0<\beta<1$.

2.4. Definition. The ρ-Laplace transform of a CF g: $[0,+\infty] \longrightarrow R$ is defined as [37]

$$L_\rho\{g(\zeta)\}(s) = \int_0^\infty e^{-s\zeta^\rho/\rho} g(\zeta) \frac{d\zeta}{\zeta^{1-\rho}}. \tag{7}$$

The fractional generalized Caputo derivative of ρ-Laplace transformation of a CF g is given by [37]

$$L_\rho\left\{D^{\beta,\rho} g(\zeta)\right\}(s) = s^\beta L_\rho\{g(\zeta)\} - \sum_{k=0}^{n-1} s^{\beta-k-1}\left(I^{\beta,\rho}\beta^n g\right)(0). \tag{8}$$

2.5. Definition. The generalized Mittag-Leffler function is defined by

$$E_{\beta,\rho}(z) = \sum_{k=0}^\infty \frac{z^\beta}{\Gamma(\beta k + \gamma)}, \tag{9}$$

where $\beta>0$, $\gamma>0$, and $E_\beta(z) = E_{\beta,1}(z)$.

3. The Rod Map of the Proposed Method

Consider the general partial differential equation given as

$$D_\tau^\gamma \mathcal{V}(\zeta,\tau) + M\mathcal{V}(\zeta,\tau) + N\mathcal{V}(\zeta,\tau) = h(\zeta,\tau), \quad \tau > 0,\ 0 < \gamma \leq 1,$$
$$\mathcal{V}(\zeta,0) = g(\zeta), \quad \nu \in \mathfrak{R}. \tag{10}$$

Applying ρ-Laplace transformation of (10), we get

$$L_\rho[D_\tau^\gamma \mathcal{V}(\zeta,\tau) + M\mathcal{V}(\zeta,\tau) + N\mathcal{V}(\zeta,\tau)] = L_\rho[h(\zeta,\tau)], \quad \tau > 0,\ 0 < \gamma \leq 1,$$
$$\mu(\zeta,\tau) = \frac{1}{s} g(\zeta) + \frac{1}{s^\beta} L_\rho[h(\zeta,\tau)] - \frac{1}{s^\beta} L_\rho[M\mathcal{V}(\zeta,\tau) + N\mathcal{V}(\zeta,\tau)]. \tag{11}$$

Now, applying the inverse ρ-Laplace transform, we get

$$\mathcal{V}(\zeta,\tau) = F(\zeta,\tau) - L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\{M\mathcal{V}(\zeta,\tau) + N\mathcal{V}(\zeta,\tau)\}\right], \tag{12}$$

where

$$F(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s} g(\zeta) + \frac{1}{s^\beta} L_\rho[h(\zeta,\tau)]\right]$$
$$= g(\nu) + L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho[h(\zeta,\tau)]\right]. \tag{13}$$

Now, the perturbation procedure in terms of power series with parameter p is presented as

$$\mathcal{V}(\zeta,\tau) = \sum_{\kappa=0}^{\infty} p^\kappa \mathcal{V}_\kappa(\zeta,\tau), \tag{14}$$

where p is the perturbation parameter and $p \in [0,1]$.

The nonlinear term can be defined as

$$N\mathcal{V}(\zeta,\tau) = \sum_{\kappa=0}^{\infty} p^\kappa H_\kappa(\mathcal{V}_\kappa), \tag{15}$$

where H_n are He's polynomials in terms of $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ and can be calculated as

$$H_n(\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n) = \frac{1}{\gamma(n+1)} D_p^\kappa \left[N\left(\sum_{\kappa=0}^{\infty} p^\kappa \mathcal{V}_\kappa\right)\right]_{p=0}, \tag{16}$$

where $D_p^\kappa = \partial^\kappa / \partial p^\kappa$.

Substituting (15) and (16) in (12), we get

$$\sum_{\kappa=0}^{\infty} p^\kappa \mathcal{V}_\kappa(\zeta,\tau) = F(\zeta,\tau) - p \times \left[L_\rho^{-1}\left\{\frac{1}{s^\beta} L_\rho\left\{M \sum_{\kappa=0}^{\infty} p^\kappa \mathcal{V}_\kappa(\zeta,\tau) + \sum_{\kappa=0}^{\infty} p^\kappa H_\kappa(\mathcal{V}_\kappa)\right\}\right\}\right]. \tag{17}$$

The coefficients comparison on both sides of p, we have

$$p^0\colon \mathcal{V}_0(\zeta,\tau) = F(\zeta,\tau),$$
$$p^1\colon \mathcal{V}_1(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho(M\mathcal{V}_0(\zeta,\tau) + H_0(\mathcal{V}))\right],$$
$$p^2\colon \mathcal{V}_2(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho(M\mathcal{V}_1(\zeta,\tau) + H_1(\mathcal{V}))\right],$$
$$\vdots$$
$$p^\kappa\colon \mathcal{V}_\kappa(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho(M\mathcal{V}_{\kappa-1}(\zeta,\tau) + H_{\kappa-1}(\mathcal{V}))\right],$$
$$\kappa > 0,\ \kappa \in N. \tag{18}$$

The $\mathcal{V}\kappa(\zeta,\tau)$ component can be determined easily which quickly leads us to the convergent series. We can get $p \longrightarrow 1$:

$$\mathcal{V}(\zeta,\tau) = \lim_{M\longrightarrow\infty} \sum_{\kappa=1}^{M} \mathcal{V}_\kappa(\zeta,\tau). \tag{19}$$

4. Numerical Implementations

Example 1. Consider the fifth-order nonlinear KdV equation

$$D_\tau^\beta \mathcal{V} + \mathcal{V}_\zeta + \mathcal{V}^2 \mathcal{V}_{2\zeta} - \mathcal{V}_\zeta \mathcal{V}_{2\zeta}$$
$$- 20\mathcal{V}^2 \mathcal{V}_{3\zeta} + \mathcal{V}_{5\zeta} = 0, \quad 0 < \beta \leq 1, \tag{20}$$

with the IC

$$\mathcal{V}(\zeta,\tau) = \frac{1}{\zeta}. \tag{21}$$

Applying the ρ-Laplace transform on (20), we get

$$L_\rho \mathscr{V}(\zeta,\tau)] = \frac{1}{s\zeta} - \frac{1}{s^\beta} L_\rho \mathscr{V}_\zeta + \mathscr{V}^2\mathscr{V}_{2\zeta} + \mathscr{V}_\zeta\mathscr{V}_{2\zeta} - 20\mathscr{V}^2\mathscr{V}_{3\zeta} + \mathscr{V}_{5\zeta}]. \tag{22}$$

Next, using the inverse of ρ-Laplace transform of (22),

$$[\mathscr{V}(\zeta,\tau)] = \frac{1}{\zeta} - L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[\mathscr{V}_\zeta + \mathscr{V}^2\mathscr{V}_{2\zeta} + \mathscr{V}_\zeta\mathscr{V}_{2\zeta} - 20\mathscr{V}^2\mathscr{V}_{3\zeta} + \mathscr{V}_{5\zeta}\right]\right]. \tag{23}$$

Now, we apply HPM

$$\sum_{n=0}^{\infty} p^n \mathscr{V}_n(\zeta,\tau) = \frac{1}{\zeta} - p\left[L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[\left(\sum_{n=0}^{\infty} p^n H_n(\mathscr{V})\right) + \left(\sum_{n=0}^{\infty} p^n \mathscr{V}_n(\zeta,\tau)\right)_\zeta + \left(\sum_{n=0}^{\infty} p^n \mathscr{V}_n(\zeta,\tau)\right)_{5\zeta}\right]\right]\right], \tag{24}$$

where $H_n(x)$ represents the nonlinear function of He's polynomial. For the first few components, we present He's polynomials

$$\begin{aligned}
H_0(\mathscr{V}) &= \mathscr{V}_0^2(\mathscr{V}_0)_{2\zeta} + (\mathscr{V}_0)_\zeta(\mathscr{V}_0)_{2\zeta} - 20\mathscr{V}_0^2(\mathscr{V}_0)_{3\zeta},\\
H_1(\mathscr{V}) &= \mathscr{V}_0^2(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_o)_{2\zeta} + (\mathscr{V}_o)_\zeta(\mathscr{V}_1)_{2\zeta} + (\mathscr{V}_o)_{2\zeta}(\mathscr{V}_1)_\zeta - 20\mathscr{V}_0^2(\mathscr{V}_1)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_o)_{3\zeta},\\
H_2(\mathscr{V}) &= \mathscr{V}_0^2(\mathscr{V}_2)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_o)_{2\zeta} + \mathscr{V}_1^2(\mathscr{V}_o)_{2\zeta} + (\mathscr{V}_o)_\zeta(\mathscr{V}_2)_{2\zeta} + (\mathscr{V}_1)_\zeta(\mathscr{V}_1)_{2\zeta}\\
&\quad + (\mathscr{V}_o)_{2\zeta}(\mathscr{V}_2)_\zeta - 20\mathscr{V}_0^2(\mathscr{V}_2)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_1)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_o)_{3\zeta} - 20\mathscr{V}_1^2(\mathscr{V}_o)_{3\zeta},\\
H_3(\mathscr{V}) &= \mathscr{V}_0^2(\mathscr{V}_3)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_2)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_3(\mathscr{V}_o)_{2\zeta} + \mathscr{V}_1^2(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_1\mathscr{V}_2(\mathscr{V}_o)_{2\zeta}\\
&\quad + (\mathscr{V}_o)_\zeta(\mathscr{V}_3)_{2\zeta} + (\mathscr{V}_1)_\zeta(\mathscr{V}_2)_{2\zeta} + (\mathscr{V}_1)_{2\zeta}(\mathscr{V}_2)_\zeta + (\mathscr{V}_0)_{2\zeta}(\mathscr{V}_3)_\zeta - 20\mathscr{V}_0^2(\mathscr{V}_3)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_2)_{3\zeta}\\
&\quad - 40\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_1)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_3(\mathscr{V}_o)_{3\zeta} - 20\mathscr{V}_1^2(\mathscr{V}_1)_{3\zeta} - 40\mathscr{V}_1\mathscr{V}_2(\mathscr{V}_o)_{3\zeta},\\
H_4(\mathscr{V}) &= \mathscr{V}_0^2(\mathscr{V})_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_3)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_2)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_3(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_0\mathscr{V}_4(\mathscr{V}_o)_{2\zeta} + \mathscr{V}_1^2(\mathscr{V}_2)_{2\zeta}\\
&\quad + 2\mathscr{V}_1\mathscr{V}_2(\mathscr{V}_1)_{2\zeta} + 2\mathscr{V}_1\mathscr{V}_3(\mathscr{V}_o)_{2\zeta} + \mathscr{V}_2^2(\mathscr{V}_o)_{2\zeta} + (\mathscr{V}_o)_\zeta(\mathscr{V}_4)_{2\zeta} + (\mathscr{V}_1)_\zeta(\mathscr{V}_3)_{2\zeta} + (\mathscr{V}_2)_{2\zeta}(\mathscr{V}_2)_\zeta\\
&\quad + (\mathscr{V}_1)_{2\zeta}(\mathscr{V}_3)_\zeta + (\mathscr{V}_o)_{2\zeta}(\mathscr{V}_4)_\zeta - 20\mathscr{V}_0^2(\mathscr{V}_4)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_1(\mathscr{V}_3)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_2(\mathscr{V}_2)_{3\zeta} - 40\mathscr{V}_0\mathscr{V}_3(\mathscr{V}_1)_{3\zeta}\\
&\quad - 40\mathscr{V}_0\mathscr{V}_4(\mathscr{V}_o)_{3\zeta} - 20\mathscr{V}_1^2(\mathscr{V}_2)_{3\zeta} - 40\mathscr{V}_1\mathscr{V}_2(\mathscr{V}_1)_{3\zeta} - 40\mathscr{V}_1\mathscr{V}_3(\mathscr{V}_o)_{3\zeta} - 20\mathscr{V}_2^2(\mathscr{V}_o)_{2\zeta},\\
&\ \vdots
\end{aligned} \tag{25}$$

Comparing the P-like coefficients, we have

$$\begin{aligned}
P^0&: \mathscr{V}_0(\zeta,\tau) = \frac{1}{\zeta},\\
P^1&: \mathscr{V}_1(\zeta,\tau) = -L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[H_0(\mathscr{V}) + (\mathscr{V}_0)_\zeta + (\mathscr{V}_0)_{5\zeta}\right]\right] = \frac{(\tau^\rho/\rho)^\beta}{\zeta^2\Gamma(\beta+1)},\\
P^2&: \mathscr{V}_2(\zeta,\tau) = -L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[H_1(\mathscr{V}) + (\mathscr{V}_1)_\zeta + (\mathscr{V}_1)_{5\zeta}\right]\right] = \frac{(\tau^\rho/\rho)^{2\beta}}{\zeta^3\Gamma(2\beta+1)},\\
P^3&: \mathscr{V}_3(\zeta,\tau) = -L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[H_2(\mathscr{V}) + (\mathscr{V}_2)_\zeta + (\mathscr{V}_2)_{5\zeta}\right]\right] = \frac{(\tau^\rho/\rho)^{3\beta}}{\zeta^4\Gamma(3\beta+1)},\\
P^4&: \mathscr{V}_4(\zeta,\tau) = -L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[H_3(\mathscr{V}) + (\mathscr{V}_3)_\zeta + (\mathscr{V}_3)_{5\zeta}\right]\right] = \frac{(\tau^\rho/\rho)^{4\beta}}{\zeta^5\Gamma(4\beta+1)},\\
P^5&: \mathscr{V}_5(\zeta,\tau) = -L_\rho^{-1}\left[\frac{1}{s^\beta} L_\rho\left[H_4(\mathscr{V}) + (\mathscr{V}_4)_\zeta + (\mathscr{V}_4)_{5\zeta}\right]\right] = \frac{(\tau^\rho/\rho)^{5\beta}}{\zeta^6\Gamma(5\beta+1)}.\\
&\ \vdots
\end{aligned} \tag{26}$$

The analytical solution of $\mathscr{V}(\zeta,\tau)$ is defined as

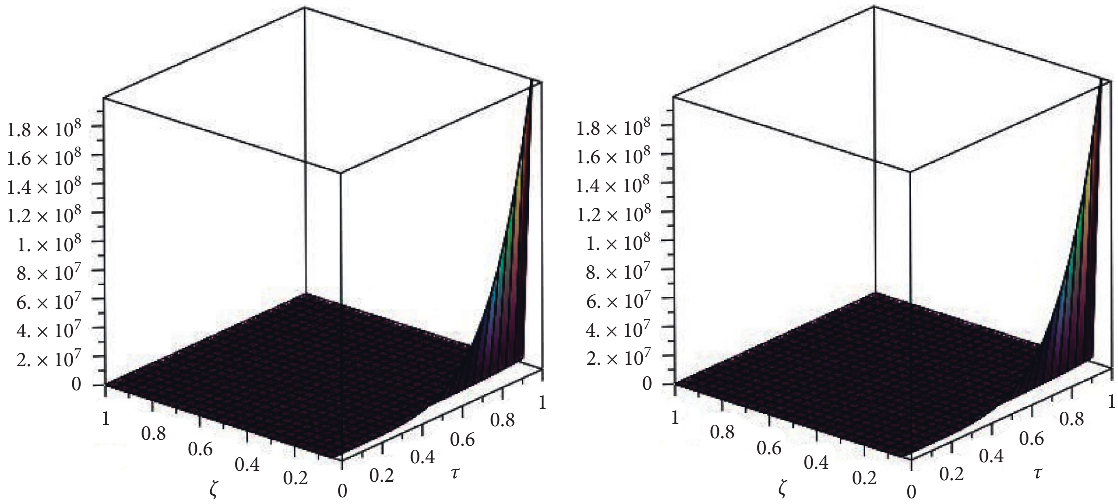
$$\begin{aligned}
\mathscr{V}(\zeta,\tau) &= \sum_{i=0}^{\infty} \mathscr{V}(\zeta,\tau)_i = \frac{1}{\zeta} + \frac{(\tau^\rho/\rho)^\beta}{\zeta^2\Gamma(\beta+1)} + \frac{(\tau^\rho/\rho)^{2\beta}}{\zeta^3\Gamma(2\beta+1)}\\
&\quad + \frac{(\tau^\rho/\rho)^{3\beta}}{\zeta^4\Gamma(3\beta+1)} + \frac{(\tau^\rho/\rho)^{4\beta}}{\zeta^5\Gamma(4\beta+1)} + \frac{(\tau^\rho/\rho)^{5\beta}}{\zeta^6\Gamma(5\beta+1)} + \cdots.
\end{aligned} \tag{27}$$

Then, put $\beta = 1$ in (27):

$$\mathscr{V}(\zeta,\tau) = \sum_{i=0}^{\infty} \mathscr{V}_i(\zeta,\tau) = \frac{1}{\zeta} + \frac{\tau}{\zeta^2} + \frac{\tau^2}{\zeta^3} + \frac{\tau^3}{\zeta^4} + \cdots. \tag{28}$$

The exact result is $\mathscr{V}(\zeta,\tau) = 1/\zeta - \tau$.

In Figure 1, the three-dimensional figures of ρ-HPTM and exact results in graphs (a) and (b) respectively at $\beta = 1$ and the close contact of the exact and ρ-HPTM solutions are investigated. In Figure 2, represent that various fractional



(a) (b)

Figure 1: Graph of (a) exact and (b) analytic solutions of $\beta = 1$ of Example 1.

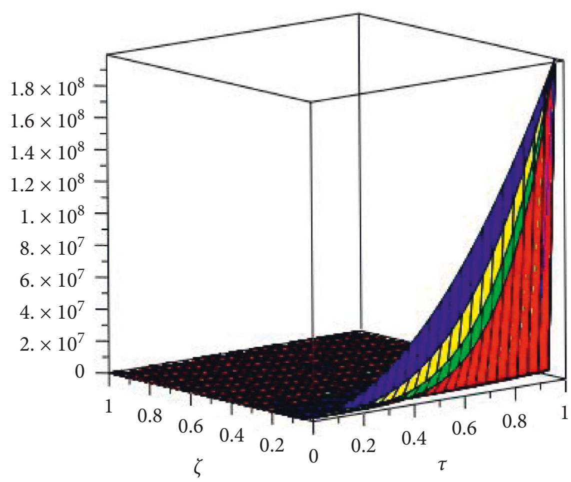


Figure 2: Figure of various fractional orders of Example 1.

order of ρ-HPTM results at $\beta = 1, 0.8, 0.6, 0.4$. The non-classical results are investigated to be converge to an integer-order result of the given problem.

Example 2. Consider the fifth-order nonlinear fraction KdV equation

$$D_\tau^{\beta}\mathscr{V} + \mathscr{V}\mathscr{V}_{\zeta} - \mathscr{V}\mathscr{V}_{3\zeta} + \mathscr{V}_{5\zeta} = 0, \quad 0 < \beta \le 1, \tag{29}$$

with the IC

$$\mathscr{V}(\zeta,\tau) = e^{\zeta}. \tag{30}$$

Applying the ρ-Laplace transform on (29), we get

$$L_\rho[\mathscr{V}(\zeta,\tau)] = \frac{1}{s}e^{\zeta} + \frac{1}{s^{\beta}}L_\rho\left[\mathscr{V}\mathscr{V}_{3\zeta} - \mathscr{V}\mathscr{V}_{\zeta} - \mathscr{V}_{5\zeta}\right]. \tag{31}$$

Next, using the inverse of ρ-Laplace transform of (31),

$$\mathscr{V}(\zeta,\tau) = e^{\zeta} + L_\rho^{-1}\left[\frac{1}{s^{\beta}}L_\rho\left\{\mathscr{V}\mathscr{V}_{3\zeta} - \mathscr{V}\mathscr{V}_{\zeta} - \mathscr{V}_{5\zeta}\right\}\right]. \tag{32}$$

Now, we apply HPM

$$\sum_{n=0}^{\infty} p^n \mathscr{V}_n(\zeta,\tau) = e^{\zeta} + p\left[L_\rho^{-1}\left\{\frac{1}{s^{\beta}}L_\rho\left(\left(\sum_{n=0}^{\infty} p^n H_n(\mathscr{V})\right) - \left(\sum_{n=0}^{\infty} p^n \mathscr{V}_n(\zeta,\tau)\right)\right)_{5\zeta}\right\}\right], \tag{33}$$

where $H_n(x)$ represents the nonlinear term of He's polynomial. For the first few components, we present He's polynomials

$$
\begin{aligned}
H_0(\mathscr{V}) &= \mathscr{V}_o(\mathscr{V}_0)_{3\zeta} - \mathscr{V}_0(\mathscr{V}_0)_\zeta,\\
H_1(\mathscr{V}) &= \mathscr{V}_1(\mathscr{V}_0)_{3\zeta} + \mathscr{V}_0(\mathscr{V}_1)_{3\zeta} - \mathscr{V}_1(\mathscr{V}_0)_\zeta - \mathscr{V}_0(\mathscr{V}_1)_\zeta,\\
H_2(\mathscr{V}) &= \mathscr{V}_2(\mathscr{V}_0)_{3\zeta} + \mathscr{V}_1(\mathscr{V}_1)_{3\zeta} + \mathscr{V}_0(\mathscr{V}_2)_{3\zeta}\\
&\quad - \mathscr{V}_2(\mathscr{V}_0)_\zeta - \mathscr{V}_1(\mathscr{V}_1)_\zeta - \mathscr{V}_0(\mathscr{V}_2)_\zeta,\\
H_3(\mathscr{V}) &= \mathscr{V}_3(\mathscr{V}_0)_{3\zeta} + \mathscr{V}_2(\mathscr{V}_1)_{3\zeta} + \mathscr{V}_1(\mathscr{V}_2)_{3\zeta}\\
&\quad + \mathscr{V}_0(\mathscr{V}_3)_{3\zeta} - \mathscr{V}_3(\mathscr{V}_0)_\zeta - \mathscr{V}_2(\mathscr{V}_1)_\zeta\\
&\quad - \mathscr{V}_1(\mathscr{V}_2)_\zeta - \mathscr{V}_0(\mathscr{V}_3)_\zeta,\\
H_4(\mathscr{V}) &= \mathscr{V}_4(\mathscr{V}_0)_{3\zeta} + \mathscr{V}_3(\mathscr{V}_1)_{3\zeta} + \mathscr{V}_2(\mathscr{V}_2)_{3\zeta}\\
&\quad + \mathscr{V}_1(\mathscr{V}_3)_{3\zeta} + \mathscr{V}_0(\mathscr{V}_4)_{3\zeta} - \mathscr{V}_4(\mathscr{V}_0)_\zeta\\
&\quad - \mathscr{V}_3(\mathscr{V}_1)_\zeta - \mathscr{V}_2(\mathscr{V}_2)_\zeta - \mathscr{V}_1(\mathscr{V}_3)_\zeta - \mathscr{V}_0(\mathscr{V}_4)_\zeta,\\
&\vdots
\end{aligned}
\tag{34}
$$

Comparing the P-like coefficients, we have

$$
\begin{aligned}
p^0&: \mathscr{V}_0(\zeta,\tau) = e^\zeta,\\
p^1&: \mathscr{V}_1(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{H_0(\mathscr{V}) - (\mathscr{V}_0)_{5\zeta}\right\}\right] = -\frac{(\tau^\rho/\rho)^\beta}{\Gamma(\beta+1)}e^\zeta,\\
p^2&: \mathscr{V}_2(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{H_1(\mathscr{V}) - (\mathscr{V}_1)_{5\zeta}\right\}\right] = \frac{(\tau^\rho/\rho)^{2\beta}}{\Gamma(2\beta+1)}e^\zeta,\\
p^3&: \mathscr{V}_3(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{H_2(\mathscr{V}) - (\mathscr{V}_2)_{5\zeta}\right\}\right] = -\frac{(\tau^\rho/\rho)^{3\beta}}{\Gamma(3\beta+1)}e^\zeta,\\
p^4&: \mathscr{V}_4(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{H_3(\mathscr{V}) - (\mathscr{V}_3)_{5\zeta}\right\}\right] = \frac{(\tau^\rho/\rho)^{4\beta}}{\Gamma(4\beta+1)}e^\zeta,\\
p^5&: \mathscr{V}_5(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{H_4(\mathscr{V}) - (\mathscr{V}_4)_{5\zeta}\right\}\right] = -\frac{(\tau^\rho/\rho)^{5\beta}}{\Gamma(5\beta+1)}e^\zeta.\\
&\vdots
\end{aligned}
\tag{35}
$$

Therefore, the analytic solution of $\mathscr{V}(\zeta,\tau)$ is defined as

$$
\begin{aligned}
\mathscr{V}(\zeta,\tau) = \sum_{i=0}^{\infty}\mathscr{V}_i(\zeta,\tau) = e^\zeta\Bigg(1 - \frac{(\tau^\rho/\rho)^\beta}{\Gamma(\beta+1)} + \frac{(\tau^\rho/\rho)^{2\beta}}{\Gamma(2\beta+1)}\\
- \frac{(\tau^\rho/\rho)^{3\beta}}{\Gamma(3\beta+1)} + \frac{(\tau^\rho/\rho)^{4\beta}}{\Gamma(4\beta+1)} - \frac{(\tau^\rho/\rho)^{5\beta}}{\Gamma(5\beta+1)} + \cdots\Bigg).
\end{aligned}
\tag{36}
$$

Then, $\beta = 1$ for (36), and we get

$$
\mathscr{V}(\zeta,\tau) = \sum_{i=0}^{\infty}\mathscr{V}_i(\zeta,\tau) = e^\zeta\left(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} - \frac{\tau^5}{5!} + \cdots\right).
\tag{37}
$$

The exact solution is $\mathscr{V}(\zeta,\tau) = e^{\zeta-\tau}$.

In Figure 3, the three-dimensional figures of ρ-HPTM and exact results in graphs (a) and (b) respectively at $\beta = 1$ and the close contact of the exact and ρ-HPTM solutions are investigated. In Figure 4, represent that various fractional order of ρ-HPTM results at $\beta = 1, 0.8, 0.6, 0.4$. The non-classical results are investigated to be converge to an integer-order result of the given problem.

Example 3. Consider nonlinear fractional-order Kawahara equation

$$
D_\tau^\beta\mathscr{V} + \mathscr{V}\mathscr{V}_\zeta + \mathscr{V}_{3\zeta} - \mathscr{V}_{5\zeta} = 0, \quad 0 < \beta \le 1,
\tag{38}
$$

with the IC

$$
\mathscr{V}(\zeta,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right).
\tag{39}
$$

Applying the ρ-Laplace transform on (38), we get

$$
L_\rho\mathscr{V}(\zeta,\tau)] = \frac{1}{s}\frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right) + \frac{1}{s^\beta}L_\rho\left[\mathscr{V}_{5\zeta} - \mathscr{V}_{3\zeta} - \mathscr{V}\mathscr{V}_\zeta\right].
\tag{40}
$$

Next, using the inverse of ρ-Laplace transform of (40),

$$
\mathscr{V}(\zeta,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right) + L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left[\mathscr{V}_{5\zeta} - \mathscr{V}_{3\zeta} - \mathscr{V}\mathscr{V}_\zeta\right]\right].
\tag{41}
$$

Now, we apply HPM

$$
\begin{aligned}
\sum_{n=0}^{\infty}p^n\mathscr{V}_n(\zeta,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right) + p\Bigg[L_\rho^{-1}\Bigg\{\frac{1}{s^\beta}L_\rho\Bigg(\left(\sum_{n=0}^{\infty}p^n\mathscr{V}_n(\zeta,\tau)\right)_{5\zeta}\\
-\left(\sum_{n=0}^{\infty}p^n\mathscr{V}_n(\zeta,\tau)\right)_{3\zeta} - \left(\sum_{n=0}^{\infty}p^nH_n(\mathscr{V})\right)\Bigg)\Bigg\}\Bigg],
\end{aligned}
\tag{42}
$$

where $H_n(\mathscr{V})$ represent the nonlinear terms of He's polynomial. For the first few components, we present He's polynomials

$$
\begin{aligned}
H_0(\mathscr{V}) &= \mathscr{V}_0(\mathscr{V}_0)_\zeta,\\
H_1(\mathscr{V}) &= \mathscr{V}_0(\mathscr{V}_1)_\zeta + \mathscr{V}_1(\mathscr{V}_0)_\zeta,\\
H_2(\mathscr{V}) &= \mathscr{V}_0(\mathscr{V}_2)_\zeta + \mathscr{V}_1(\mathscr{V}_1)_\zeta + \mathscr{V}_2(\mathscr{V}_0)_\zeta,\\
H_3(\mathscr{V}) &= \mathscr{V}_0(\mathscr{V}_3)_\zeta + \mathscr{V}_1(\mathscr{V}_2)_\zeta + \mathscr{V}_2(\mathscr{V}_1)_\zeta + \mathscr{V}_3(\mathscr{V}_0)_\zeta,\\
H_4(\mathscr{V}) &= \mathscr{V}_0(\mathscr{V}_4)_\zeta + \mathscr{V}_1(\mathscr{V}_3)_\zeta + \mathscr{V}_2(\mathscr{V}_2)_\zeta\\
&\quad + \mathscr{V}_3(\mathscr{V}_1)_\zeta + \mathscr{V}_4(\mathscr{V}_0)_\zeta,\\
&\vdots
\end{aligned}
\tag{43}
$$

Comparing the P-like coefficients, we get

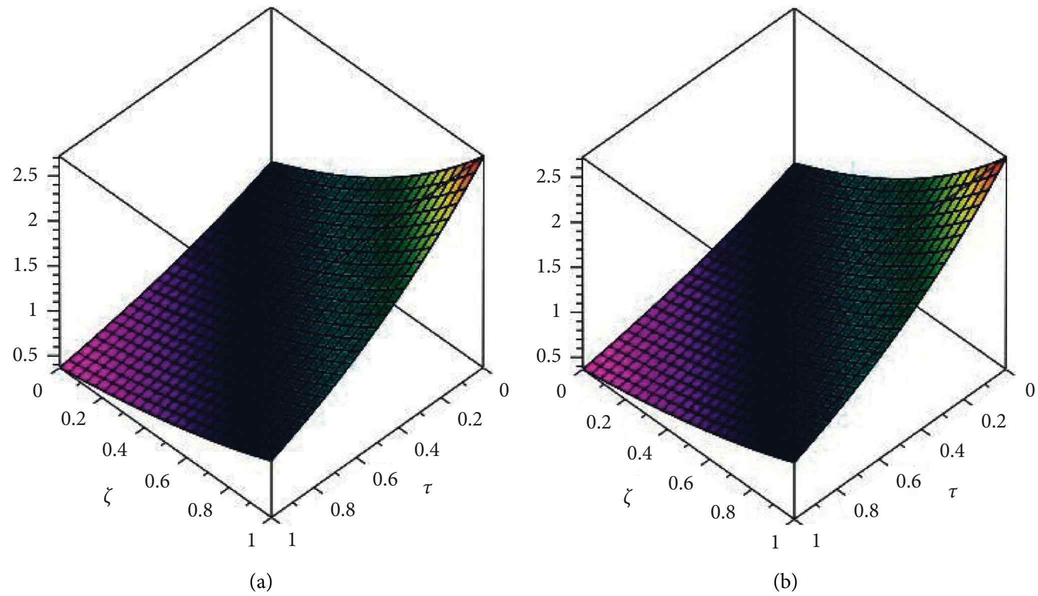
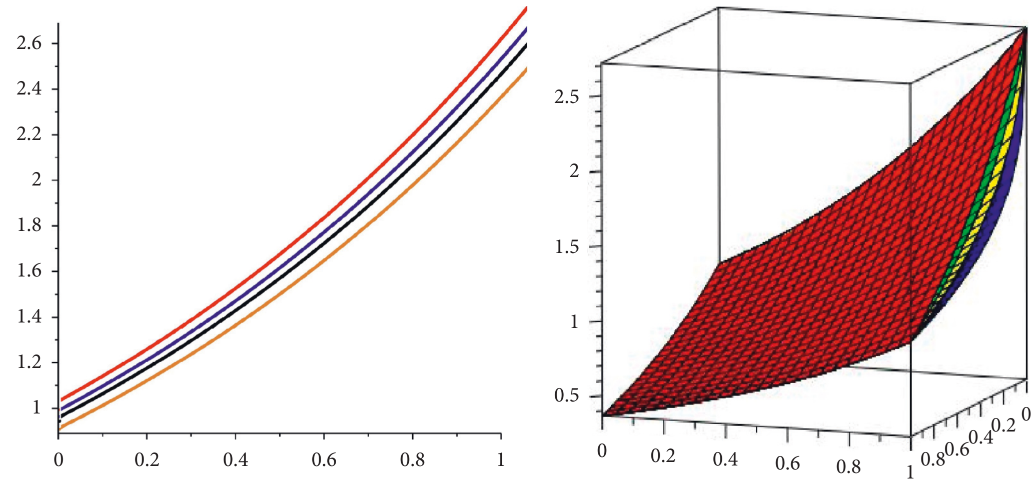


Figure 3: Graph of (a) exact and (b) analytic solutions of $\beta = 1$ of Example 2.

$$\begin{aligned}
p^0&: \mathscr{V}_0(\zeta,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right),\\
p^1&: \mathscr{V}_1(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left[(\mathscr{V}_0)_{5\zeta}-(\mathscr{V}_0)_{5\zeta}-H_0(\mathscr{V})\right]\right] = -\frac{100}{377\sqrt{13}}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\frac{(\tau^\rho/\rho)^\beta}{\Gamma(\beta+1)},\\
p^2&: \mathscr{V}_2(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left[(\mathscr{V}_1)_{5\zeta}-(\mathscr{V}_1)_{3\zeta}-H_1(\mathscr{V})\right]\right]\\
&= -\frac{21687}{10\times10^7\sqrt{13}}\operatorname{sech}^6\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\left[-3+2\cosh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\right]\frac{(\tau^\rho/\rho)^{2\beta}}{\Gamma(2\beta+1)},\\
p^3&: \mathscr{V}_3(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left\{(\mathscr{V}_2)_{5\zeta}-(\mathscr{V}_2)_{3\zeta}-H_2(\mathscr{V})\right\}\right]\\
&= -\frac{461962}{10\times10^7\sqrt{13}}\operatorname{sech}^7\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[-13\sinh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)+2\sinh\left(\frac{3(\zeta-t\phi)}{2\sqrt{`13}}\right)\right]\frac{(\tau^\rho/\rho)^{3\beta}}{\Gamma(3\beta+1)},\\
p^4&: \mathscr{V}_4(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left[(\mathscr{V}_3)_{5\zeta}-(\mathscr{V}_3)_{3\zeta}-H_3(\mathscr{V})\right]\right]\\
&= -\frac{3784854}{10\times10^7\sqrt{13}}\operatorname{sech}^8\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[-49s\cosh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)+4\cosh\left(\frac{2(\zeta-t\phi)}{2\sqrt{`13}}\right)+52\right]\frac{(\tau^\rho/\rho)^{4\beta}}{\Gamma(4\beta+1)},\\
p^5&: \mathscr{V}_5(\zeta,\tau) = L_\rho^{-1}\left[\frac{1}{s^\beta}L_\rho\left[(\mathscr{V}_4)_{5\zeta}-(\mathscr{V}_4)_{3\zeta}-H_4(\mathscr{V})\right]\right]\\
&= -\frac{3.22496310\times10^7}{\sqrt{13}}\operatorname{sech}^9\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[171\sinh\left(\frac{3(\zeta-\phi)}{2\sqrt{13}}\right)-8\sinh\left(\frac{5(\zeta-\phi)}{2\sqrt{`13}}\right)\right.\\
&\left.-661\sinh\left(\frac{5(\zeta-t\phi)}{2sqrt13}\right)\right]\frac{(\tau^\rho/\rho)^{5\beta}}{\Gamma(5\beta+1)}.\\
&\vdots
\end{aligned} \tag{44}$$



(a) (b)

Figure 4: Figure of (a) and (b) at various fractional-order of Example 2.

The analytic solution $\mathscr{V}(\zeta,\tau)$ is achieved as

$$\mathscr{V}(\zeta,\tau)=\sum_{i=0}^{\infty}\mathscr{V}_i(\zeta,\tau),$$

$$\begin{aligned}\mathscr{V}(\zeta,\tau)=&\frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)-\frac{100}{377\sqrt{13}}\operatorname{sech}^4\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\frac{(\tau^{\rho}/\rho)^{\beta}}{\Gamma(\beta+1)}\\&-\frac{21687}{10\times10^7\sqrt{13}}\operatorname{sech}^6\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\left[-3+2\cosh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\right]\frac{(\tau^{\rho}/\rho)^{2\beta}}{\Gamma(2\beta+1)}\\&-\frac{461962}{10\times10^7\sqrt{13}}\operatorname{sech}^7\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[-13\sinh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)+2\sinh\left(\frac{3(\zeta-\phi)}{2\sqrt{'13}}\right)\right]\frac{(\tau^{\rho}/\rho)^{3\beta}}{\Gamma(3\beta+1)}\\&-\frac{3784854}{10\times10^7\sqrt{13}}\operatorname{sech}^8\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[-49s\cosh\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)+4\cosh\left(\frac{2(\zeta-\phi)}{2\sqrt{'13}}\right)+52\right]\frac{(\tau^{\rho}/\rho)^{4\beta}}{\Gamma(4\beta+1)}\\&-\frac{3.22496310\times10^7}{\sqrt{13}}\operatorname{sech}^9\left(\frac{\zeta-\phi}{2\sqrt{13}}\right)\times\left[171\sinh\left(\frac{3(\zeta-\phi)}{2\sqrt{13}}\right)-8\sinh\left(\frac{5(\zeta-\phi)}{2\sqrt{'13}}\right)\right.\\&\left.-661\sinh\left(\frac{5(\zeta-\phi)}{2\sqrt{13}}\right)\right]\frac{(\tau^{\rho}/\rho)^{5\beta}}{\Gamma(5\beta+1)}+\cdots.\end{aligned}\tag{45}$$

The exact solution is $\mathscr{V}(\zeta,\tau)=105/169\operatorname{sech}^4[1/2\sqrt{13}\,(\zeta+36\tau/169-\phi)]$.

In Figure 5, the three-dimensional figures of ρ-HPTM and exact results in graphs (a) and (b) respectively at $\beta=1$

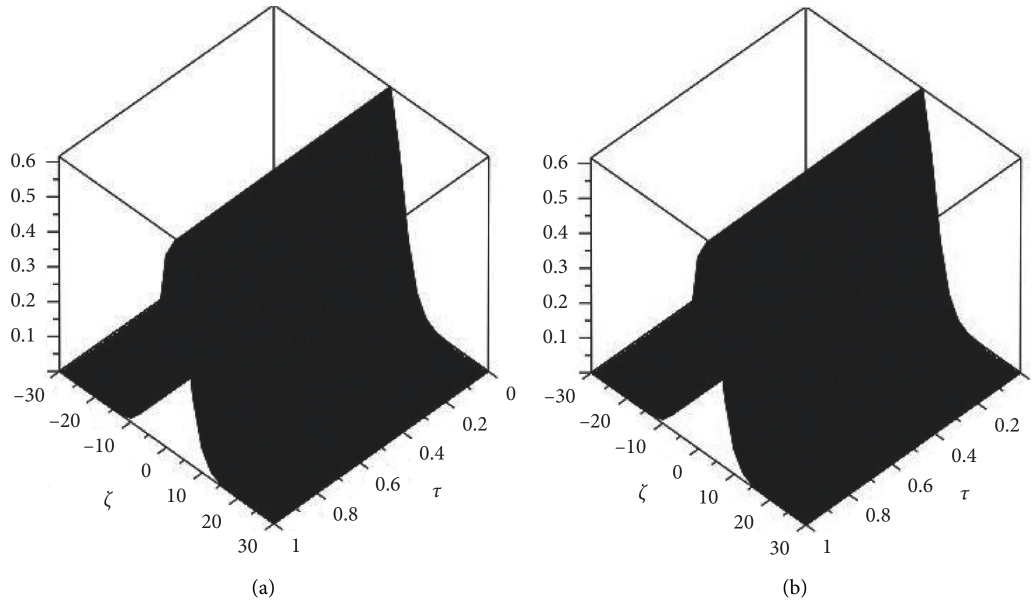


Figure 5: Graph of (a) exact and (b) analytical results of $\beta = 1$ of Example 3.

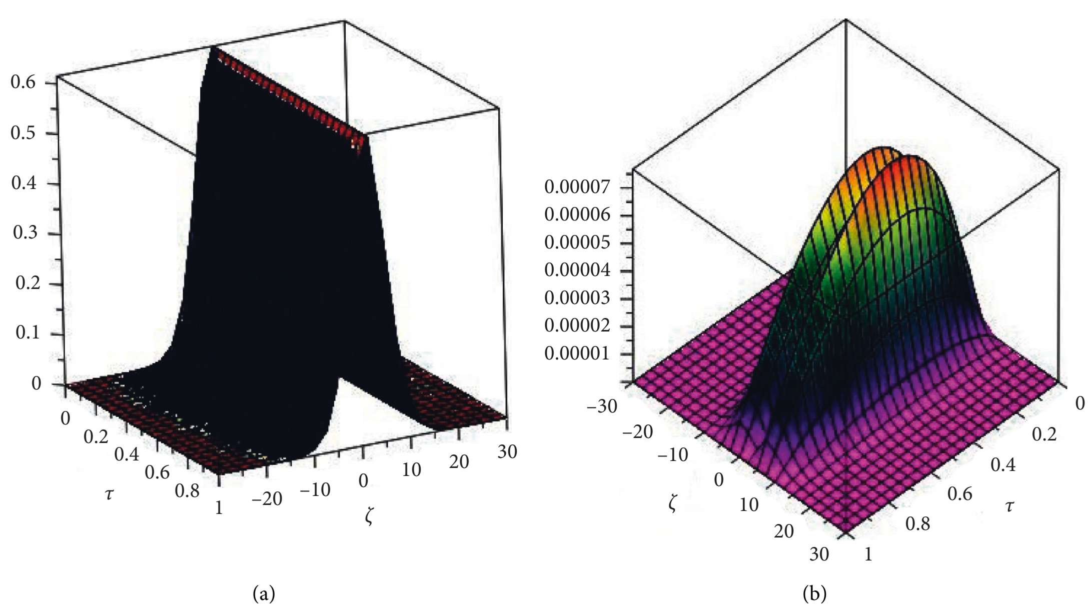


Figure 6: Figure of (a) at various fractional-order of β and (b) error graph of Example 3.

and the close contact of the exact and ρ-HPTM solutions are investigated. In Figure 6, represent that various fractional order of ρ-HPTM results at $\beta = 1, 0.8, 0.6, 0.4$. The nonclassical results are investigated to be converge to an integer-order result of the given problem.

5. Conclusions

This paper determined the fractional-order Kawahara and fifth-order KdV equations, applying the ρ-homotopy perturbation transform method. The present method is used to describe the results for specific examples. The ρ-HPTM result is highly congruent with the precise solution of the suggested problems. Additionally, the proposed method estimated the results of the cases using fractional-order derivatives. The graphical examination of the resulting fractional-order results proved their convergence to integer-order outcomes. Additionally, the ρ-HPTM technique is straightforward, simple, and computationally efficient; the suggested method can be adapted to solve additional fractional-order partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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