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Research Article

Existence and Stability of α – Harmonic Maps

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In this paper, we first study the α–energy functional, Euler-Lagrange operator, and α-stress-energy tensor. Second, it is shown that the critical points of the α–energy functional are explicitly related to harmonic maps through conformal deformation. In addition, an α–harmonic map is constructed from any smooth map between Riemannian manifolds under certain assumptions. Next, we determine the conditions under which the fibers of horizontally conformal α–harmonic maps are minimal submanifolds. Then, the stability of any α–harmonic map on Riemannian manifold with nonpositive curvature is studied. Finally, the instability of α–harmonic maps from a compact manifold to a standard unit sphere is investigated.

1. Introduction

Eells and Sampson introduced the concept of harmonic maps between Riemannian manifolds [1]. If ψ: $(M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, then the energy functional of ψ is denoted by $E(\psi)$ and defined as follows:

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dV_g, \tag{1}$$

where dV_g is the volume element of (M, g). The Euler-Lagrange equation corresponding to E_ψ is given by

$$\tau(\psi) \coloneqq Tr_g \nabla d\psi = 0, \tag{2}$$

where ∇ is the induced connection on the pull-back bundle $\psi^{-1}TN$. The section $\tau(\psi) \in \Gamma(\psi^{-1}TN)$ described in (2) is known as the tension field of ψ.

The authors showed in [1] that any smooth map from a compact Riemannian manifold to a Riemannian manifold with nonpositive sectional curvature can be deformed into a harmonic map. This is well-known as the *fundamental existence theorem for harmonic maps*. Following that, several researchers conducted a study on these maps [2–6]. Harmonic maps have been examined in numerous theories in mathematical physics, such as liquid crystal, ferromagnetic material, superconductors [7, 8].

Sacks and Uhlenbeck introduced perturbed energy functional that satisfies the Palais–Smale condition in their pioneering paper [9] in 1981 to prove the existence of harmonic maps from a closed surface and thus obtained so-called α-harmonic maps, as critical points of perturbed functional, to approximate harmonic maps. For $\alpha > 1$, the *α–energy functional* of the map ψ is denoted by $E_\alpha(\psi)$ and defined as follows:

$$E_\alpha(\psi) \coloneqq \int_M \left(1 + |d\psi|^2\right)^\alpha dV_g, \tag{3}$$

where $|d\psi|^2$ denotes the Hilbert–Schmidt norm of the differential map $d\psi \in \Gamma(T^*M \otimes \psi^{-1}TN)$ with respect to g and h. Noting that, for $\alpha > 1$, E_α satisfies Morse theory and Ljusternik–Schnirelman theory [10]. Furthermore, the critical points of E_α are said to be the *α-harmonic maps*. Sacks and Uhlenbeck developed an existence theory for harmonic mappings of orientable surfaces into Riemannian manifolds using the critical maps of E_α. Furthermore, they demonstrated that the convergence of the critical points of

E_α is sufficient to construct at least one harmonic map of the sphere into any Riemannian manifold [9]. Many studies have recently been conducted on these maps. For example, the authors of [11] investigated the convergence behaviour of a sequence of α–harmonic mappings u_α with $E_\alpha(u_\alpha) < C$ from a compact surface (M, g) into a compact Riemannian manifold (N, h) without boundary. It is worth noticing that this sequence converges weakly to a harmonic map. Furthermore, in [10], it is studied the energy identity and necklessness for a sequence of Sacks–Uhlenbeck maps during blowing up.

Following the concepts of [1, 4, 8–14], we study the stability, existence, and structure of α-harmonic maps, as well as their practical applications, in this work. The existence of α–harmonic maps in an arbitrary class of homotopy and the conditions under which the fibers of α–harmonic maps are minimal submanifolds are explored in particular. Furthermore, the stability of harmonic mappings from a Riemannian manifold to a nonpositive Riemannian curvature and unit standard sphere is explored. Sections 3–4.1 present our key findings. This manuscript is organized as follows: Section 1 investigates the concepts of ɑ–energy functional and ɑ–harmonic maps, and it provides an explicit relationship between ɑ–harmonic maps and harmonic maps via conformal deformation. Section 2 looks at the existence of ɑ–harmonic mappings. It is demonstrated that, under certain assumptions, every smooth map between Riemannian manifolds can be deformed into an ɑ–harmonic map. The ɑ–stress energy tensor and its practical applications are discussed in Section 3. It is also shown that the vanishing of the divergence of the ɑ–stress energy tensor of ψ is identical to the vanishing of the ɑ–hormonicity of ψ. Furthemore, the criteria that cause the fibres of horizontally conformal ɑ–harmonic mappings to be minimal submanifolds are then examined. In Section 4, the Jacobi operator and Green's theorem are used to derive the second variation formula of the ɑ–energy functional. The stability of ɑ–harmonic maps is explored in the last part. Any ɑ–harmonic map from a Riemannian manifold to a Riemannian manifold with nonpositive curvature is proved to be stable in this sense. The instability of ɑ–harmonic mappings from a compact manifold to a unit standard sphere is also explored.

1.1. α-Harmonic Maps. In this section, first, we study the notion of α–harmonic maps. Then, the certain relation between α-harmonic maps and harmonic maps through conformal deformation is given.

Let $\psi\colon (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. Throughout this paper, it is considered that (M, g) is a compact Riemannian manifold. Furthermore, the Levi–Civita connections on M and N are denoted by ∇^M and ∇^N, respectively. Moreover, the induced connection on the pullback bundle $\psi^{-1}TN$ is denoted by ∇ and defined by $\nabla_Y V = \nabla^N_{d\psi(Y)} V$, for any smooth vector field $Y \in \chi(M)$ and section $V \in \Gamma(\psi^{-1}TN)$.

Let α be a positive constant such that $\alpha > 1$. Then, the α–energy functional of ψ is defined as follows:

$$E_\alpha(\psi) := \int_M \left(1 + |d\psi|^2\right)^\alpha \mathrm{d}V_g. \tag{4}$$

The critical points of E_α are called α–harmonic maps. By Green's theorem, the corresponding Euler-Lagrange equation of the α–energy functional, E_α, is given by

$$\begin{aligned}\tau_\alpha((\psi)) &:= 2\alpha\left(1 + |\mathrm{d}\psi|^2\right)^{\alpha-1}\tau(\psi)\\ &\quad + \mathrm{d}\psi\left(\mathrm{grad}\left(2\alpha\left(1 + |\mathrm{d}\psi|^2\right)^{\alpha-1}\right)\right.\\ &= 0.\end{aligned} \tag{5}$$

The section $\tau_\alpha(\psi) \in \Gamma(\psi^{-1}TN)$ is said to be the α–tension field of ψ.

By (2) and (5), and [[12], Theorem 2.1], in which $F(t) = (1 + 2t)^\alpha$, we have the following:

Theorem 1. *Let $\psi\colon (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. Then, ψ is α–harmonic if and only if it has a vanishing α–tension field.*

Definition 1. Let $\psi\colon (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. It is called nondegenerate if the induced tangent map $\psi_* = \mathrm{d}\psi$ is nondegenerate, i.e., $\ker \mathrm{d}\psi = \varnothing$.

Example 1. Let $\alpha = 3$ and $\psi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a smooth map defined as follows:

$$\psi(x_1, x_2) = (2x_1 + x_2, x_1 - x_2, -4x_1 + 3x_2). \tag{6}$$

By equation (5) and Definition 1, it can be seen that ψ is a nondegenerate α–harmonic map.

Now, an explicit relation between α–harmonic maps and harmonic maps through conformal deformation is given, which provides examples of [[12], Proposition 2.2], where $F(t) = (1 + 2t)^\alpha$.

Proposition 1. *Let $\psi\colon (M^m, g) \longrightarrow (N^n, h)$ be a nondegenerate smooth map with $m > 2$. Then, ψ is an α–harmonic map if and only if the map ψ is harmonic with respect to the conformally related metric $\overline{g}$ defined by*

$$\overline{g} = \left\{(2\alpha)^{2/(m-2)}\left(1 + \mid d\psi \mid^2\right)^{2\alpha - 2/m-2}\right\} g. \tag{7}$$

Proof 1. Assume that $\overline{g}$ be a Riemannian metric for some smooth positive function μ on M conformally associated to g by $\overline{g} = \mu^2 g$. Denote the tension fields of the smooth map ψ with respect to g and $\overline{g}$ by $\tau(\psi)$ and $\overline{\tau}(\psi)$, respectively. By (2), it can be obtained that

$$\overline{\tau}(\psi) = \frac{1}{\mu^m}\left\{\mu^{m-2}\tau(\psi) + \mathrm{d}\psi\left(\mathrm{grad}\mu^{m-2}\right)\right\}. \tag{8}$$

Setting

$$\mu = \left\{(2\alpha)^{1/m-2}\left(1 + |\mathrm{d}\psi|^2\right)^{\alpha - 1/m-2}\right\}. \tag{9}$$

By (5), (7), and (8), we get

$$\begin{aligned}\mu^m\overline{\tau}(\psi) &= \mu^{m-2}\tau(\psi) + d\psi\left(\mathrm{grad}\mu^{m-2}\right)\\ &= 2\alpha\left(1+|d\psi|^2\right)^{\alpha-1} + d\psi\left(\mathrm{grad}\left(2\alpha\left(1+|d\psi|^2\right)^{\alpha-1}\right)\right)\\ &= \tau_\alpha(\psi).\end{aligned} \tag{10}$$

The Proposition 1 follows from (10). □

2. The Existence of α – Harmonic Map

In this section, it is assumed that (M, g) and (N, h) are compact Riemannian manifolds, and $\mathscr{H}$ is a homotopy class of a smooth given map ψ: $(M^m, g) \longrightarrow (N^n, h)$. The following theorem is due to Hong.

Theorem 2 (see [4]). *Suppose that ψ: $(M, g) \longrightarrow (N, h)$ is a harmonic map. Then, for any $\hat{\varepsilon} > 0$, there exists a smooth metric $\hat{g}$ conformally equivalent to g such that ψ: $(M, \hat{g}) \longrightarrow (N, h)$ is harmonic and $|d\psi|_{\hat{g}} \le \hat{\varepsilon}$, where $|d\psi|_{\hat{g}}$ is the Hilbert–Schmidt norm with respect to $\hat{g}$ and h.*

By Theorem 2, the following theorem follows and provides examples of [[12], Theorem 3.5], in which $F(t) = (1 + 2t)^\alpha$ and F–harmonic become α–harmonic.

Theorem 3. *Let $\psi \in \mathscr{H}$ and ψ: $(M^m, g) \longrightarrow (N^n, h)$ be a harmonic map. Then, there is a smooth metric $\overline{g}$ on M conformally equivalent to g such that ψ: $(M, \overline{g}) \longrightarrow (N, h)$ is α–harmonic if $m > 2\alpha$.*

Proof 2. Assume that ε be a positive constant and $h(t) = (1 + 2t)^\alpha$. Let $k(t) = 1/2\left\{(t/2\alpha)^{1/\alpha-1} - 1\right\}$ be the inverse function of $h'(t)$ on $[0, \varepsilon)$. Then, we have

$$h'(k(t)) = t, \tag{11}$$

$$k'(t)h''(k(t)) = 1, \tag{12}$$

on $[0, \varepsilon)$. Setting

$$y(u) = \frac{k\left(u^{m-2}\right)}{u^2}. \tag{13}$$

The derivative of y with respect to u can be obtained as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u^3}\left\{\frac{(m-2)u^{m-2}}{4\alpha(\alpha-1)}\left(\frac{u^{m-2}}{2\alpha}\right)^{2-\alpha/\alpha-1} - \left(\left(\frac{u^{m-2}}{2\alpha}\right)^{1/\alpha-1} - 1\right)\right\}. \tag{14}$$

Due to the fact that k is the inverse function of h' and $m > 2\alpha$, we get

$$\begin{aligned}&\frac{dy}{du}\left(\left(h'(t)\right)^{1/m-2}\right)\\ &= \frac{\{(2m-4\alpha)t + (m-2)\}}{2(\alpha-1)\{2\alpha(1+2t)\}^{3(\alpha-1)/m-2}} \neq 0,\end{aligned} \tag{15}$$

for $t \in [0, \varepsilon)$. Then, $\mathrm{d}y/\mathrm{d}u \neq 0$, on u between $(h'(0))^{1/m-2}$ and $(h'(\varepsilon))^{1/m-2}$. Thus, there exists a smooth function $\theta(y)$ on $[0, \varepsilon')$ for some positive constant ε', such that

$$y = \frac{k\left((\theta(y))^{m-2}\right)}{(\theta(y))^2}. \tag{16}$$

Moreover, by (11), we have

$$h'\left(k\left((\theta(y))^{m-2}\right)\right) = (\theta(y))^{m-2}. \tag{17}$$

By using (16) and (17), we get

$$h'\left(y(\theta(y))^2\right) = (\theta(y))^{m-2}. \tag{18}$$

Since $h'(t)$ is a positive function on $[0, \varepsilon')$, by (18), it can be seen that θ is a positive function on $[0, \varepsilon')$.

Now, by Theorem 5, for $\hat{\varepsilon} := \sqrt{\varepsilon'}$, there exists a smooth metric $\hat{g}$ conformally equivalent to g such that ψ: $(M, \hat{g}) \longrightarrow (N, h)$ is harmonic and $|d\psi|^2_{\hat{g}}/2 < \varepsilon'$. Due to the harmonicity of ψ: $(M, \hat{g}) \longrightarrow (N, h)$, the tension field of ψ associated to the metrics $\hat{g}$ and h vanishes, $\hat{\tau}(\psi) = Tr_{\hat{g}} d\psi = 0$. Assume that $\overline{g} = \mu^{-2}\hat{g}$ for a smooth positive function μ: $M \longrightarrow \mathbb{R}$, by (8), we get

$$\begin{aligned}0 &= \hat{\tau}(\psi)\\ &= \frac{1}{\mu^m}\left\{\mu^{m-2}\overline{\tau}(\psi) + d\psi\left(\mathrm{grad}_{\overline{g}}\mu^{m-2}\right)\right\}.\end{aligned} \tag{19}$$

Due to the fact that $|d\psi|^2_{\hat{g}}/2 < \varepsilon'$, the positive function μ can be defined as follows:

$$\mu := \theta\left(\frac{|d\psi|^2_{\hat{g}}}{2}\right). \tag{20}$$

By (18) and (20), it yields that

$$\begin{aligned}\mu^{m-2} = \left(\theta\left(\frac{|d\psi|^2_{\hat{g}}}{2}\right)\right)^{m-2} &= h'\left(\frac{|d\psi|^2_{\hat{g}}}{2}\left(\theta\left(\frac{|d\psi|^2_{\hat{g}}}{2}\right)\right)^2\right)\\ &= h'\left(\frac{|d\psi|^2_{\hat{g}}}{2}\mu^2\right) = h'\left(\frac{|d\psi|^2_{\overline{g}}}{2}\right)\\ &= 2\alpha\left(1+|d\psi|_{\overline{g}}\right)^{\alpha-1}.\end{aligned} \tag{21}$$

By substituting (21) in (19) and using (5), we get

$$\begin{aligned}\overline{\tau}_\alpha(\psi) &= 2\alpha(x)\left(1+|d\psi|_{\overline{g}}^{\,2}\right)^{\alpha(x)-1}\overline{\tau}(\psi)\\ &\quad + d\psi\left(\mathrm{grad}_{\overline{g}}(2\alpha(x)\left(1+|d\psi|_{\overline{g}}^{\,2}\right)^{\alpha(x)-1}\right) = 0.\end{aligned} \tag{22}$$

And this completes the proof. □

2.1. The α–Stress-Energy Tensor. Let $\psi\colon (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. The stress-energy tensor of ψ associated with the α–energy functional (in the sequel, we call the α-stress-energy tensor of ψ, in short) is denoted by $S_\alpha(\psi)$ and defined as follows:

$$\begin{aligned} S_\alpha(\psi) &= \left(1 + \,|\, |d\psi|^2\right)^\alpha g \\ &\quad - 2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1}\psi^* h. \end{aligned} \tag{23}$$

Remark 1. The stress-energy tensor sometimes called the stress-energy-momentum tensor or the energy-momentum tensor is a tensor quantity in physics that describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It is an attribute of matter, radiation, and nongravitational force fields. This density and flux of energy and momentum are the sources of the gravitational field in the Einstein field equations of general relativity, just as mass density is the source of such a field in Newtonian gravity [3].

In general relativity, the symmetric stress-energy tensor acts as the source of spacetime curvature and is the current density associated with gauge transformations of gravity which are general curvilinear coordinate transformations [14].

The following Proposition 2 and Remark 2 provide examples of [[12], Proposition 4.1], in which $S_F = S_\alpha$ and $\tau_F = \tau_\alpha$.

Proposition 2. *Based on the above notations, we get*

$$(\mathrm{div} S_\alpha(\psi))(Y) = -h(\tau_\alpha(\psi), d\psi(Y)), \tag{24}$$

for all $Y \in \chi(M)$.

Proof 3. Choose a local orthonormal frame $\{e_i\}_{i=1}^m$ on M with $\nabla_{e_i} e_j|_p = 0$, at a point $p \in M$. Let Y be a smooth vector field on M. According to the definition of divergence operator on a Riemannian manifold, at point p, we have

$$\begin{aligned} &(\mathrm{div} S_\alpha(\psi))(Y) \\ &= \sum_{i=1}^m \left(\nabla^M_{e_i} S_\alpha(\psi)\right)(e_i, Y) \\ &= -h\left(d\psi\left(\mathrm{grad}\left(2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1}\right)\right), d\psi(Y)\right) \\ &\quad - 2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1} \sum_{j=1}^m \left\{h\left(d\psi\left(e_j\right), \nabla_Y d\psi\left(e_j\right)\right)\right\} \\ &\quad - 2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1} \sum_{i=1}^m \left\{h\left(\nabla_{e_i} d\psi(e_i), d\psi(Y)\right) + h\left(d\psi(e_i), \nabla_{e_i} d\psi(Y) - d\psi\left(\nabla^M_{e_i} Y\right)\right)\right\}. \end{aligned} \tag{25}$$

At point p, the last term of the right-hand side of 24 can be obtained as follows:

$$\begin{aligned} &\nabla_{e_i} d\psi(Y) - d\psi\left(\nabla^M_{e_i} Y\right) \\ &\quad = \nabla_Y d\psi(e_i) + d\psi[e_i, Y] - d\psi\left(\nabla^M_{e_i} Y\right) \\ &\quad = \nabla_Y d\psi(e_i) + d\psi\left(\nabla^M_{e_i} Y - \nabla^M_Y e_i\right) - d\psi\left(\nabla^M_{e_i} Y\right) \\ &\quad = \nabla_Y d\psi(e_i), \end{aligned} \tag{26}$$

where we use

$$\begin{aligned} \nabla^M_Y e_i &= \sum_{j=1}^m \nabla^M_{g\left(e_j, Y\right) e_j} e_i \\ &= \sum_{j=1}^m g\left(e_j, Y\right) \nabla^M_{e_j} e_i = 0, \end{aligned} \tag{27}$$

at the point p, for the third equality. Furthermore,

$$\begin{aligned} \nabla_{e_i} d\psi(e_i) &= \nabla_{e_i} d\psi(e_i) - d\psi\left(\nabla^M_{e_i} e_i\right) \\ &= \tau(\psi). \end{aligned} \tag{28}$$

By substituting (26) and (28) in the last equation of (25), we get

$$\begin{aligned} &(\mathrm{div} S_\alpha(\psi))(Y) \\ &\quad = -2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1}) h(\tau(\psi), d\psi(Y)) \\ &\qquad - h\left(d\psi\left(\mathrm{grad}\left(2\alpha\left(1 + |d\psi|^2\right)^{\alpha-1}\right), d\psi(Y)\right)\right. \\ &\quad = -h(\tau_\alpha(\psi), d\psi(Y)), \end{aligned} \tag{29}$$

and hence completes the proof. □

Remark 2. Based on Proposition 2, it can be seen that if ψ is a submersion and $\mathrm{div}\, S_\alpha(\psi) = 0$, then ψ is an α–harmonic map. Conversely, if ψ is an α–harmonic map, then $\mathrm{div}\, S_\alpha(\psi) = 0$.

3. Horizontal Conformal α – Harmonic Maps

Let $\psi\colon (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $d\psi_p \neq 0$ for each $p \in M$. The vertical space at p with respect to ψ is denoted by V_p and defined by $V_p = \ker d\psi_p$. The orthogonal complement of V_p in T_pM is denoted by H_p and called the horizontal space at p. For any $X \in T_pM$, the tangent vector X can be decomposed as follows:

$$X = X^H + X^V, \tag{30}$$

where $X^V \in V_p$ and $X^H \in H_p$. The map ψ is called a horizontally conformal map if there exists a positive function $\mu \in C^\infty(M)$ such that

$$h(d\psi(X), d\psi(Y)) = \mu^2 g(X,Y), \tag{31}$$

for any $X, Y \in H_p$ and $p \in M$. The function μ is said to be the dilation of ψ.

Remark 3. In Relativistic Astrophysics, conformal mappings are used as a mathematical mechanism to obtain exact solutions to the Einstein field equations in general relativity. The behaviour of the space-time geometry quantities is given under a conformal transformation, and the Einstein field equations are exhibited for a perfect fluid distribution matter configuration [3].

The following Theorem 4 and Corollary 1 provide examples of [[12], Theorem 5.1].

Theorem 4. *Let $\psi: (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal α–harmonic map with dilation μ where $m > n$ and $n > 2\alpha$. Then, the fibers of ψ are minimal submanifold if and only if gradμ^2 is vertical.*

Proof 4. For $p \in M$, choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ near p, such that $e_1, \ldots, e_n$ are horizontal vectors and $e_{n+1}, \ldots, e_m$ are vertical tangent vectors. Due to the fact that ψ is horizontally conformal with dilation μ, we get $|d\psi|^2 = n\mu^2$. By (23), it is obtained that

$$\begin{aligned} S_\alpha(\psi) &= \left(1 + n\mu^2\right)^\alpha g \\ &\quad - 2\alpha\left(1 + n\mu^2\right)^{\alpha-1} \psi^* h. \end{aligned} \tag{32}$$

By Proposition 2 and considering the fact that the map ψ is α–harmonic, we get

$$\begin{aligned} 0 &= \operatorname{div}(S_\alpha(\psi))(e_j) \\ &= \sum_{i=1}^m \left(\nabla_{e_i} S_\alpha(\psi)\right)(e_i, e_j) \\ &= n\alpha\left(1 + n\mu^2\right)^{\alpha-1} e_j(\mu^2) \\ &\quad - \sum_{i=1}^n e_i\left\{2\alpha\mu^2\left(1 + n\mu^2\right)^{\alpha-1} h\left(d\psi(e_i), d\psi(e_j)\right)\right\} \\ &\quad + 2\alpha\left(1 + n\mu^2\right)^{\alpha-1} \sum_{i=1}^m \left\{h\left(d\psi\left(\nabla^M_{e_i} e_i\right), d\psi(e_j)\right) + h\left(d\psi(e_i), d\psi\left(\nabla^M_{e_i} e_j\right)\right)\right\}. \end{aligned} \tag{33}$$

For $j\,(1 \le j \le n)$, we have

$$\begin{aligned} 0 &= \sum_{i=1}^n e_i\left(g\left(e_i, e_j\right)\right) \\ &= \sum_{i=1}^n \left\{g\left(\nabla^M_{e_i} e_i, e_j\right) + g\left(e_i, \nabla^M_{e_i} e_j\right)\right\} \\ &= \sum_{i=1}^n \left\{g\left(\left(\nabla^M_{e_i} e_i\right)^H, e_j\right) + g\left(e_i, \left(\nabla^M_{e_i} e_j\right)^H\right)\right\} \\ &= \frac{1}{\mu^2} \sum_{i=1}^n \left\{h\left(d\psi\left(\nabla^M_{e_i} e_i\right)^H, d\psi(e_j)\right) \right. \\ &\quad \left. + h\left(d\psi(e_i), d\psi\left(\nabla^M_{e_i} e_j\right)^H\right)\right\} \\ &= \frac{1}{\mu^2} \sum_{i=1}^n h\left\{\left(d\psi\left(\nabla^M_{e_i} e_i\right), d\psi(e_j)\right)\right. \\ &\quad \left. + h\left(d\psi(e_i), d\psi\left(\nabla^M_{e_i} e_j\right)\right)\right\}. \end{aligned} \tag{34}$$

By making use of (33) and (34), for $1 \le j \le n$, we get

$$\begin{aligned} 0 &= n\alpha\left(1 + n\mu^2\right)^{\alpha-1} e_j(\mu^2) - e_j\left(2\alpha\mu^2\left(1 + n\mu^2\right)^{\alpha-1}\right) \\ &\quad + 2\alpha\left(1 + n\mu^2\right)^{\alpha-1} \sum_{i=n+1}^m \left\{h\left(d\psi\left(\nabla^M_{e_i} e_i\right), d\psi(e_j)\right)\right. \\ &\quad \left. + h\left(d\psi(e_i), d\psi\left(\nabla^M_{e_i} e_j\right)\right)\right\} \\ &= n\alpha\left(1 + n\mu^2\right)^{\alpha-1} e_j(\mu^2) - 2\alpha\left(1 + n\mu^2\right)^{\alpha-1}) e_j(\mu^2) \\ &\quad - 2n\mu^2\alpha(\alpha-1)\left(1 + n\mu^2\right)^{\alpha-2} e_j(\mu^2) \\ &\quad + 2\alpha\left(1 + n\mu^2\right)^{\alpha-1} \sum_{i=n+1}^m h\left(d\psi\left(\nabla^M_{e_i} e_i\right)^H, d\psi(e_j)\right) \\ &= \alpha\left(1 + n\mu^2\right)^{\alpha-2} \left[(n-2) + n\mu^2(n-2\alpha)\right] e_j(\mu^2) \\ &\quad + 2\alpha\mu^2\left(1 + n\mu^2\right)^{\alpha-1} \sum_{i=n+1}^m g\left(\nabla^M_{e_i} e_i, e_j\right). \end{aligned} \tag{35}$$

By (35) and the definition of the mean curvature vector H, of the fiber of ψ, given as follows:

$$H = \frac{1}{m-n}\sum_{j=1}^{n}\sum_{i=n+1}^{m} g\left(\nabla^M_{e_i} e_i, e_j\right)e_j, \tag{36}$$

it is obtained that

$$\begin{aligned}&\alpha\left(1+n\mu^2\right)^{\alpha-2}\left[(n-2)+n\mu^2(n-2\alpha)\right]\left(\operatorname{grad}\mu^2\right)^H\\&\quad+2(m-n)\alpha\left(1+n\mu^2\right)^{\alpha-1}H=0.\end{aligned} \tag{37}$$

From (37) and the assumptions of this theorem, it can be seen that

$$\alpha\left(1+n\mu^2\right)^{\alpha-2}\left[(n-2)+n\mu^2(n-2\alpha)\right]>0. \tag{38}$$

From (37) and (38), Theorem 4 follows. □

Corollary 1. *Under the assumptions of Theorem 4, the fibers of ψ are minimal submanifold if and only if the horizontal distribution has mean curvature grad $\mu^2/2\mu^2$.*

Proof 5. By (33), for $j\,(n+1\le j\le m)$, we get

$$0=\alpha\left(1+n\mu^2\right)^{\alpha-2}\left\{ne_j\left(\mu^2\right)+2\sum_{i=1}^{n}h\left(\mathrm{d}\psi(e_i),\mathrm{d}\psi\left(\nabla^M_{e_i}e_j\right)\right)\right\}. \tag{39}$$

For $i\,(1\le i\le n)$, it is obtained that

$$\begin{aligned}&h\left(\mathrm{d}\psi(e_i),\mathrm{d}\psi\left(\nabla^M_{e_i}e_j\right)\right)\\&\quad=h\left(\mathrm{d}\psi(e_i),\mathrm{d}\psi\left(\nabla^M_{e_i}e_j\right)^H\right)\\&\quad=\mu^2 g\left(e_i,\left(\nabla^M_{e_i}e_j\right)^H\right)\\&\quad=\mu^2 g\left(e_i,\nabla^M_{e_i}e_j\right)\\&\quad=-\mu^2 g\left(\nabla^M_{e_i}e_i,e_j\right).\end{aligned} \tag{40}$$

By (39) and (40), we get

$$ne_j\left(\mu^2\right)-2\mu^2\sum_{i=1}^{n}g\left(\nabla^M_{e_i}e_i,e_j\right)=0. \tag{41}$$

Thus, the mean curvature $\overline{H}$, of the horizontal distribution, is obtained as follows:

$$\begin{aligned}\overline{H}&=\frac{1}{n}\sum_{j=n+1}^{m}\sum_{i=1}^{n}g\left(\nabla^M_{e_i}e_i,e_j\right)e_j\\&=\frac{1}{2\mu^2}\sum_{j=n+1}^{m}e_j\left(\mu^2\right)e_j\\&=\frac{1}{2\mu^2}\left(\operatorname{grad}\mu^2\right)^V.\end{aligned} \tag{42}$$

From Theorem 4 and equation (42), Corollary 1 follows. □

4. Second Variational Formula

In this section, the second variation formula of the α–energy functional is calculated by using the Jacobi operator and Stokes' theorem, illuminating [[12], Theorem 6.1] in which $E_F = E_\alpha$.

Remark 4. In physics, the Jacobi operator plays a key role to solve the nonlinear boundary value of Troesch's equation. This equation arises in the investigation of the confinement of a plasma column by radiation pressure and also in the theory of gas porous electrodes [14, 15].

Theorem 5. *(The second variation formula) Let ψ: $(M, g)\longrightarrow(N,h)$ be an α–harmonic map and $\left\{\psi_{t,s}\colon M\longrightarrow N\right\}_{-\varepsilon<s,t<\varepsilon}$ be a 2-parameter smooth variation of ψ such that $\psi_{0,0}=\psi$. Then,*

$$\frac{\partial^2}{\partial t\,\partial s}E_\alpha(\psi)|_{t=s=0}=-\int_M h\left(J_\alpha(v),\omega\right), \tag{43}$$

where

$$\begin{aligned}v&=\frac{\partial\psi_{t,s}}{\partial t}|_{s=t=0},\\\omega&=\frac{\partial\psi_{t,s}}{\partial s}|_{s=t=0},\end{aligned} \tag{44}$$

and $J_\alpha(v)\in\Gamma\left(\psi^{-1}TN\right)$ is given by

$$\begin{aligned}J_\alpha(v)&=2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\mathrm{Tr}_g R^N(v,\mathrm{d}\psi)\mathrm{d}\psi\\&\quad+4\alpha(\alpha-1)\mathrm{Tr}_g\nabla\langle\nabla v,\mathrm{d}\psi\rangle\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-2}\mathrm{d}\psi\\&\quad+2\alpha\mathrm{Tr}_g\nabla\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla v,\end{aligned} \tag{45}$$

where R^N is the curvature tensor on (N,h), and $\langle,\rangle$ denotes the inner product on $T^*M\otimes\psi^{-1}TN$.

Proof 6. Let $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ be a product manifold that is equipped with the product metric, and let the natural extension of $\partial/\partial t$ on $(-\varepsilon,\varepsilon)$, $\partial/\partial s$ on $(-\varepsilon,\varepsilon)$, and $\partial/\partial x$ on M to the product manifold $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ is denoted by $\partial/\partial t, \partial/\partial s$, and $\partial/\partial x$ again, respectively. Assume that Ψ: $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M\longrightarrow N$ be a smooth map defined by $\Psi(t,s,x)=\psi_{t,s}(x)$. The same notation ∇ shall be used for the induced connection on $\Psi^{-1}TN$. Consider an orthonormal frame $\{e_i\}$ with respect to g on M. Then, by (4), we have

$$\begin{aligned}&\frac{\partial^2}{\partial t\,\partial s}E_\alpha\left(\psi_{t,s}\right)|_{s=t=0}\\&\quad=\int_M\frac{\partial^2}{\partial t\,\partial s}\left(1+\left|\mathrm{d}\psi_{t,s}\right|^2\right)^\alpha|_{s=t=0}\mathrm{d}V_g.\end{aligned} \tag{46}$$

By computing the second derivative, we get

$$\begin{aligned}
&\frac{\partial^2}{\partial t\,\partial s}\left(1+\left|\mathrm{d}\psi_{t,s}\right|^2\right)^{\alpha}\\
&=\left(\frac{\partial A}{\partial s}\right)h\left(\nabla_{\frac{\partial}{\partial t}} d\psi_{t,s}\left(e_i\right),\mathrm{d}\psi_{t,s}\left(e_i\right)\right)\\
&\quad+A\left\{h\left(\nabla_{\partial/\partial s}\mathrm{d}\psi_{t,s}\left(e_i\right),\nabla_{\partial/\partial s}\mathrm{d}\psi_{t,s}\left(e_i\right)\right)+h\left(\nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\mathrm{d}\psi_{t,s}\left(e_i\right),\mathrm{d}\psi_{t,s}\left(e_i\right)\right)\right\},
\end{aligned}\tag{47}$$

where $A=2\alpha\left(1+\left|\mathrm{d}\psi_{t,s}\right|^2\right)^{\alpha-1}$. Due to the fact that

$$\frac{\partial A}{\partial s}\Big|_{s=t=0}=Bh\left(\nabla_{e_i}\omega,\mathrm{d}\psi\left(e_i\right)\right),\tag{48}$$

where $B=4\alpha(\alpha-1)\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-2}$, and the first term of the right-hand side of (47) can be obtained as follows:

$$\begin{aligned}
&\frac{\partial A}{\partial s}h\left(\nabla_{\partial/\partial t}\mathrm{d}\psi_{t,s}\left(e_i\right),\mathrm{d}\psi_{t,s}\left(e_i\right)\right)\Big|_{s=t=0}\\
&=B\langle\nabla v,\mathrm{d}\psi\rangle h\left(\nabla_{e_i}\omega,\mathrm{d}\psi\left(e_i\right)\right)\\
&=e_i\left(h\left(\omega,B\langle\nabla v,\mathrm{d}\psi\rangle\mathrm{d}\psi\left(e_i\right)\right)\right)\\
&\quad-h\left(\omega,\nabla_{e_i}\left\{B\langle\nabla v,\mathrm{d}\psi\rangle\mathrm{d}\psi\left(e_i\right)\right\}\right).
\end{aligned}\tag{49}$$

Furthermore, by calculating the second term of the right-hand side of (47), we get

$$\begin{aligned}
&Ah\left(\nabla_{\partial/\partial t}\mathrm{d}\psi_{t,s}\left(e_i\right),\nabla_{\partial/\partial s}\mathrm{d}\psi_{t,s}\left(e_i\right)\right)\Big|_{s=t=0}\\
&=e_i\left(h\left(2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla_{e_i}v,\omega\right)\right)\\
&\quad-h\left(\nabla_{e_i}\left(2\alpha\nabla_{e_i}\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla_{e_i}v\right),\omega\right).
\end{aligned}\tag{50}$$

By the definition and properties of the curvature tensor of (N,h) and the compatibility of ∇ with the metric h, the last term of the right-hand side of (47) can be obtained as follows:

$$\begin{aligned}
&Ah\left(\nabla_{/\frac{\partial}{\partial s}}\nabla_{\partial/\partial t}\mathrm{d}\psi_{t,s}\left(e_i\right),\mathrm{d}\psi_{t,s}\left(e_i\right)\right)\Big|_{s=t=0}\\
&=-2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}h\left(R^N\left(v,\mathrm{d}\psi\left(e_i\right)\right)\mathrm{d}\psi\left(e_i\right),\omega\right)\\
&\quad+e_i\left(h\left(\nabla_{\partial/\partial s}\mathrm{d}\psi_{t,s}\left(\frac{\partial}{\partial t}\right),A\,d\psi_{t,s}\left(e_i\right)\right)\right)\Big|_{s=t=0}\\
&\quad-h\left(\nabla_{\partial/\partial s}\mathrm{d}\psi_{t,s}\left(\frac{\partial}{\partial t}\right),\nabla_{e_i}\left(A\,d\psi_{t,s}\left(e_i\right)\right)\right)\Big|_{s=t=0}.
\end{aligned}\tag{51}$$

From equations (46), (47), (49), (50), and (44) and considering Green's theorem and α–harmonicity of ψ, Theorem 5 follows. □

4.1. Stability of α-Harmonic Maps. In this section, the stability of α–harmonic maps is studied. First, we showed that any α–harmonic map from a Riemannian manifold to a Riemannian manifold with nonpositive curvature is stable. Then, the stability of α–harmonic maps from a compact manifold to a unit standard sphere is investigated.

Remark 5. The stability of harmonic maps plays a key role in mathematical physics and mechanics, [5]. For instance, the linearized Vlasov–Maxwell equations are used to investigate harmonic stability properties for planar wiggler free-electron laser (FEL). It is worth noting that the analysis is carried out in the Compton regime for a tenuous electron beam propagating in the z direction through the constant amplitude planar wiggler magnetic field $B^0=-B_{\omega}\cos k_0 z\hat{e}_x$ [16].

Definition 2. Under the assumptions of Theorem 5, setting

$$I(v,\omega)=\frac{\partial^2}{\partial t\,\partial s}E_{\alpha}\left(\psi_{t,s}\right)\Big|_{s=t=0}.\tag{52}$$

Then, ψ is said to be stable α–harmonic if $I(v,v)\geq 0$ for any compactly supported vector field v along ψ.

From Theorem 5, we obtain the following corollary and provide examples of [[12], Theorem 6.2], in which F–harmonic map is α–harmonic map.

Corollary 2. *Let N be a Riemannian manifold with nonpositive Riemannian curvature. Then, any α–harmonic map ψ: $(M,g)\longrightarrow(N,h)$ is stable.*

Proof 7. Setting

$$\delta(X)=2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}h\left(\nabla_X v,\omega\right),\tag{53}$$

$$\begin{aligned}\eta(X)&=4\alpha(\alpha-1)\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-2}\\&\quad\cdot\langle\nabla v,\mathrm{d}\psi\rangle h\left(\mathrm{d}\psi(X),\omega\right),\end{aligned}\tag{54}$$

for any $X\in\chi(M)$. Then, we have

$$\begin{aligned}
&-h\left(2\alpha\mathrm{Tr}_g\nabla\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla v,\omega\right)\\
&=-\sum_{i=1}^{m}h\left(2\alpha\nabla_{e_i}\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla_{e_i}v,\omega\right)\\
&=-\sum_{i=1}^{m}\left\{e_i\left(h\left(2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla_{e_i}v,\omega\right)\right)\right.\\
&\quad\left.+h\left(2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\nabla_{e_i}v,\nabla_{e_i}\omega\right)\right\}\\
&=-\mathrm{div}\delta+2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}\langle\nabla v,\nabla\omega\rangle.
\end{aligned}\tag{55}$$

On the other hand, by (54), we get

$$\begin{aligned}
&-h\Big(\mathrm{Tr}_g\nabla\langle\nabla v,\mathrm{d}\psi\rangle B\mathrm{d}\psi,\omega\Big)\\
&=\sum_{i=1}^{m}\Big\{-h\Big(\nabla_{e_i}\langle\nabla v,\mathrm{d}\psi\rangle B\mathrm{d}\psi(e_i),\omega\Big)\Big\}\\
&=\sum_{i=1}^{m}\{-e_i\left(h\left(\langle\nabla v,\mathrm{d}\psi\rangle B\mathrm{d}\psi(e_i),\omega\right)\right)\\
&\quad+h\Big(\langle\nabla v,\mathrm{d}\psi\rangle B\mathrm{d}\psi(e_i),\nabla_{e_i}\omega\Big)\}\\
&=-\mathrm{div}\eta+B\langle\nabla v,\mathrm{d}\psi\rangle\langle\nabla\omega,\mathrm{d}\psi\rangle,
\end{aligned}\tag{56}$$

where $B=4\alpha(\alpha-1)(1+|\mathrm{d}\psi|^2)^{\alpha-2}$. By substituting (55) and (56) in (45) and using the Divergence theorem and Definition 2, we get

$$\begin{aligned}
&I(v,v)\\
&=\int_M\Big\{4\alpha(\alpha-1)\Big(1+|\mathrm{d}\psi|^2\Big)^{\alpha-2}\langle\nabla v,\mathrm{d}\psi\rangle^2\\
&\quad-2\alpha\Big(1+|\mathrm{d}\psi|^2\Big)^{\alpha-1}h\Big(\mathrm{Tr}_gR^N(v,\mathrm{d}\psi)\mathrm{d}\psi,v\Big)\\
&\quad+2\alpha\Big(1+|\mathrm{d}\psi|^2\Big)^{\alpha-1}|\nabla v|^2\}\mathrm{d}V_g.
\end{aligned}\tag{57}$$

By (57) and the assumptions, Corollary 2 follows.

Now, the stability of α−harmonic maps from a compact without boundary Riemannian manifold to a standard unit sphere is studied. As in [12], an extrinsic average variational method of Wei ([17, 18]) is employed.

Consider a unit standard sphere $\mathbb{S}^n$ as a submanifold in $(n+1)$−dimensional Euclidean space $\mathbb{R}^{n+1}$. Denote the Levi-Civita connections on $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$ by ∇^S and ∇^R, respectively. At $p\in\mathbb{S}^n$, any vector V in $\mathbb{R}^{n+1}$ can be decomposed as follows:

$$V=V^{\top}+V^{\perp},\tag{58}$$

where $V^{\perp}=\langle V,p\rangle p$ is the normal part to $\mathbb{S}^n$, and $V^{\top}$ is the tangent part to $\mathbb{S}^n$. The second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is denoted by B and defined as follows:

$$B(X,Y)=-\langle X,Y\rangle p,\tag{59}$$

where X,Y are tangent vectors of $\mathbb{S}^n$ at p and $\langle,\rangle$ is the Euclidean metric on $\mathbb{R}^{n+1}$. Furthermore, the shape operator A^W corresponding to a normal vector field W on $\mathbb{S}^n$ is defined by

$$A^W(X)=-\Big(\nabla_X^R W\Big)^{\top},\tag{60}$$

where X is a tangent vector field on $\mathbb{S}^n$. Noting that, the second fundamental form and the shape operator of $\mathbb{S}^n$ are satisfied by the following equation:

$$\begin{aligned}
\langle B(X,Y),W\rangle&=\langle A^W(X),Y\rangle\\
&=-\langle X,Y\rangle\langle p,W\rangle,
\end{aligned}\tag{61}$$

for any tangent vectors X and Y of $\mathbb{S}^n$ at p.

From the sectional curvature of $\mathbb{S}^n$ and using (59), (60), and (61), the following lemma is obtained. □

Lemma 1. *Let $\psi\colon (M,g)\longrightarrow\mathbb{S}^n$ be a smooth map, and let Λ is a parallel vector field in $\mathbb{R}^{n+1}$. Then, at $p\in\mathbb{S}^n$, we have*

(1) $\nabla_X\Lambda^{\top}=A^{\Lambda^{\perp}}(d\psi(X))$,

(2) $\langle\nabla_X\Lambda^{\top},d\psi(X)\rangle=-|d\psi(X)|^2\langle p,\Lambda\rangle$,

(3) $|\nabla_X\Lambda^{\top}|^2=|d\psi(X)|^2\langle p,\Lambda\rangle^2$,

(4) $\langle R^S(\Lambda^{\top},d\psi(X))d\quad\psi(X),\Lambda^{\top}\rangle=|d\psi(X)|^2\quad|\Lambda^{\top}|^2-\langle d((X)),\Lambda\rangle^2$,

where ∇ and R^S denote the induced connection on $\psi^{-1}T\mathbb{S}^n$ and the curvature tensor of $\mathbb{S}^n$, respectively.

Proof 8. By (58) and (60), we have

$$\begin{aligned}
\nabla_X\Lambda^{\top}&=\nabla^S_{\mathrm{d}\psi(X)}\Lambda^{\top}=\Big(\nabla^R_{\mathrm{d}\psi(X)}\Lambda^{\top}\Big)^{\top}\\
&=\Big(\nabla^R_{\mathrm{d}\psi(X)}\Lambda-\Lambda^{\perp}\Big)^{\top}=-\Big(\nabla^R_{\mathrm{d}\psi(X)}\Lambda^{\perp}\Big)^{\top}\\
&=A^{\Lambda^{\perp}}(\mathrm{d}\psi(X)).
\end{aligned}\tag{62}$$

Making use of (61) and (62), we get

$$\begin{aligned}
\langle\nabla_X\Lambda^{\top},\mathrm{d}\psi(X)\rangle&=\langle A^{\Lambda^{\perp}}(\mathrm{d}\psi(X)),\mathrm{d}\psi(X)\rangle\\
&=-|\mathrm{d}\psi(X)|^2\langle p,\Lambda^{\perp}\rangle\\
&=-|\mathrm{d}\psi(X)|^2\langle p,\Lambda\rangle\\
\left|\nabla_X\Lambda^{\top}\right|^2&=\langle\nabla_X\Lambda^{\top},\nabla_X\Lambda^{\top}\rangle\\
&=\langle A^{\Lambda^{\perp}}(\mathrm{d}\psi(X)),A^{\Lambda^{\perp}}(\mathrm{d}\psi(X))\rangle\\
&=-\langle A^{\Lambda^{\perp}}(\mathrm{d}\psi(X)),\mathrm{d}\psi(X)\rangle\langle p,\Lambda^{\perp}\rangle\\
&=\langle\mathrm{d}\psi(X),\mathrm{d}\psi(X)\rangle\langle p,\Lambda^{\perp}\rangle^2\\
&=|\mathrm{d}\psi(X)|^2\langle p,\Lambda\rangle^2.
\end{aligned}\tag{63}$$

Due to the fact that the sectional curvature of $\mathbb{S}^n$ is constant, we get

$$\begin{aligned}
&\langle R^S(\Lambda^{\top},\mathrm{d}\psi(X))\mathrm{d}\psi(X),\Lambda^{\top}\rangle\\
&=|\mathrm{d}\psi(X)|^2\left|\Lambda^{\top}\right|^2-\mathrm{d}\psi(X),\Lambda^{\top 2}\\
&=|\mathrm{d}\psi(X)|^2\left|\Lambda^{\top}\right|^2-\mathrm{d}\psi(X),\Lambda^2,
\end{aligned}\tag{64}$$

and hence completes the proof. □

Theorem 6. *Let M be a compact without boundary manifold and let $\psi\colon (M,g)\longrightarrow\mathbb{S}^n$ be an α−harmonic map such that $|d\psi|^2<n-2/2\alpha-n$. Then, ψ is unstable.*

Proof 9. Let $\{\Lambda_k\}_{k=1}^{n+1}$ be a parallel orthonormal frame field in $\mathbb{R}^{n+1}$ and $\{e_i\}_{i=1}^{m}$ be a local orthonormal frame field on M, and let $R^{\mathbb{S}}$ and ∇ denote the curvature tensor of $\mathbb{S}^n$ and the induced connection on $\psi^{-1}(T\mathbb{S}^n)$, respectively. By (57), we have

$$\begin{aligned}
&\sum_{k=1}^{n+1} I\left(\Lambda_k^{\top}, \Lambda_k^{\top}\right)\\
&\quad= \int_M \left\{ B \sum_{k=1}^{n+1}\sum_{i=1}^{m} \langle \nabla_{e_i}\Lambda_k^{\top}, \mathrm{d}\psi(e_i)\rangle^2 \right.\\
&\qquad + 2\alpha\left(1 + |\mathrm{d}\psi|^2\right)^{\alpha-1} \sum_{k=1}^{n+1}\sum_{i=1}^{m}\left\{\left|\nabla_{e_i}\Lambda_k^{\top}\right|^2\right.\\
&\qquad - \langle R^S\left(\Lambda_k^{\top}, \mathrm{d}\psi(e_i)\right)\mathrm{d}\psi(e_i), \Lambda_k^{\top}\rangle\}\mathrm{d}V_g,
\end{aligned} \tag{65}$$

where $B = 4\alpha(\alpha-1)(1+|\mathrm{d}\psi|^2)^{\alpha-2}$. According to Lemma 1's second assertion, the first term of the right-hand side of (65) can be obtained as follows:

$$\begin{aligned}
&\sum_{k=1}^{n+1}\sum_{i=1}^{m} \langle \nabla_{e_i}\Lambda_k^{\top}, \mathrm{d}\psi(e_i)\rangle^2\\
&\quad= \sum_{k=1}^{n+1}\sum_{i=1}^{m}\left(-|\mathrm{d}\psi(e_i)|^2\langle p, \Lambda_k\rangle\right)^2\\
&\quad= \sum_{k=1}^{n+1}|\mathrm{d}\psi|^4\langle p, \Lambda_k\rangle^2\\
&\quad= |\mathrm{d}\psi|^4|p|^2,\\
&\quad= |\mathrm{d}\psi|^4.
\end{aligned} \tag{66}$$

By virtue of Lemma 1's third claim, the second term of the right-hand side of (65) can be calculated as follows:

$$\begin{aligned}
&\sum_{k=1}^{n+1}\sum_{i=1}^{m}\left|\nabla_{e_i}\Lambda_k^{\top}\right|^2\\
&\quad= \sum_{k=1}^{n+1}\sum_{i=1}^{m}|\mathrm{d}\psi(e_i)|^2\langle p, \Lambda_k\rangle^2\\
&\quad= \sum_{k=1}^{n+1}|\mathrm{d}\psi|^2\langle p, \Lambda_k\rangle^2\\
&\quad= |\mathrm{d}\psi|^2|p|^2\\
&\quad= |\mathrm{d}\psi|^2.
\end{aligned} \tag{67}$$

Since the sectional curvature of $\mathbb{S}^{n+1}$ is constant and using Lemma 1's fourth assertion, the last term of the right-hand side of (65) can be calculated as follows:

$$\begin{aligned}
&\sum_{k=1}^{n+1}\sum_{i=1}^{m}\langle R^S\left(\Lambda_k^{\top}, \mathrm{d}\psi(e_i)\right)\mathrm{d}\psi(e_i), \Lambda_k^{\top}\rangle\\
&\quad= \sum_{k=1}^{n+1}\sum_{i=1}^{m}|\mathrm{d}\psi(e_i)|^2\left|\Lambda_k^{\top}\right|^2 - \langle \mathrm{d}\psi(e_i), \Lambda_k\rangle^2\\
&\quad= -|\mathrm{d}\psi|^2 + |\mathrm{d}\psi|^2\sum_{k=1}^{n+1}\left|\Lambda_k - \Lambda_k^{\perp}\right|^2\\
&\quad= -|\mathrm{d}\psi|^2 + |\mathrm{d}\psi|^2\sum_{k=1}^{n+1}\left|\Lambda_k - \langle\Lambda_k, p\rangle p\right|^2\\
&\quad= -|\mathrm{d}\psi|^2 + |\mathrm{d}\psi|^2\sum_{k=1}^{n+1}\left(|\Lambda_k|^2 - \langle\Lambda_k, p\rangle^2\right)\\
&\quad= -|\mathrm{d}\psi|^2 + |\mathrm{d}\psi|^2\left\{(n+1) - |p|^2\right\}\\
&\quad= (n-1)|\mathrm{d}\psi|^2.
\end{aligned} \tag{68}$$

By substituting (66), (67), and (68) in (65), we get

$$\begin{aligned}
&\sum_{k=1}^{n+1} I\left(\Lambda_k^{\top}, \Lambda_k^{\top}\right)\\
&\quad= \int_M \left\{4\alpha(\alpha-1)\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-2}|\mathrm{d}\psi|^4\right.\\
&\qquad + 2\alpha\left(1+|\mathrm{d}\psi|^2\right)^{\alpha-1}(2-n)|\mathrm{d}\psi|^2\}\mathrm{d}V_g\\
&\quad= \int_M C\left\{(2-n) + (2\alpha - n)|\mathrm{d}\psi|^2\mathrm{d}V_g,\right.
\end{aligned} \tag{69}$$

where $C = 2\alpha(1+|\mathrm{d}\psi|^2)^{\alpha-1}|\mathrm{d}\psi|^2$. By (69) and the assumptions, it is obtained that

$$\sum_{k=1}^{n+1} I\left(\Lambda_k^{\top}, \Lambda_k^{\top}\right) < 0. \tag{70}$$

Then, ψ is unstable and hence completes the proof. □

Data Availability

No data were used to support this study.

Disclosure

An earlier version of the manuscript has been presented in [19] as a preprint.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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