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Research Article

The Perfect Roman Domination Number of the Cartesian Product of Some Graphs

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A perfect Roman dominating function on a graph G is a function $f\colon V(G) \longrightarrow \{0,1,2\}$ for which every vertex v with $f(v) = 0$ is adjacent to exactly one neighbor u with $f(u) = 2$. The weight of f is the sum of the weights of the vertices. The perfect Roman domination number of a graph G, denoted by $\gamma_R^p(G)$, is the minimum weight of a perfect Roman dominating function on G. In this paper, we prove that if G is the Cartesian product of a path P_r and a path P_s, a path P_r and a cycle C_s, or a cycle C_r and a cycle C_s, where $r, s > 5$, then $\gamma_R^p(G) \leq (2/3)|G|$.

1. Introduction and Preliminaries

All graphs considered in this work are simple, finite, and undirected. Let $G = (V, E)$ be a graph. We denote the cardinality of V by $|G|$. Two vertices $u, v \in V$ are adjacent when $uv \in E$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u | uv \in E\}$ while the closed neighborhood of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The number of vertices of a path is its length. We denote a path of length n by P_n. We denote the cycle graph with n vertices by C_n.

A dominating set of a graph G is a subset D of V where each vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D. The domination number is the minimum cardinality of a dominating set of G, and it is usually denoted by $\gamma(G)$. There is a large literature that covered the domination number. For basic definitions and concepts relating to this subject, we refer the reader to[1].

A perfect dominating set of a graph G is a subset S of V where each vertex $v \in V(G)$ satisfies $|N[v] \cap S| = 1$. The perfect domination number is the minimum cardinality of a perfect domination set of G, and it is usually denoted by $\gamma^p(G)$. The study of perfect domination has received much attention in the literature, see for example [2, 3].

A Roman dominating function on a graph G, denoted by RD-function, is a function $f\colon V(G) \longrightarrow \{0,1,2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. For any vertex v, the weight of v is its value under the function f while the weight of f, denoted by $w(f)$, is the sum $\sum_{v\in V(G)} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a RD-function, i.e.,

$$\gamma_R(G) = \min\{w(f) | f \text{ is a RD} - \text{function on } G\}. \qquad (1)$$

Roman domination has been studied well and there are many research papers on this subject, such as [4, 5]. There are some variations on domination number and Roman domination number have been appeared in the literature such as total, week, and perfect [6–10]. In this paper, we continue the investigation of perfect Roman domination.

A perfect Roman dominating function on a graph G is a function $f\colon V(G) \longrightarrow \{0,1,2\}$ for which every vertex v with $f(v) = 0$ is adjacent to exactly one neighbor u with $f(u) = 2$. We denote a perfect Roman dominating function by PRD-function. The weight of f, denoted by $w(f)$, is the sum $\sum_{v\in V(G)} f(v)$. The perfect Roman domination number of a graph G, denoted by $\gamma_R^p(G)$, is the minimum weight of a PRD-function, i.e.,

$$\gamma_R^p(G) = \min\{w(f) | f \text{ is a PRD} - \text{function on } G\}. \quad (2)$$

The investigation of perfect Roman domination was initiated by Henning et al. in [7] on trees and then Henning and Klostermeyer considered this subject on regular graphs [11]. More recent work on perfect Roman domination can be found in [12–15].

Roman domination in product graphs has become an attractive topic in the study of domination and much work has been done in this area such as Cartesian product [16], lexicographic product [6, 17], rooted product [8, 18], and direct product [18]. Perfect Roman domination in product graphs has been considered for corona product in [13]. In this work, we studied perfect Roman domination in the Cartesian product of paths and paths, paths and cycles, and cycles and cycles.

Let H_1 and H_2 be two graphs. The Cartesian product graph of H_1 and H_2, denoted by $H_1 \square H_2$, is the graph with $V(H_1) \times V(H_2)$ as its set of vertices, and two vertices (u, v), $(u', v') \in H_1 \square H_2$ are adjacent if either

(1) $u = u'$ and $vv' \in E(H_2)$ or

(2) $v = v'$ and $uu' \in E(H_1)$

The graph $P_r \square P_s$ is a grid graph with r columns and s rows, see Figure 1.

We denote the vertex in row i and column j by $a_{i,j}$. The graph $P_r \square C_s$ is a cylinder grid graph which is a grid graph, with r columns and s rows, and some extra edges between the vertices of the first and the last rows, see Figure 2. The graph $C_r \square C_s$ is a torus grid graph which is a cylinder grid graph with some extra edges between the vertices of the first and last columns, see Figure 3. So $V(P_r \square P_s) = V(P_r \square C_s) = V(C_r \square C_s)$ and $E(P_r \square P_s) \subseteq E(P_r \square C_s) \subseteq E(C_r \square C_s)$.

2. Discussion

In this section, we present an upper bound for the perfect Roman domination number of a grid graph, a cylinder grid graph, and a torus grid graph.

Theorem 1. *Let $r, s > 5$. If $G \in \{P_r \square P_s,\ P_r \square C_s,\ C_r \square C_s\}$, $\gamma_R^p(G) \le (2/3)|G|$.*

Proof. The statement will be a result from the following three cases. □

Case 1. $r = 3k$ or $s = 3k$ for some integer k. If $r = 3k$, label each vertex in column number $2 + 3m, m \in \{0, 1, \ldots, k-1\}$ with 2, and label the remainder vertices with 0. See Figure 4. It is not hard to see that this labeling produces a perfect Roman domination function of weight equals to $(2/3)|G|$ where $G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$. In a similar way, if $s = 3k$, label each vertex in row number $2 + 3m, m \in \{0, 1, \ldots, k-1\}$ with 2, and label the remainder vertices with 0.

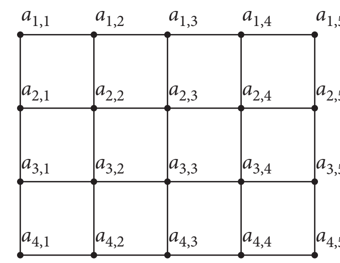

Figure 1: Grid graph $P_5 \square P_4$.

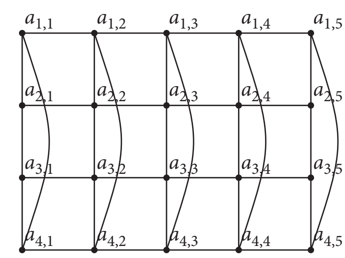

Figure 2: Cylinder grid graph $P_5 \square C_4$.

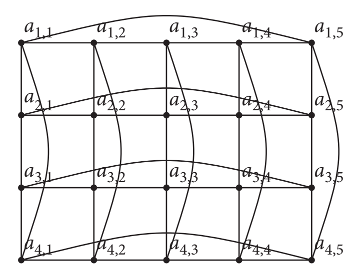

Figure 3: Torus grid graph $C_5 \square C_4$.

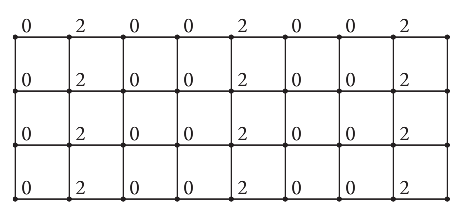

Figure 4: $r = 3k$.

We need the following function for the remaining cases. Define a function $f: V(G) \longrightarrow \{0, 1, 2\}$ as follows:

$$f(a_{i,j}) = \begin{cases} 2, & \text{if } i \equiv 0 \mod 3 \text{ and } j \equiv 3 \mod 6, \\ 2, & \text{if } i \equiv 1 \mod 3 \text{ and } j \equiv 1 \mod 6, \\ 2, & \text{if } i \equiv 2 \mod 3 \text{ and } j \equiv 5 \mod 6, \\ 1, & \text{if } i \equiv 0 \mod 3 \text{ and } j \equiv 0 \mod 6, \\ 1, & \text{if } i \equiv 1 \mod 3 \text{ and } j \equiv 4 \mod 6, \\ 1, & \text{if } i \equiv 2 \mod 3 \text{ and } j \equiv 2 \mod 6, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

The function f has a pattern recurring every six columns, and it also has a pattern recurring every three rows, as shown in Figure 5. It is not hard to see that every vertex $a_{i,j}$

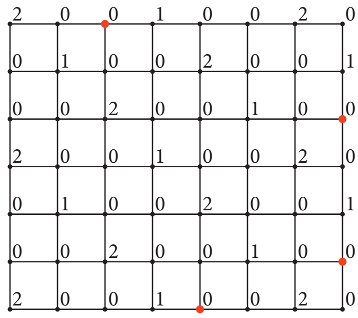

Figure 5: The function f.

with $f(a_{i,j}) = 0$ has exactly one neighbor labelled 2, except some vertices in the first and last rows and possibly in the first and last columns. So we need to modify f slightly depending on the values of r and s.

Case 2. $r = 3k+1$. If s is a multiple of three, we are in a situation symmetric to Case 1, and we are done. So we may assume that s is not a multiple of three. We divide this case to two subcases.

Case 2.1. $s = 3l+1$ for some integer l. If $r \equiv 1 \mod 6$, define a function $f'\colon V(G) \longrightarrow \{0,1,2\}$ such that

$$f'(a_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \equiv 3 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 5 \mod 6, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{4}$$

Then, f' is a PRD-function on G where $G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 6. Note that if $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 2$. If $j \equiv 2 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3)$. If $j \equiv 3 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 1$. If $j \equiv 4 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3) + 1$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 1$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3)$. Therefore,

$$\begin{aligned} w(f') &= \left(\frac{9}{3}(s-1) + 5\right)\frac{k}{2} + \frac{2}{3}(s-1) + 2 \\ &= (3(s-1)+5)\left(\frac{r-1}{6}\right) + \frac{2}{3}(s-1) + 2 \\ &= \frac{1}{2}sr + \frac{1}{6}s + \frac{1}{3}r + 1 \\ &< \frac{1}{2}sr + \frac{1}{6}sr \\ &= \frac{2}{3}sr = \frac{2}{3}|G|. \end{aligned} \tag{5}$$

Inequality (5) follows from the fact that $r, s \geq 7$.

If $r \equiv 4 \mod 6$, define a function $f'\colon V(G) \longrightarrow \{0,1,2\}$ such that

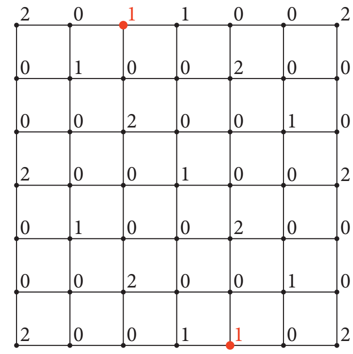

Figure 6: The function f' where $r = 3k+1$, $s = 3l+1$ and $r \equiv 1 \mod 6$.

$$f'(a_{i,j}) = \begin{cases} 2, & \text{if } i = 1 \text{ and } j = r, \\ 1, & \text{if } i = s \text{ and } j \equiv 5 \mod 6, \\ 1, & \text{if } i = 1,\ j \equiv 3 \mod 6 \text{ and } j \neq r-1, \\ 1, & \text{if } i \equiv 2 \mod 3,\ i \neq 2 \text{ and } j = r, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{6}$$

Then, f' is a PRD-function on G where $G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 7. If $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 2$. If $j \equiv 2 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3)$. If $j \equiv 3 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 1$. If $j \equiv 4 \mod 6$ and $j \neq r$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3) + 1$. If $j = r$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 1$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-1)/3) + 1$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-1)/3)$. Thus,

$$\begin{aligned} w(f') &= (3(s-1)+5)\left(\frac{r-4}{6}\right) + \frac{7}{3}(s-1) + 4 \\ &= \frac{1}{2}sr + \frac{1}{3}s + \frac{1}{3}r + \frac{1}{3} \\ &< \frac{1}{2}sr + \frac{1}{6}sr \\ &= \frac{2}{3}sr = \frac{2}{3}|G|. \end{aligned} \tag{7}$$

Inequality (7) follows from the fact that $r, s \geq 7$.

Case 2.2. $s = 3l+2$ for some integer l. Assume that $r \equiv 1 \mod 6$. Define a function $f'\colon V(G) \longrightarrow \{0,1,2\}$ such that

$$f'(a_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \equiv 3 \mod 6, \\ 1, & \text{if } i = 1 \text{ and } j \equiv 5 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 1 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 3 \mod 6, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{8}$$

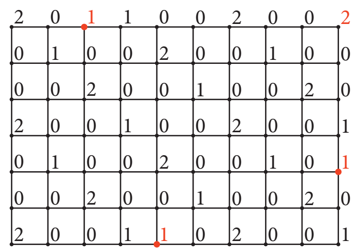

Figure 7: The function f' where $r = 3k + 1$, $s = 3l + 1$ and $r \equiv 4 \mod 6$.

Then, f' is a PRD-function on G where $G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 8.
If $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 2 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 3 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 2$. If $j \equiv 4 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3)$. Therefore,

$$\begin{aligned} w(f') &= (3(s-2) + 10)\left(\frac{r-1}{6}\right) + \frac{2}{3}(s-2) + 3 \\ &= \frac{1}{2}sr + \frac{1}{6}s + \frac{2}{3}r + 1, \\ &< \frac{1}{2}sr + \frac{1}{6}sr = \frac{2}{3}sr. \end{aligned} \tag{9}$$

Inequality (9) follows from the fact that $r \geq 7$ and $s \geq 8$.
Assume that $r \equiv 4 \mod 6$. Define a function $f' \colon V(G) \longrightarrow \{0, 1, 2\}$ such that

$$f'(a_{i,j}) = \begin{cases} 2, & \text{if } i = 1 \text{ and } j = r, \\ 2, & \text{if } i = s \text{ and } j = r, \\ 1, & \text{if } i = 1 \text{ and } j \equiv 5 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 1 \mod 6, \\ 1, & \text{if } i = 1,\ j \equiv 3 \mod 6 \text{ and } j \neq r-1, \\ 1, & \text{if } i = s,\ j \equiv 3 \mod 6 \text{ and } j \neq r-1, \\ 1, & \text{if } i \equiv 2 \mod 3,\ i \notin \{2, s\} \text{ and } j = r, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{10}$$

Then, f' is a PRD-function on G where $G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 9. If $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 2 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 3 \mod 6$ and $j \neq r-1$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 2$. If $j = r-1$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3)$. If $j \equiv 4 \mod 6$ and $j \neq r$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j = r$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3)$. Therefore,

$$\begin{aligned} w(f') &= (3(s-2) + 10)\frac{r-4}{6} + 2(s-2) + 7 \\ &= \frac{1}{2}sr + \frac{2}{3}r + \frac{1}{3}, \\ &< \frac{1}{2}sr + \frac{1}{6}sr = \frac{2}{3}sr. \end{aligned} \tag{11}$$

Inequality (11) follows from the fact that $r, s > 5$.

Remark 1. In both cases, i.e., $r \equiv 1 \mod 6$ and $r \equiv 4 \mod 6$, the function f' is also a PRD-function on the Cartesian product graph$C_r \square P_s$.

Case 3. $r = 3k + 2$. If $s = 3l + 1$ then we are in a situation symmetric to Case 2 (see Remark 1). So we may assume that $s = 3l + 2$.

Assume that $r \equiv 2 \mod 6$. Define a function $f' \colon V(G) \longrightarrow \{0, 1, 2\}$ such that

$$f'(a_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \equiv 3 \mod 6, \\ 1, & \text{if } i = 1 \text{ and } j \equiv 5 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 1 \mod 6, \\ 1, & \text{if } i = s \text{ and } j \equiv 3 \mod 6, \\ 2, & \text{if } i \equiv 1 \mod 3 \text{ and } j = r, \\ 0, & \text{if } i \equiv 2 \mod 3 \text{ and } j = r, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{12}$$

Then, f' is a PRD-function on G where$G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 10. If $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 2 \mod 6$ and $j \neq r$ then $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$, and if $j = r$ then $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 2$. If $j \equiv 3 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 2$. If $j \equiv 4 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3)$. Therefore,

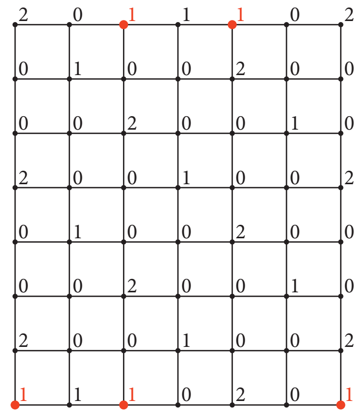

FIGURE 8: The function f' where $r = 3k+1$, $s = 3l+2$ and $r \equiv 1 \mod 6$,

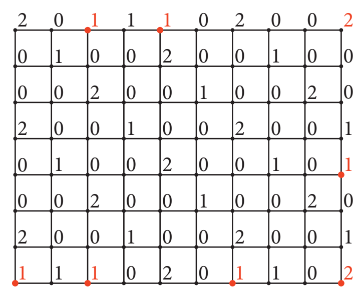

FIGURE 9: The function f' where $r = 3k+1$, $s = 3l+2$ and $r \equiv 4 \mod 6$.

$$w(f') = (3(s-2)+10)\frac{r-2}{6} + \frac{4}{3}(s-2) + 5$$

$$= \frac{1}{2}sr + \frac{1}{3}s + \frac{2}{3}r + 1, \tag{13}$$

$$< \frac{1}{2}sr + \frac{1}{6}sr = \frac{2}{3}sr.$$

Inequality (13) follows from the fact that $r, s \geq 8$. Assume that $r \equiv 5 \mod 6$. Define a function $f': V(G) \longrightarrow \{0,1,2\}$ such that

$$f'(a_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \equiv 3 \mod 6, \\ 1, & \text{if } i = 1,\ j \equiv 5 \mod 6 \text{ and } j \neq r, \\ 0, & \text{if } i \equiv 1 \mod 3 \text{ and } j = r-1, \\ 1, & \text{if } i \equiv 2 \mod 3 \text{ and } j = r-1, \\ 1, & \text{if } i = s \text{ and } j \equiv 3 \mod 6, \\ 2, & \text{if } i \equiv 1 \mod 3 \text{ and } j = r, \\ 0, & \text{if } i \equiv 2 \mod 3,\ i \neq s \text{ and } j = r, \\ 1, & \text{if } i = s \text{ and } j = r, \\ f(a_{i,j}), & \text{otherwise.} \end{cases} \tag{14}$$

Then, f' is a PRD-function on G where$G \in \{P_r \square P_s, P_r \square C_s, C_r \square C_s\}$, see Figure 11. If $j \equiv 1 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 2 \mod 6$,

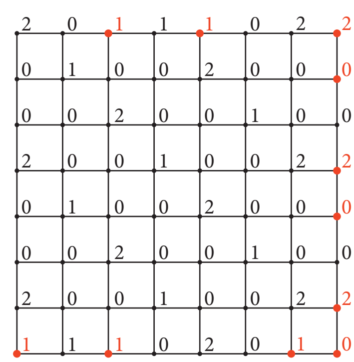

FIGURE 10: The function f' where $r = 3k+2$, $s = 3l+2$ and $r \equiv 2 \mod 6$.

$\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 3 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 2$. If $j \equiv 4 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3) + 1$. If $j \equiv 5 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = 2((s-2)/3) + 3$. If $j \equiv 0 \mod 6$, $\sum_{i=1}^{s} f'(a_{i,j}) = ((s-2)/3)$. Therefore,

$$w(f') = (3(s-2)+10)\frac{r-5}{6} + \frac{8}{3}(s-2) + 10$$

$$= \frac{1}{2}sr + \frac{1}{6}s + \frac{2}{3}r + \frac{4}{3} \tag{15}$$

$$< \frac{1}{2}sr + \frac{1}{6}sr = \frac{2}{3}sr.$$

Inequality (15) follows from the fact that $r, s > 5$.

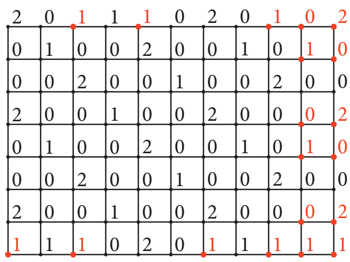

Figure 11: The function f' where $r = 3k + 2$, $s = 3l + 2$ and $r \equiv 5 \mod 6$.

3. Conclusions

In this paper, we discussed the perfect Roman domination number of the Cartesian product of a path and a path, a path and a cycle, and cycle and a cycle. We found that in these three graphs the perfect Roman domination number is not greater than 2/3 of the cardinality of its vertices. Concerning the further plans for our work, we will investigate the perfect Roman domination number of the Cartesian product of trees.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare there are no conflicts of interest.

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