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Research Article

Solving Fractional Generalized Fisher–Kolmogorov–Petrovsky–Piskunov's Equation Using Compact-Finite Different Methods Together with Spectral Collocation Algorithms

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The main target of this work is presenting two efficient accurate algorithms for solving numerically one of the most important models in physics and engineering mathematics, Fisher–Kolmogorov–Petrovsky–Piskunov's equation (Fisher-KPP) with fractional order, where the derivative operator is defined and studied by the fractional derivative in the sense of Liouville–Caputo (LC). There are two main processes; in the first one, we use the compact finite difference technique (CFDT) to discretize the derivative operator and generate a semidiscrete time derivative and then implement the Vieta–Lucas spectral collocation method (VLSCM) to discretize the spatial fractional derivative. The presented approach helps us to transform the studied problem into a simple system of algebraic equations that can be easily resolved. Some theoretical studies are provided with their evidence to analyze the convergence and stability analysis of the presented algorithm. To test the accuracy and applicability of our presented algorithm a numerical simulation is given.

1. Introduction

Nowadays, mathematics is playing a main role in all industries and the huge economics; also, in most applications in our daily life and as a logical result, it was necessary to find a way to relate and connect these real-life problems by mathematical modeling. In physics, chemistry, and biology, many reaction-diffusion equations have moving wavefronts that play a common role in researching many real-life problems [1, 2]. Reaction-diffusion models are useful mathematical instruments that explain how, under the influence of two processes, the concentration of one or more substances distributed in space differs: firstly, local chemical reactions in which the substances are converted into each other, and secondly, the diffusion, that is, the explanation for the substances. The classical Fisher-KPP's equation is one of these important models in many applications such as in engineering [3, 4]. The classical Fisher-KPP's equation [5] is not the best way to estimate the important calculations in these important mathematical models; the fractional Fisher-KPP's equation will be more accurate and is better than that equation in the classical case [6].

Fractional differential equations (FDEs) which are described by the fractional derivative operators are more accurate than the classical ones to model most of the phenomena in our daily life. The role played by the fractional derivative is more than great; this branch of mathematics helps us to model and understand many physical and daily phenomena such as in physics [7], chemistry [8], and biology [9]. FDEs also are excellent path to model many phenomena [10–14]. However, the problem is that most fractional- and variable-order differential equations have no exact solutions,

so numerical and approximate techniques are the semi-unique ways to solve these types of the FDEs, and as a result, there are many numerical techniques presented to study these daily phenomena in our real life, such as the finite difference method [15], the finite element method [16], the wavelet collocation method [17], the homotopy perturbation method [18], the residual power series method [19], the variational iteration method [20], and many other techniques [21–23].

One of the most useful tools to simulate differential equation (partial, fractional, and variable-order) is the spectral methods [24]. The most famous advantage of these methods is their capability to generate accurate outcomes with a very small degrees error of freedom [25, 26]. The orthogonality property of some important polynomials as Vieta–Lucas polynomials is used to approximate functions on the interval $[a, b]$ [27, 28]. These polynomials have a main and important role in these methods for FDEs [29, 30].

The outline of the paper is as follows:

(i) Section 2 is devoted to present the formulation of the problem (fractional Fisher-KPP's equations)

(ii) Section 3 is devoted to define the Vieta–Lucas and shifted Vieta–Lucas polynomials

(iii) Section 4 is devoted to approximate the fractional derivative and study its convergence analysis

(iv) Section 5 is devoted to apply the CFDT-VLSCM technique for the F-KPP's equation

(v) Section 6 is devoted to discuss the stability analysis of the proposed numerical scheme

(vi) Section 7 is devoted to give the numerical simulation

(vii) Section 8 gives the conclusions

2. Formulation of the Problem: Fractional Fisher-KPP's Equations

The importance (due to its many applications in physics and engineering) of Fisher–Kolmogorov–Petrovsky–Piskunov's equation can be considered as one of the category of reaction-diffusion equations [3, 4]. Moreover, we can say that it is one of the simplest semilinear reaction-diffusion equations due to the inhomogeneous term that lies in it. In ecology, physiology, combustion, crystallization, plasma physics, and general phase transition issues, certain equations exist, for example, Fisher proposed this equation in 1937 [5] and explored its travelling wave solutions.

There are two different names of the well-known Fisher-KPP's equation; the first name is Fisher's equation, which was named by the statistician and biologist Ronald Fisher. Also, this PDE is known by KPP's equation related to the three scientists: A. Kolmogorov, I. Petrovsky, and N. Piskunov. In our work, we will consider the Fisher-KPP's equation which takes the classical form (Fisher's paper, [5]):

$$\frac{\partial \phi}{\partial t} - \gamma \frac{\partial^2 \phi}{\partial x^2} = N(\phi), \tag{1}$$

where $N(\phi)$ is called the reaction term and the diffusive constant is denoted by $0 \le \gamma \le 1$. In this work, we will study the linear fractional-order Fisher-KPP's equation which depends on the formula of the reaction term $N(\phi)$.

2.1. Linear Fisher-KPP's Equation. In some chemical and biological reactions, the reaction term $N(\phi)$ may be expressed as a linear function of ϕ in the form $N(\phi) = r(1 - \phi(x,t))$; in that case, the resulting equation is called fractional-order Fisher-KPP's equation (F-KPP) which takes the form:

$$\phi_t = \gamma D_x^{\mu} \phi + r(1 - \phi(x,t)), \quad 1 < \mu \le 2,\ 0 < x \le 1, \tag{2}$$

where the reactive constant is written as $0 \le r \le 1$, under the boundary and initial conditions:

$$\begin{aligned} \phi(0,t) &= g(t), \\ \phi(1,t) &= 0, \end{aligned} \tag{3}$$

$$\phi(x,0) = 0. \tag{4}$$

The exact analytical solution for the above equation in the special case $\mu = 2$ is

$$\phi(x,t) = 1 - \frac{\cosh(x)}{\cosh(1)} - \frac{16}{\pi^2} \sum_{j=1}^{\infty} \frac{(-1)^j \cos(0.5\pi(2j-1)x)}{(2j-1)\left(\pi^2(2j-1)^2 + 4\right)} e^{-\left(1+0.25\pi^2(2j-1)^2\right)t}. \tag{5}$$

There are many research papers which study numerically this system such as [31–33].

The fractional derivative in the proposed model (2) is described in the sense of Liouville–Caputo which is defined as follows.

Definition 1 (see [34]). The fractional-order derivative in the sense of Liouville–Caputo, denoted by D_x^{μ} of order $\mu \in \mathbb{R}^+$ for a function $\theta(x,t)$, is defined as

$$D_x^{\mu}\theta(x,tt) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^x (x-\eta)^{m-\mu-1} \frac{\partial^m \theta(\eta,t)}{\partial \eta^m} d\eta, & m-1 < \mu \le m,\ m \in \mathbb{N}, \\ \dfrac{\partial^m \theta(x,t)}{\partial x^m}, & \mu = m. \end{cases} \tag{6}$$

3. Vieta–Lucas and the Shifted Vieta–Lucas Polynomials

In this section of the paper, we are presenting the basic definitions of Vieta–Lucas and the shifted Vieta–Lucas polynomials and its notations and properties that we will use in our study, and they are necessary to reach our goal [29].

3.1. Vieta–Lucas Polynomials. We are researching for a class of orthogonal polynomials in this section of the paper, which lies at the heart of our research. Using the recurrence relations and analytical formula of these polynomials, they can be generated to construct a new family of orthogonal polynomials that will be known as Vieta–Lucas polynomials.

The Vieta–Lucas polynomials, $\mathrm{VL}_m(z)$ of degree $m \in \mathbb{N}_0$, are defined by the following relation [29]:

$$\mathrm{VL}_m(z) = 2\cos(m\psi), \quad \psi = \arccos\left(\frac{z}{2}\right), \ \psi \in [0,\pi], \ |z| \le 2. \tag{7}$$

It is easy like other famous functions; one can prove that these Vieta–Lucas polynomials satisfy the following recurrence formula:

$$\begin{aligned} \mathrm{VL}_m(z) &= z\mathrm{VL}_{m-1}(z) - \mathrm{VL}_{m-2}(z), \\ m &= 2,3,\ldots, \ \mathrm{VL}_0(z) = 2, \ \mathrm{VL}_1(z) = z. \end{aligned} \tag{8}$$

The analytical form of the polynomials $\mathrm{VL}_m(z)$ can be given by

$$\mathrm{VL}_m(z) = \sum_{j=0}^{\lceil m/2 \rceil} (-1)^j \frac{m\Gamma(m-j)}{\Gamma(j+1)\Gamma(m+1-2j)} z^{m-2j}, \quad m = 2,3,\ldots, \tag{9}$$

where $\lceil m/2 \rceil$ is the well-known ceiling function.

$\mathrm{VL}_m(z)$ are orthogonal polynomials on the interval $[-2,2]$ with respect to the weight function $1/\sqrt{4-z^2}$, so we have the following orthogonality property:

$$\begin{aligned} \langle \mathrm{VL}_m(z), \mathrm{VL}_n(z) \rangle &= \int_{-2}^{2} \frac{\mathrm{VL}_m(z)\mathrm{VL}_n(z)}{\sqrt{4-z^2}} \mathrm{d}z \\ &= \begin{cases} 0, & m \neq n \neq 0, \\ 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0. \end{cases} \end{aligned} \tag{10}$$

3.2. Shifted Vieta–Lucas Polynomials. Using the transformation $z = 4x - 2$, we can generate from the family of Vieta–Lucas polynomials, a new class of orthogonal polynomials on the interval $[0,1]$, which are the orthogonal family of the shifted Vieta–Lucas polynomials, and it will be denoted by $\mathrm{VL}_m^*(x)$ and can be obtained as follows:

$$\mathrm{VL}_m^*(x) = \mathrm{VL}_m(4x-2) = \mathrm{VL}_{2m}(2\sqrt{x}). \tag{11}$$

The shifted Vieta–Lucas polynomials $\mathrm{VL}_m^*(x)$ satisfy the following recurrence relation:

$$\mathrm{VL}_{m+1}^*(x) = (4x-2)\mathrm{VL}_m^*(x) - \mathrm{VL}_{m-1}^*(x), \quad m = 1,2,\ldots, \tag{12}$$

where $\mathrm{VL}_0^*(x) = 2$and$\mathrm{VL}_1^*(x) = 4x-2$. Also, we find $\mathrm{VL}_m^*(0) = 2(-1)^m$ and $\mathrm{VL}_m^*(1) = 2, m = 0,1,2,\ldots$. The analytical formula for $\mathrm{VL}_m^*(x)$ is given by

$$\mathrm{VL}_m^*(x) = 2m \sum_{j=0}^{m} (-1)^j \frac{4^{m-j}\Gamma(2m-j)}{\Gamma(j+1)\Gamma(2m-2j+1)} x^{m-j}, \quad m = 2,3,\ldots. \tag{13}$$

The shifted Vieta–Lucas polynomials $\mathrm{VL}_m^*(x)$ are orthogonal polynomials on the interval $[0,1]$ with respect to the weight function $1/\sqrt{x-x^2}$, and so we have the following orthogonality property:

$$\begin{aligned} \langle \mathrm{VL}_m^*(x), \mathrm{VL}_n^*(x) \rangle &= \int_0^1 \frac{\mathrm{VL}_m^*(x)\mathrm{VL}_n^*(x)}{\sqrt{x-x^2}} \mathrm{d}x \\ &= \begin{cases} 0, & n \neq m \neq 0, \\ 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0. \end{cases} \end{aligned} \tag{14}$$

Let $v(x)$ be a function in the space $L^2[0,1]$; then, using the shifted Vieta–Lucas polynomials $\mathrm{VL}_m^*(x)$, $v(x)$ can be written as follows:

$$v(x) = \sum_{j=0}^{\infty} \kappa_j \mathrm{VL}_j^*(x), \tag{15}$$

where κ_j are the values that we should evaluate to express the function $v(x)$ in terms of the shifted Vieta–Lucas polynomials $\mathrm{VL}_m^*(x)$. Consider the first $m+1$ terms only of (15); then, we can write

$$v_m(x) = \sum_{j=0}^{m} \kappa_i \mathrm{VL}_j^*(x), \tag{16}$$

such that $\kappa_j, j = 0,2,\ldots,m$, can be evaluated using the following formula:

$$\begin{aligned} \kappa_0 &= \frac{1}{2\pi} \int_0^1 \frac{v(x)}{\sqrt{x-x^2}} \mathrm{d}x, \\ \kappa_j &= \frac{1}{2\pi} \int_0^1 \frac{v(x)\mathrm{VL}_j^*(x)}{\sqrt{x-x^2}} \mathrm{d}x. \end{aligned} \tag{17}$$

4. An Approximate of the Fractional Derivative and the Convergence Analysis

This section is devoted to present an approximate formula of the fractional derivative via shifted Vieta–Lucas polynomials and study the convergence analysis by computing the error estimate of the proposed approximation.

Theorem 1. *The LC-fractional-order derivative for the function $v_m(x)$ which is defined in (16) can be computed by the following approximate formula [7]:*

$$D^{\gamma}\left(v_m(x)\right) = \sum_{j=\lceil\gamma\rceil}^{m}\sum_{s=0}^{j-\lceil\gamma\rceil}\kappa_j\chi_{j,s}^{(\gamma)}x^{j-s-\gamma}, \tag{18}$$

where

$$\chi_{j,s}^{(\gamma)} = (-1)^s\frac{4^{j-s}(2j)\Gamma(2j-s)\Gamma(j-s+1)}{\Gamma(s+1)\Gamma(2j-2s+1)\Gamma(j-s+1-\gamma)}. \tag{19}$$

Theorem 2 (see [7]). *Assume that $v(x)$ belongs to the space of all Lebesgue-square integrable on the interval $[0,1]$ with respect to the function $1/\sqrt{x-x^2}$, and assume that the second derivative of $v(x)$ is a bounded function by the constant L is an upper bound. Then, $v(x)$ can be written in terms of the shifted Vieta–Lucas polynomials as an infinite linear combination of $VL_m^*(x)$, and $v_m(x)$ contains $m+1$ terms only of this expression. Also, this series converges uniformly to the function $v(x)$ as $m \longrightarrow \infty$. Moreover, the coefficients given in (16) are bounded, i.e.,*

$$\left|\kappa_j\right| \le \frac{L}{4j\left(j^2-1\right)}, \quad j > 2. \tag{20}$$

Theorem 3 (see [7]). *Assume that $v(x)$ satisfies the hypothesis of the above theorem, and let the weight function in the orthogonality relation of the shifted Vieta–Lucas polynomials on $[0,1]$ be the function $w(x) = 1/\sqrt{x-x^2}$; then, the norm of the error estimate ($L_w^2[0,1]$-norm) is given by*

$$\left\|v(x) - v_m(x)\right\|_w < \frac{L}{12\sqrt{m^3}}. \tag{21}$$

Theorem 4 (see [7]). *Let $v(x)$ be an m-times continuously differentiable on $[0,1]$, and the most square-suitable approximation for $v(x)$ is $v_m(x)$ which is expressed in (16); then, we have the following absolute error bound:*

$$\left\|v(x) - v_m(x)\right\| \le \frac{\Delta\Lambda^{m+1}}{(m+1)!}\sqrt{\pi}, \quad \text{where } \Delta = \max_{x\in[0,1]} v^{(m+1)}(x), \text{ and } \Lambda = \max\{1-x_0, x_0\}. \tag{22}$$

5. An Application of the CFDT-VLSCM Technique for F-KPP's Equation

Now, we are ready to derive the numerical scheme for Fisher-KPP's equations (2)–(4) via the CFDT-VLSCM technique. For this aim, we will assume that $M, F \in \mathbb{Z}^+$ with $\tau_i = (i-1)\Delta\tau$, $i = 1, 2, \ldots, M+1$, where $\Delta\tau = T_f/M$, and assume that the roots of the shifted Vieta–Lucas polynomial $\mathrm{VL}_{F+1-\lceil\mu\rceil}(x)$ are $\left\{x_q\right\}_{q=1}^{F+1-\lceil\mu\rceil}$. If $\phi(x,t) \in \mathbb{C}^3(0,1)$, then, by using the well-known Taylor's expansion, we can obtain

$$\frac{\partial\phi\left(x_q,t_i\right)}{\partial t} = \frac{\phi_q^i - \phi_q^{i-1}}{\Delta\tau} - \frac{\Delta\tau}{2}\frac{\partial^2\phi\left(x_q,t_i\right)}{\partial t^2} + O\left((\Delta\tau)^2\right). \tag{23}$$

Using (23) after evaluating the F-KPP's (2) at (x_q, t_i), we can obtain

$$\begin{aligned}\frac{\phi_q^i - \phi_q^{i-1}}{\Delta\tau} - \frac{\Delta\tau}{2}\frac{\partial^2\phi\left(x_q,t_i\right)}{\partial t^2} + O\left((\Delta\tau)^2\right)\\ = \gamma D_x^{\mu}\phi\left(x_q,t_i\right) + r\left(1-\phi\left(x_q,t_i\right)\right).\end{aligned} \tag{24}$$

To get a discretization of the second derivative of the function ϕ at (x_q,t_i) (*i.e.*, $\partial^2\phi(x_q,t_i)/\partial t^2$), differentiate the first-order discretization (2) with respect to t. By inserting the last result (24), simplifying the resulted form, and rewriting $\phi(x_q,t_i)$ as Φ_q^i, we can derive a semidiscrete numerical scheme for Fisher-KPP's equation (3):

$$\begin{aligned}\frac{\Phi_q^i - \Phi_q^{i-1}}{\Delta\tau} &= \gamma D_x^{\mu}\Phi_q^i + r - r\Phi_q^i - \frac{\Delta\tau}{2}\left[\gamma\frac{D_x^{\mu}\Phi_q^i - D_x^{\mu}\Phi_q^{i-1}}{\Delta\tau} - r\frac{\Phi_q^i - \Phi_q^{i-1}}{\Delta\tau}\right] + E^i(x)(\Delta\tau)^2\\ \text{Or } A_1\Phi_q^i - A_2D_x^{\mu}\Phi_q^i &= A_2D_x^{\mu}\Phi_q^{i-1} + A_3\Phi_q^{i-1} + r\Delta\tau + E^i(x)(\Delta\tau)^3,\end{aligned} \tag{25}$$

such that the resulting truncation term is denoted by $E^i(x)$ and, A_1, A_2, and A_3 are given by

$$\begin{aligned}A_1 &= \frac{2+r\Delta\tau}{2},\\ A_2 &= \frac{\gamma\Delta\tau}{2},\\ A_3 &= \frac{2-r\Delta\tau}{2}.\end{aligned} \tag{26}$$

Now, our aim is devoted to the occurrence of the required full discrete for Fisher-KPP's (2), and this will be done if we can get a formula such as (9) for the LC-fractional derivatives $D_x^{\mu}\Phi_q^i$ and $D_x^{\mu}\Phi_q^{i-1}$. Moreover, we approximate the solution $\phi(x,t)$ using the shifted Vieta–Lucas collocation approach as

$$\phi_F(x,t) = \sum_{m=0}^{F}\phi_m(t)\mathrm{VL}_m^*(x). \tag{27}$$

By using the link between (9), (25), and (27) and by remarking that $\Phi_q^i = \phi_F(x_q, t_i)$ and ϕ_m^i is the coefficient in t_i, we can obtain

$$A_1 \sum_{m=0}^{F} \phi_m^i \mathrm{VL}_m^*(x_p) - A_2 \sum_{m=\lceil\mu\rceil}^{F} \sum_{s=0}^{m-\lceil\mu\rceil} \phi_m^i \chi_{m,s}^{(\mu)} x_q^{m-s+\mu} = r\Delta\tau + A_2 \sum_{m=\lceil\mu\rceil}^{F} \sum_{s=0}^{m-\lceil\mu\rceil} \phi_m^{i-1} \chi_{m,s}^{(\mu)} x_q^{m-s+\mu} + A_3 \sum_{m=0}^{F} \phi_m^{i-1} \mathrm{VL}_m^*(x_q). \tag{28}$$

By using (3) with (27) and applying the facts $\mathrm{VL}_m^*(0) = 2(-1)^m$ and $\mathrm{VL}_m^*(1) = 2$, $m = 0, 1, 2, \ldots$, we can obtain

$$\sum_{m=0}^{F} 2(-1)^m \phi_m^i = g(t_i) = g^i, \qquad \sum_{m=0}^{F} 2\phi_m^i = 0. \tag{29}$$

Now, we have a system of $(F+1)$ of linear algebraic equations resulting from $\lceil\mu\rceil$ in (29) with (28), and we can solve numerically this system to compute the unknowns $\phi_m^i, m = 0, 1, 2, \ldots, F$ and $i = 1, 2, \ldots, M+1$.

The above system of equations (28) and (29) can be written in a special case $F = 3$ and x_1 and x_2 are the roots of the shifted Vieta–Lucas polynomial $\mathrm{VL}_2^*(x)$, i.e., $x_1 = 0.146447 \text{and} x_2 = 0.853553$, in a matrix form as follows:

$$\begin{pmatrix} A_1 & A_1\sigma_1 & -A_2\beta_1 & A_1\sigma_2 - A_2\beta_2 \\ A_1 & A_1\sigma_{11} & -A_2\beta_{11} & A_1\sigma_{22} - A_2\beta_{22} \\ 2 & -2 & 2 & -2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}^i = \begin{pmatrix} A_3 & A_3\sigma_1 & A_2\beta_1 & A_3\sigma_2 + A_2\beta_2 \\ A_3 & A_3\sigma_{11} & A_2\beta_{11} & A_3\sigma_{22} + A_2\beta_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}^{i-1} + \begin{pmatrix} r\Delta\tau \\ r\Delta\tau \\ g^i \\ 0 \end{pmatrix}^i, \tag{30}$$

where

$$\begin{aligned} \sigma_1 &= \mathrm{VL}_1^*(x_1), \\ \sigma_2 &= \mathrm{VL}_3^*(x_1), \\ \sigma_{11} &= \mathrm{VL}_1^*(x_2), \\ \sigma_{22} &= \mathrm{VL}_3^*(x_2), \\ \beta_1 &= \chi_{2,0}^{(\mu)} x_1^{2-\mu}, \\ \beta_2 &= \chi_{2,0}^{(\mu)} x_1^{2-\mu} + \chi_{3,0}^{(\mu)} x_1^{3-\mu} + \chi_{3,1}^{(\mu)} x_1^{2-\mu}, \\ \beta_{11} &= \chi_{2,0}^{(\mu)} x_2^{2-\mu}, \\ \beta_{22} &= \chi_{2,0}^{(\mu)} x_2^{2-\mu} + \chi_{3,0}^{(\mu)} x_2^{3-\mu} + \chi_{3,1}^{(\mu)} x_2^{2-\mu}. \end{aligned} \tag{31}$$

Now, let $\Phi^i = (\phi_0^i, \phi_1^i, \phi_2^i, \phi_3^i)^T \text{and} C^i = (r\Delta\tau, r\Delta\tau, g^i, 0)^T$; then, the above system can be rewritten in the following matrix form:

$$A\Phi^i = B\Phi^{i-1} + C^i, \quad \text{or } \Phi^i = A^{-1}B\Phi^{i-1} + A^{-1}C^i. \tag{32}$$

For $i = 1$, using the initial condition (3), we can evaluate ϕ^0 which is the initial solution of the linear system (32), and this helps us to get the other numerical solutions at each step of the time.

6. The Stability Analysis of the Proposed Numerical Scheme

Now, we are going to study the stability analysis of the derived numerical scheme (25). For this aim, let $\overline{\mathbb{D}} \subset \mathbb{R}^2$ be an open bounded region, and let

$$H^n(\overline{\mathbb{D}}) = \left\{ g(x) \in L_2(\overline{\mathbb{D}}) : g^{(n)}(x) \in L_2(\overline{\mathbb{D}}) \right\}. \tag{33}$$

Rewrite the resulting numerical scheme (25) in the following form:

$$\Phi^k - \Omega_{1a} D_x^\mu \Phi^k = \Omega_2 \Phi^{k-1} + \Omega_{1a} D_x^\mu \Phi^{k-1} + \Omega_3, \tag{34}$$

where $k = 1, 2, \ldots, L+1$; the constants Ω_1, Ω_2, and Ω_3 are

$$\Omega_1 = \frac{\gamma\Delta\tau}{2 + r\Delta\tau}, \qquad \Omega_2 = \frac{2 - r\Delta\tau}{2 + r\Delta\tau}, \qquad \Omega_3 = \frac{2r\Delta\tau}{2 + r\Delta\tau}. \tag{35}$$

Lemma 1 (see [35]). *If $\theta(x), \varphi(x) \in H^{\mu/2}(\overline{\mathbb{D}})$, then, for $1 < \mu < 2$, we have the following relations:*

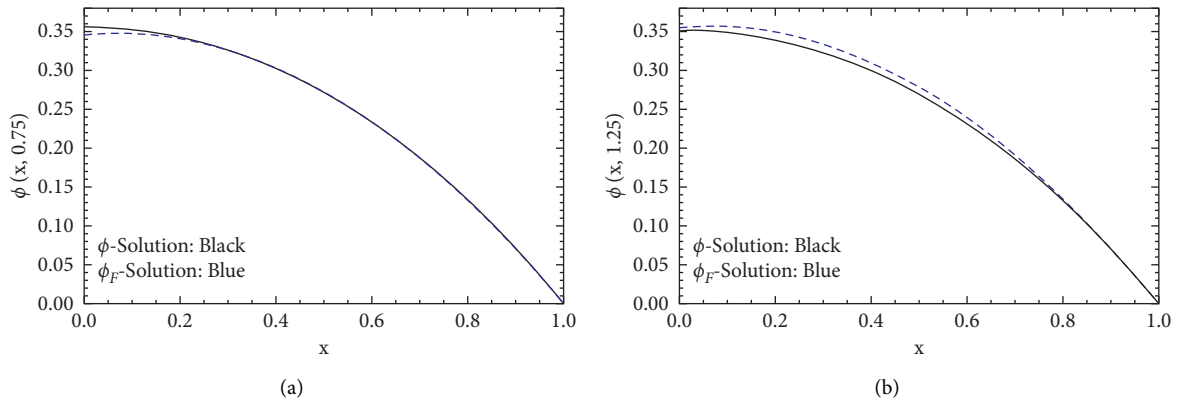

Figure 1: The approximate and exact solutions at $F = 4$ via $T_f = 0.75\,(a)$ and $T_f = 1.25\,(b)$.

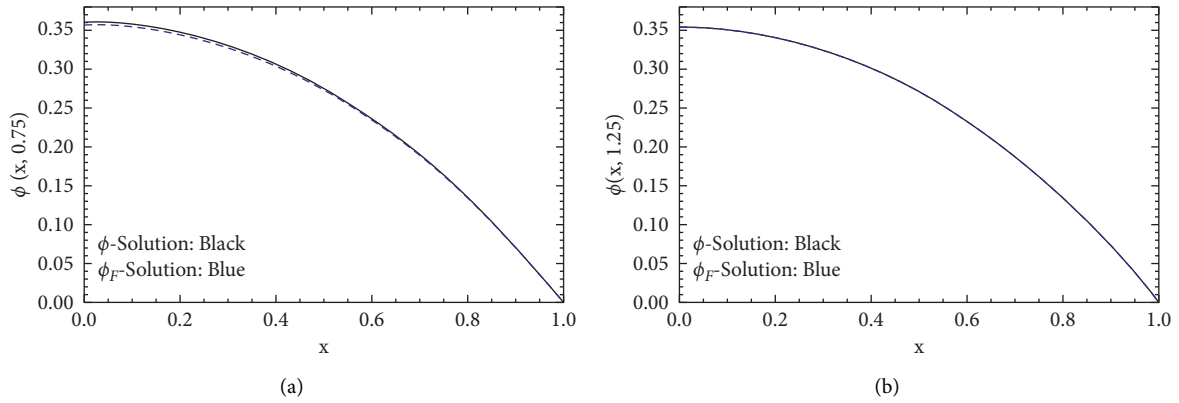

Figure 2: The approximate and exact solutions at $F = 7$ via $T_f = 0.75\,(a)$ and $T_f = 1.25\,(b)$.

$$\begin{aligned}\langle {}_aD_x^{\mu}\theta, \varphi\rangle &= \langle {}_aD_x^{\mu/2}\theta, {}_xD_b^{\mu/2}\varphi\rangle,\\ \langle {}_xD_b^{\mu}\theta, \varphi\rangle &= \langle {}_xD_b^{\mu/2}\theta, {}_aD_x^{\mu/2}\varphi\rangle,\end{aligned} \tag{36}$$

where $\langle , \rangle$ is the usual inner product defined on the Hilbert space $L_2(\overline{\mathbb{D}})$.

Lemma 2 (see [35]). *If $\mu \in \mathbb{R}^+$, then*

$$\langle {}_aD_x^{\mu}\theta, {}_xD_b^{\mu}\theta\rangle = \cos(\pi\mu)\left\|{}_aD_x^{\mu}\theta\right\|^2_{L_2(\overline{\mathbb{D}})} = \cos(\pi\mu)\left\|{}_xD_b^{\mu}\theta\right\|^2_{L_2(\overline{\mathbb{D}})}. \tag{37}$$

Lemma 3. *If $\theta(x) \in H^{\mu}(\overline{\mathbb{D}})$ and if the fractional derivative lies also in $H^{\mu}(\overline{\mathbb{D}})$, then*

$$\forall 1 < \mu < 2 \exists \Delta\tau \ll 1: \quad \left\|\theta(x) + \Omega_1 {}_aD_x^{\mu}\theta(x)\right\| \le \|\theta(x)\|. \tag{38}$$

Lemma 4. *If $\Phi^k \in H^1(\overline{\mathbb{D}})$, $k = 1, 2, \ldots, M+1$, is an approximate solution of (34), then we have*

$$\left\|\Phi^k\right\| \le \Omega_2\left\|\Phi^{k-1}\right\| + \Omega_3. \tag{39}$$

Theorem 5 (see [15]). *The resulting numerical scheme (25) through applying the CFDT-VLSCM technique for F-KPP's (2) is unconditionally stable.*

Theorem 6 (see [15]). *Let $\varepsilon^k = \phi(x, t_k) - \Phi^k, k = 1, 2, \ldots, M+1$, be the resulting error from (25); then, the error upper bound is estimated as follows:*

$$\left\|\varepsilon^k\right\| \le \rho_x (\Delta\tau)^2, \tag{40}$$

where ρ_x is a constant (the maximum of the truncation error term).

7. Numerical Simulation

In this section, we are going to achieve the above technique to process and solve the proposed model with distinct values of fractional order μ and the order of approximation F. The exact solution was also compared at $\mu = 2$. In addition, the effect of a diffusion constant, γ, and the reactive constant, r, are given. Finally, the adequacy of the aforementioned technique is checked by calculating the residual error function (REF) with distinct values of μ and F. In Figures 1–5, we plotted the approximate solution for the given model by implementing the presented algorithm.

In Figure 1, we give a comparison of the approximate and exact solutions at $\mu = 2, F = 4, \Delta\tau = 0.002$, and $\gamma = r = 1$, via different values of $T_f = 0.75\,(a)$ and $T_f = 1.25\,(b)$. Figure 2 is the same Figure 1 but with $F = 7$. In Figure 3, we give the behavior of the approximate solution via different values of the fractional order, $\mu = 2.0, 1.75, 1.5, 1.25$, at

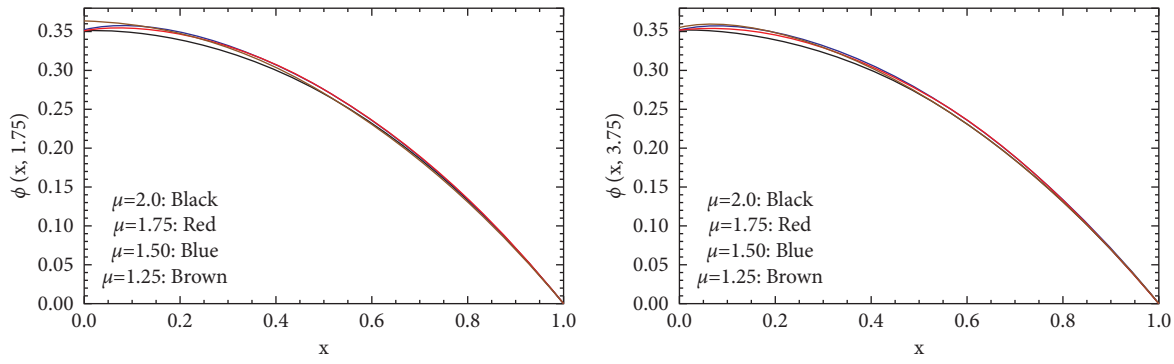

FIGURE 3: The approximate solution at $F = 6$ via different values of μ with $T_f = 1.75\,(a)$ and $T_f = 3.75\,(b)$.

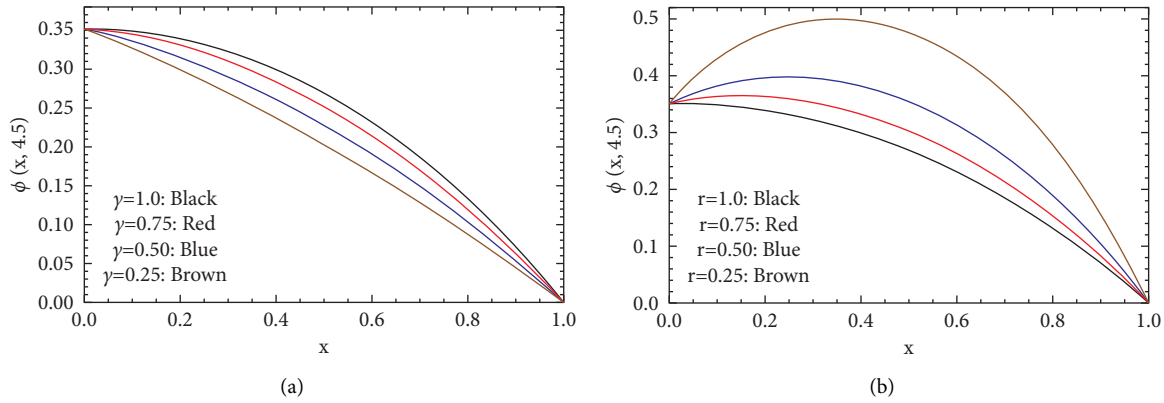

FIGURE 4: The approximate solution at $F = 6$and$T_f = 4.5$ via different values of $\gamma\,(a)$ and $r\,(b)$.

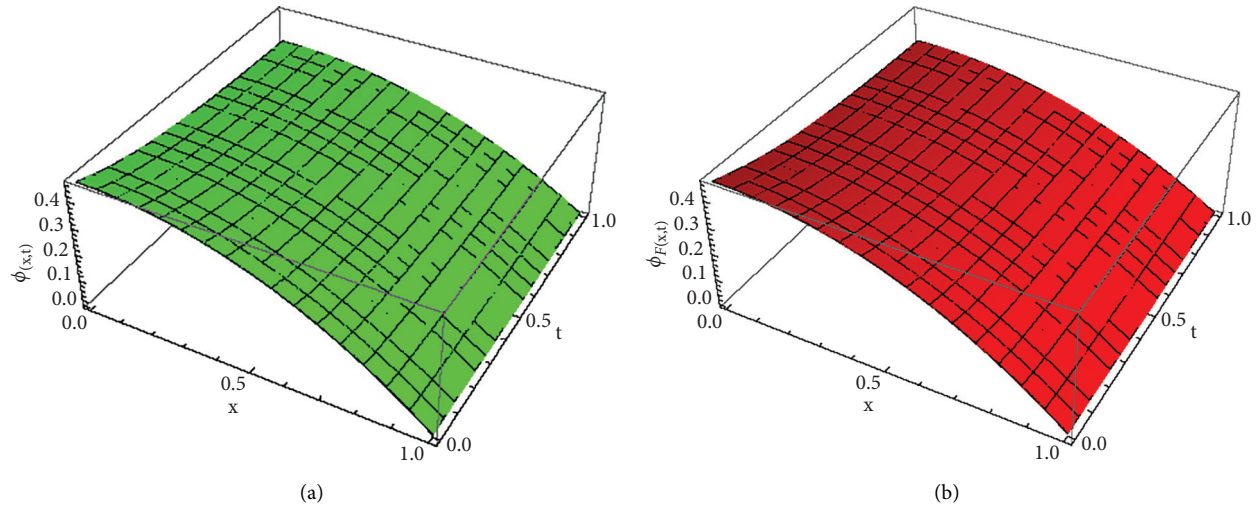

FIGURE 5: 3D plot of the approximate and exact solutions at $F = 5$ via $\mu = 2\,(a)$ and $\mu = 1.9\,(b)$.

$F = 6, \Delta\tau = 0.01,$ and$\gamma = r = 1$, with $T_f = 1.75\,(a)$ and $T_f = 3.75\,(b)$. In Figure 4, we present the approximate solution at $F = 6, T_f = 4.5,$ and$\Delta\tau = 0.005$, via different values of the diffusive constant $\gamma = 0.25, 0.5, 0.75, 1\,(a)$ and the reactive constant $r = 0.25, 0.5, 0.75, 1\,(b)$. Finally, Figure 5 presents 3D plot of the approximate solution, $\phi(x,t)$ at $\mu = 2.0$, and the exact solution, $\phi_F(x,t)$ at $\mu = 1.9$, with $F = 5$ and $\Delta\tau = 0.003$.

From these figures, we see that the behavior of the approximate solution depends on the values of μ and F, and this ensures that the proposed technique is applied in a good way to solve the given model in the case of fractional

Table 1: A comparison of REF at $T_f = 1.5$ via different values of μ and F.

	Current method, REF, at			Current method, REF, at		
x	$\mu = 1.75$	$\mu = 1.85$	$\mu = 1.95$	$F = 4$	$F = 7$	$F = 10$
0.0	5.15720E-06	6.74512E-07	1.96325E-08	4.65412E-07	2.25874E-08	8.32145E-09
0.2	2.85214E-07	4.63254E-09	3.85214E-09	7.65412E-07	5.95124E-08	5.32145E-08
0.4	5.85214E-06	8.62541E-09	4.85210E-10	1.02145E-07	7.95280E-08	8.75821E-09
0.6	4.85231E-08	2.32541E-09	5.96325E-10	1.98745E-06	2.85214E-08	3.95214E-10
0.8	7.85214E-08	8.66421E-09	1.74125E-11	6.85214E-07	0.95214E-09	2.85214E-10
1.0	6.12050E-09	1.20140E-10	9.90241E-11	0.02145E-07	3.96325E-08	4.32541E-10

derivatives. Also, we found that the diffusion constant and the reactive constant are clearly affected by the behavior of the presented solution. Likewise, these results bring to light the reasonability convergence of the proposed method for a given problem and consistent with which was predicted in the theoretical study through the proved lemmas and theorems. Additionally, to validate our solutions with $\tau = 0.05, T_f = 0.5, \text{and} \gamma = r = 1$, we calculate the REF via different values for $\mu = 1.75, 1.85, 1.95$ and $F = 4, 7, 10$ in Table 1. From Table 1 and all Figures 1–5, it is clear that the overall errors can be reduced by adding new terms from series (27).

Because of the values in Table 1, we can see the proposed method by using the shifted Vieta–Lucas polynomial is best than the shifted Chebyshev polynomials [15].

8. Conclusions

In this study, we have studied the fractional Fisher-KPP's equation. The model was proposed via the Liouville–Caputo sense fractional derivative because it only needs initial conditions described in terms of integer-order derivatives. The solution of the proposed problem was achieved via efficient numerical techniques, the Vieta–Lucas spectral collocation based on the scheme of the compact FDM. The unconditional stability of the proposed numerical scheme is discussed and proved in the Sobolev space. From the solutions obtained using the proposed method, we can confirm that these solutions are in excellent agreement with the already existing ones and explain that this method is efficient and applicable to solve the aforesaid problem effectively. Finally, we can say the results obtained give better explanation of model dynamics. Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of our numerical method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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