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Research Article

Common Fixed Point Results via Set-Valued Generalized Weak Contraction with Directed Graph and Its Application

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In this manuscript, common fixed point results for set-valued mapping under generalized $(\psi, \phi)_1$ and $(\psi, \phi)_2$ weak contraction without using Hausdorff metric are studied endowing with a graph. To demonstrate the authenticity of the established result, a suitable example and application to integral inclusion are also discussed.

1. Introduction

The basic fixed point (F.P) result of Banach [1] has set the basis of metric fixed point theory (FPT) in a complete metric space for contraction mappings. Due to their results, FPT of specific single-valued mappings is interesting in its benefit owning constructive proofs and applications in industrial fields such as physics, computer science, engineering, image processing, telecommunication, and economics.

The theory of set-valued mapping has applications in convex optimization, control theory, economics and differential inclusions, and integral inclusion. Differential inclusions are also used to analyze modeling errors and analyze robustness to bounded perturbations to model physical phenomena such as to model differential games and Coulomb friction and impact [2, 3]. Following the Banach contraction principle (BCP), Nadler [4] presented the idea of set-valued contractions and established that in complete metric space, a set-valued contraction possesses a F.P. Consequently, many authors generalized Nadler F.P theorem in various ways.

Connecting FPT and graph theory, Echenique [5] provided a proof of Tarski fixed point result by using graphs. Espinola and Kirk [6] in 2006 applied fixed point results in graph theory. Recently, two fundamental results have appeared for FPT with a graph. The first result was given by Jachymski [7] for single-valued mappings and consequently Beg et al. [8] continued Jachymski result for set-valued mappings. Subsequently, Beg et al. [8] extended some results in [9] for set-valued mappings. Recently, Bojor [10] succeeded Jachymsky idea for Kannan contractions applying a new postulate called the weak T-connectivity of the graph. Fallahi and Aghanians formulated Chatterjea contractions using graphs in metric spaces endowed with a graph and investigated the existence of F.Ps. Kutbi and Sintunavarat established F.P analysis for set-valued o with graph approach by the generalized Hausdorff distance. Abbas and Nazir [11] got few results for pair of power graphic contraction on a metric space along with a graph. On metric space along with a graph, Abbas et al. [12] established the presence of common F.Ps of set-valued F- contraction mappings. For more details, see [13, 14].

Let $Q \neq \varnothing$, $\triangle$ represent the diagonal of $Q \times Q$ and $\mathbf{G} = (\mathbf{v}(\mathbf{G}), \mathbf{e}(\mathbf{G}))$ be a directed graph (with no parallel edges). The set $\mathbf{v}(\mathbf{G})$ denoted vertices coincides with Q, and the set $\mathbf{e}(\mathbf{G})$ is the edges of the graph furthermore $\triangle \subseteq \mathbf{e}(\mathbf{G})$. $\mathbf{e}^*(\mathbf{G})$ denotes the set all edges which is not loops, i.e., $\mathbf{e}^*(\mathbf{G}) = \mathbf{e}(\mathbf{G}) - \triangle$. If path exist between any two vertices, then a graph $\mathbf{G}$ is connected. It is weakly connected if $\widetilde{\mathbf{G}}$ is connected; here, $\widetilde{\mathbf{G}}$ is an undirected form of the graph $\mathbf{G}$.

A clique, C, in an undirected graph $\mathbf{G} = (\mathbf{v}(\mathbf{G}), \mathbf{e}(\mathbf{G}))$ is a subset of the vertices, $C \subseteq \mathbf{v}(\mathbf{G})$, such that every two distinct vertices are adjacent. This is equivalent to the condition that the induced subgraph of $\mathbf{G}$ induced by C is a complete graph.

Jachymski and Jozwik [15] used the property (P) stated that for any sequence $\{\vartheta_n\}$ in Q; if $\vartheta_n \longrightarrow \vartheta$ and $(\vartheta_n, \vartheta_{n+1}) \in \mathbf{e}(\mathbf{G})$, then $(\vartheta_n, \vartheta) \in \mathbf{e}(\mathbf{G})$.

A mapping $T\colon Q \longrightarrow CL(Q)$ is said to be upper semi-continuous if for $\vartheta_n \in Q$ and $\eta_n \in T\vartheta_n$ with $\vartheta_n \longrightarrow \vartheta_0$ and $\eta_n \longrightarrow \eta_0$ implies $\eta_0 \in T\vartheta_0$.

Assume $T_1, T_2\colon Q \longrightarrow CL(Q)$. Set

$$Q_{\{T_1,T_2\}} = \{r \in Q\colon (r, \vartheta_r) \in e(\mathbf{G}) \text{ where } \vartheta_r \in T_1(r) \cap T_2(r)\}. \tag{1}$$

Definition 1. Consider a metric space (Q, d), $\mathbf{G} = (\mathbf{v}(\mathbf{G}), \mathbf{e}(\mathbf{G}))$ be a graph such that $\mathbf{v}(\mathbf{G}) = Q$, and let $T\colon Q \longrightarrow CL(Q)$. If $(m_1, n_1) \in \mathbf{e}(\mathbf{G})$, $(u_1, v_1) \in \mathbf{e}(\mathbf{G})$ for all $u_1 \in Tm_1$ and $v_1 \in Tn_1$, then T is said to be graph-preserving.

Motivated from above in the present work, we give common F.P results for generalized $(\psi,\phi)_1$ and $(\psi,\phi)_2$ weak contraction, in metric spaces (endowed with a graph). To demonstrate the authenticity of our result, we give suitable example. Also, we discuss application to integral inclusion.

2. Fixed Point Results

In our main work, we used the following three classes:

$$\begin{aligned}
\Psi &= \left\{ \begin{array}{c} \psi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ is non-decreasing, continuous mapping such that} \\ \psi(\varsigma) = 0 \text{ if and only if } \varsigma = 0 \end{array} \right\}, \\
\Phi &= \left\{ \phi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ non-decreasing } \lim_{n\longrightarrow\infty} \phi(\varsigma_n) = 0 \Rightarrow \lim_{n\longrightarrow\infty} \varsigma_n = 0 \right\}, \\
\Psi_1 &= \left\{ \begin{array}{c} \psi_1\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^+ \\ \text{(i) } \psi_1 \text{ is continuous and non-decreasing in each coordinate} \\ \text{(ii) } \psi_1(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5) = 0 \text{ implies } \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0 \text{ and} \\ \psi_1(\varsigma, \varsigma, \varsigma, \varsigma, \varsigma) \le \varsigma \text{ for } \varsigma > 0 \end{array} \right\}.
\end{aligned} \tag{2}$$

Definition 2. Consider a complete (Q, d) metric space. Assume $T_1, T_2\colon Q \longrightarrow CL(Q)$ be two set valued mapping. Assume that for each vertex r in $\mathbf{G}$ and for each $\vartheta_r \in T_i(r), i = 1, 2$, we have $(r, \vartheta_r) \in \mathbf{e}(\mathbf{G})$. A pair (T_1, T_2) is said to be

(i) A graphic $(\psi,\phi)_1$ contraction if for any $r, t \in Q$ with $(r, t) \in \mathbf{e}(\mathbf{G})$ and $\vartheta_r \in T_i(r)$, there exists $\vartheta_t \in T_j(t)$ for $i \ne j$ and $i, j = 1, 2$ and $(\vartheta_r, \vartheta_t) \in \mathbf{e}^*(\mathbf{G})$ and

$$\psi(d(\vartheta_r, \vartheta_t)) \le \psi\left(\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t)\right), \tag{3}$$

hold where $\psi \in \Psi$, $\phi \in \Phi$ and $\psi_1 \in \Psi_1$ and

$$\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t) = \psi_1\left(d(r, t), d(r, \vartheta_r), d(t, \vartheta_t), \frac{d(r, \vartheta_t) + d(t, \vartheta_r)}{2}, \frac{d(t, \vartheta_r) + d(r, \vartheta_t)}{2} \right), \tag{4}$$

$\forall\, r, t \in Q$.

(ii) A graphic $(\psi,\phi)_2$ contraction if any $r, t \in Q$ with $(r, t) \in \mathbf{e}(\mathbf{G})$ and $\vartheta_r \in T_i(r)$, there exists $\vartheta_t \in T_j(t)$ for $i, j = 1, 2, i \ne j$; furthermore, $(\vartheta_r, \vartheta_t) \in \mathbf{e}^*(\mathbf{G})$ and

$$\psi(d(Tr, Tt) \le \psi\left(\Lambda_{\psi_2}(r, t; \vartheta_r, \vartheta_t)\right) - \phi\left(\Lambda_{\psi_2}(r, t; \vartheta_r, \vartheta_t)\right), \tag{5}$$

hold where $\psi \in \Psi$, $\phi \in \Phi$, and $\psi_2 \in \Psi_1$ and

$$\Lambda_{\psi_2}(r, t; \vartheta_r, \vartheta_t) = \psi_1\left(\alpha\, d(r, t) + \beta\, d(r, \vartheta_r) + \gamma\, d(t, \vartheta_t) + \sigma_1 \frac{d(r, \vartheta_t) + d(t, \vartheta_r)}{2} + \sigma_2 \frac{d(t, \vartheta_r) + d(r, \vartheta_t)}{2} \right). \tag{6}$$

$\forall\, r, t \in Q$ and $\alpha, \beta, \gamma, \sigma_1, \sigma_2 \ge 0$ with $\alpha + \beta + \gamma + \sigma_1 + \sigma_2 + \sigma_3 \le 1$.

Theorem 1. *Consider a complete (Q, d) metric space endowed with a directed graph $\mathbf{G}$ such that $\mathbf{v}(\mathbf{G}) = Q$ and*

$\mathbf{e}(\mathbf{G})\supseteq\Delta$. *If mapping* $T_1, T_2\colon Q \longrightarrow CL(Q)$ *form a graphic* $(\psi,\phi)_1$ *contraction, then the following statement holds:*

(i) $F_x(T_1)\neq\varnothing$ *or* $F_x(T_2)\neq\varnothing \Leftrightarrow F_x(T_1)=F_x(T_1)\neq\varnothing$

(ii) $Q_{\{T_1,T_2\}}\neq\varnothing$ *provided that* $F_x(T_1)\cap F_x(T_2)\neq\varnothing$

(iii) If $\mathbf{G}$ *is weakly connected and* $Q_{\{T_1,T_2\}}\neq\varnothing$*, then* $F_x(T_1)=F_x(T_2)\neq\varnothing$ *provided that either*

(a) T_1 *or* T_2 *is upper semicontinuous or*

(b) $\mathbf{G}$ *has property* (P) *and either* T_1 *or* T_2 *is bounded*

(iv) $F_x(T_1)\cap F_x(T_2)$ *is a clique of* $\widetilde{\mathbf{G}}\Leftrightarrow F_x(T_1)\cap F_x(T_2)$ *which is singleton.*

Proof. To prove (i), let $r^*\in T_1(r^*)$. Assume $r^*\notin T_2(r^*)$, then since (T_1,T_2) form a graphic $(\psi,\phi)_1$ contraction, there exists an $r\in T_2(r)$ with $(r^*,r)\in\mathbf{e}^*(\mathbf{G})$ such that

$$\psi(d(r^*,r))\leq\psi\left(\Lambda_{\psi_1}(r^*,r^*;r^*,r)\right)-\phi\left(\Lambda_{\psi_1}(r^*,r^*;r^*,r)\right),\tag{7}$$

where

$$\begin{aligned}\Lambda_{\psi_1}(r^*,r^*;r^*,r)&=\psi_1\left(d(r^*,r^*),d(r^*,r^*),d(r^*,r),\frac{d(r,r^*)+d(r^*,r^*)}{2},\frac{d(r^*,r^*)+d(r^*,r)}{2}\right)\\&\leq\psi_1(d(r^*,r^*),d(r^*,r^*),d(r^*,r),d(r^*,r),d(r^*,r)).\end{aligned}\tag{8}$$

Using property (i) of ψ_1, we have

$$\Lambda_{\psi_1}(r^*,r^*;r^*,r)\leq d(r^*,r^*),\tag{9}$$

which is contradiction. Hence, $r^*\in T(r^*)$ and so $F_x(T_1)\subseteq F_x(T_2)$. Similarly, $F_x(T_2)\subseteq F_x(T_1)$; therefore, $F_x(T_1)=F_x(T_2)$.

To prove (ii), let $F_x(T_1)\cap F_x(T_2)\neq\varnothing$, then there exists $r\in Q$ such that $r\in T_1(r)\cap T_2(r)$. As $\mathbf{e}(\mathbf{G})\supseteq\Delta$, we conclude that $Q_{T_1,T_2}\neq\varnothing$.

To prove (iii), let r_0 is an arbitrary point of Q. If $r_0\in T_1(r_0)$ or $r_0\in T_1(r_0)$, then the proof is completed. So, we assume that $r_0\notin T_i(r_0)$ for $i\in\{1,2\}$. Now, for $i,j\in\{1,2\} i\neq j$ if $r_1\in T_i(r_0)$, then there exists $r_2\in T_j(r_1)$ with $(r_1,r_2)\in\mathbf{e}^*(\mathbf{G})$ such that

$$\psi(d(r_1,r_2))\leq\psi\left(\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)\right)-\phi\left(\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)\right),\tag{10}$$

where,

$$\begin{aligned}\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)&=\psi_1\left(d(r_0,r_1),d(r_0,r_1),d(r_1,r_2),\frac{d(r_0,r_2)+d(r_1,r_1)}{2},\frac{d(r_1,r_1)+d(r_0,r_2)}{2}\right),\\\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)&=\psi_1\left(d(r_0,r_1),d(r_0,r_1),d(r_1,r_2),\frac{d(r_0,r_2)+d(r_1,r_1)}{2},\frac{d(r_0,r_2)}{2}\right).\end{aligned}\tag{11}$$

If $d(r_0,r_1)\leq d(r_1,r_2)$, then by simple calculation, we get

$$\begin{aligned}&\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)\leq d(r_1,r_2),\\&\psi(d(r_1,r_2))\leq\psi(d(r_1,r_2))-\phi(d(r_1,r_2)),\end{aligned}\tag{12}$$

which gives contradiction; therefore,

$$\Lambda_{\psi_1}(r_0,r_1;r_1,r_2)=d(r_0,r_1),\tag{13}$$

and further we have

$$\psi(d(r_1,r_2))\leq\psi(d(r_0,r_1))-\phi(d(r_0,r_1)).\tag{14}$$

Similarly, for the point $r_2\in T_j(r_1)$, there exists $r_3\in T_i(r_1)$ with $(r_2,r_3)\in\mathbf{e}^*(\mathbf{G})$ such that

$$\psi(d(r_2,r_3))\leq\psi\left(\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)\right)-\phi\left(\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)\right),\tag{15}$$

where

$$\begin{aligned}\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)&=\psi_1\left(d(r_1,r_2),((r_1,r_2),d(r_2,r_3),\frac{d(r_1,r_3)+d(r_2,r_2)}{2},\frac{d(r_2,r_2)+d(r_1,r_3)}{2}\right),\\\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)&=\psi_1\left(d(r_1,r_2),((r_1,r_2),d(r_2,r_3),\frac{d(r_1,r_3)}{2},\frac{d(r_1,r_3)}{1+d(r_1,r_2)}\right).\end{aligned}\tag{16}$$

If $d(r_1,r_2)\le d(r_2,r_3)$, then

$$\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)\le d(r_2,r_3), \tag{17}$$

and then

$$\psi(d(r_2,r_3))\le \psi(d(r_2,r_3))-\phi(d(r_2,r_3)), \tag{18}$$

which gives contradiction; therefore

$$\begin{aligned}\Lambda_{\psi_1}(r_1,r_2;r_2,r_3)&=d(r_1,r_2),\\ \psi(d(r_2,r_3))&\le \psi(d(r_1,r_2))-\phi(d(r_1,r_2)).\end{aligned} \tag{19}$$

Continuing this way, for $r_{2\gamma}\in T_j(r_{2\gamma-1})$, there exist $r_{2\gamma+1}\in T_j(r_{2\gamma})$ such that $(r_{2\gamma},r_{2\gamma+1})\in \mathbf{e}^*(\mathbf{G})$ such that

$$\begin{aligned}\psi\big(d\big(r_{2\gamma},r_{2\gamma+1}\big)\big)&\le \psi\big(\Lambda_{\psi_1}\big(r_{2\gamma-1},r_{2\gamma};r_{2\gamma},r_{2\gamma+1}\big)\big)\\ &\quad-\phi\big(\Lambda_{\psi_1}\big(r_{2\gamma-1},r_{2\gamma};r_{2\gamma},r_{2\gamma+1}\big)\big),\end{aligned} \tag{20}$$

where

$$\Lambda_{\psi_1}\big(r_{2\gamma-1},r_{2\gamma};r_{2\gamma},r_{2\gamma+1}\big)\le d\big(r_{2\gamma-1},r_{2\gamma}\big). \tag{21}$$

Therefore,

$$\psi\big(d\big(r_{2\gamma},r_{2\gamma+1}\big)\big)\le \psi\big(d\big(r_{2\gamma-1},r_{2\gamma}\big)\big)-\phi\big(d\big(r_{2\gamma-1},r_{2\gamma}\big)\big). \tag{22}$$

In similar pattern r for $r_{2\gamma+1}\in T_j(r_{2\gamma})$, there exist $r_{2\gamma+2}\in T_j(r_{2\gamma+1})$ such that $(r_{2\gamma+1},r_{2\gamma+2})\in \mathbf{e}^*(\mathbf{G})$ such that

$$\begin{aligned}\psi\big(d\big(r_{2\gamma+1},r_{2\gamma+2}\big)\big)&\le \psi\big(\Lambda_{\psi_1}\big(r_{2\gamma},r_{2\gamma+1};r_{2\gamma+1},r_{2\gamma+2}\big)\big)\\ &\quad-\phi\big(\Lambda_{\psi_1}\big(r_{2\gamma},r_{2\gamma+1};r_{2\gamma+1},r_{2\gamma+2}\big)\big),\end{aligned} \tag{23}$$

where

$$\Lambda_{\psi_1}\big(r_{2\gamma},r_{2\gamma+1};r_{2\gamma+1},r_{2\gamma+2}\big)\le d\big(r_{2\gamma-1},r_{2\gamma}\big). \tag{24}$$

Therefore,

$$\psi\big(d\big(r_{2\gamma},r_{2\gamma+1}\big)\big)\le \psi\big(d\big(r_{2\gamma-1},r_{2\gamma}\big)\big)-\phi\big(d\big(r_{2\gamma-1},r_{2\gamma}\big)\big). \tag{25}$$

Hence, we obtain a sequence $\{r_\gamma\}$ in Q such that for $r_\gamma\in T_j(r_{\gamma-1})$, there exist $r_{\gamma+1}\in T_j(r_\gamma)$ such that $(r_\gamma,r_{\gamma+1})\in \mathbf{e}^*(\mathbf{G})$, and

$$\psi\big(d\big(r_\gamma,r_{\gamma+1}\big)\big)\le \psi\big(d\big(r_{\gamma-1},r_\gamma\big)\big)-\phi\big(d\big(r_{\gamma-1},r_\gamma\big)\big). \tag{26}$$

Let $d_\gamma=d(r_\gamma,r_{\gamma+1})$, then the above equation implies that $d_{\gamma+1}\le d_\gamma$ for all $\gamma\ge 1$. Since $\{d_\gamma\}$ is a decreasing positive real sequence, there exists $\xi\ge 0$, such that

$$\lim_{\gamma\longrightarrow\infty} d_\gamma=\xi. \tag{27}$$

We shall show that $\xi=0$; by applying limit, we have

$$\psi(\xi)\le \psi(\xi)-\phi(\xi)<\psi(\xi), \tag{28}$$

which is a contradiction; therefore, $\xi=0$ which implies that

$$\lim_{\gamma\longrightarrow\infty} d_\gamma=0. \tag{29}$$

Now, we want show that $\{r_\gamma\}$ is Cauchy. Suppose that $\{r_\gamma\}$ is not Cauchy. Then, there exist $\epsilon>0$ and subsequences $\{r_{\gamma(\kappa)}\}$ and $\{r_{\delta(\kappa)}\}$ of $\{r_\gamma\}$ with $\gamma(\kappa)>\delta(\kappa)>\kappa$ such that

$$d\big(r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\ge \epsilon,\quad \forall\kappa\in\mathbb{N}. \tag{30}$$

Moreover, one can choose $\gamma(\kappa)$ corresponding to $\delta(\kappa)$ such that it is the smallest possible integer with $\gamma(\kappa)>\delta(\kappa)$ holding (30); then,

$$d\big(r_{\delta(\kappa)},r_{\gamma(\kappa)-1}\big)<\epsilon,\quad \forall\kappa\in\mathbb{N}, \tag{31}$$

$$\begin{aligned}\psi\big(d\big(r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\big)&\le \psi\big(\Lambda_{\psi_1}\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)};r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\big)-\phi\big(\Lambda_{\psi_1}\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)};r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\big),\\ \Lambda_{\psi_1}\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)};r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)&=\psi_1\Big(d\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)}\big),d\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)}\big)\big),d\big(r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\\ &\quad\cdot\frac{d\big(r_{\gamma(\kappa)-1},r_{\gamma(\kappa)}\big)+d\big(r_{\delta(\kappa)},r_{\delta(\kappa)}\big)}{2},\frac{d\big(r_{\delta(\kappa)},r_{\delta(\kappa)}\big)+d\big(r_{\gamma(\kappa)-1},r_{\gamma(\kappa)}\big)}{2}\Big).\end{aligned} \tag{32}$$

Applying limit $\kappa\longrightarrow\infty$, we get

$$\lim_{\kappa\longrightarrow\infty}\Lambda_{\psi_1}\big(r_{\gamma(\kappa)-1},r_{\delta(\kappa)};r_{\delta(\kappa)},r_{\gamma(\kappa)}\big)\le\epsilon. \tag{33}$$

Taking limit of (32) using (33), (29), and lower semicontinuity of ϕ, we have

$$\psi(\epsilon)\le\psi(\epsilon)-\phi(\epsilon)<\psi(\epsilon), \tag{34}$$

which is contradiction; therefore, r_γ is a Cauchy sequence. Now, if T_i is upper semicontinuous (USC), then as $r_{2\gamma}\in Q$, $r_{2\gamma+1}\in T_i(r_{2\gamma})$ with $s_{2\gamma}\longrightarrow r^*$ and $r_{2\gamma+1}\longrightarrow r^*$ as $\gamma\longrightarrow\infty$ which implies that $r^*\in T_i(r^*)$. Using (i), we get $r^*\in T_i(r^*)=T_j(r^*)$. Similarly, the result holds when T_j is upper semicontinuous (USC).

Assume that F is continuous. Since $s_{2\gamma}$ converges to r^* as $\gamma\longrightarrow\infty$ and $(r_{2\gamma},r_{2\gamma+1})\in\mathbf{e}(\mathbf{G})$, we have $(r_{2\gamma},r^*)\in\mathbf{e}^*(\mathbf{G})$. For $r_{2\gamma}\in T_j(r_{2\gamma-1})$, there exists $\vartheta_\gamma\in T_i(r^*)$ such that $(r_{2\gamma},\vartheta_\gamma)\in\mathbf{e}^*(\mathbf{G})$. As $\{r_\gamma\}$ is bounded, $\limsup_{\gamma\longrightarrow\infty}\vartheta_\gamma=\vartheta^*$ and $\liminf_{\gamma\longrightarrow\infty}\vartheta_\gamma=\vartheta_*$, both exist. Assume that $\vartheta^*\neq r^*$. Since (T_1,T_2) form a graphic F_1- contraction,

$$\psi\left(d\left(r_{2n}, r_3\right)\right) \leq \psi\left(\Lambda_{\psi_1}\left(r_{2\gamma-1}, r^*; r_{2\gamma}, \vartheta_\gamma\right)\right) - \phi\left(\Lambda_{\psi_1}\left(r_{2\gamma-1}, r^*; r_{2\gamma}, \vartheta_\gamma\right)\right), \tag{35}$$

where

$$\Lambda_{\psi_1}\left(r_{2\gamma-1}, r^*; r_{2\gamma}, \vartheta_\gamma\right) = \psi_1\left(d\left(r_{2\gamma-1}, r^*\right), d\left(r_{2\gamma-1}, r_{2\gamma}\right), d\left(r^*, \vartheta_\gamma\right), \frac{d\left(r_{2\gamma}, \vartheta_\gamma\right) + d\left(r_2, r_2\right)}{2}, \frac{d\left(r_2, r_2\right) + d\left(r_{2\gamma}, \vartheta_\gamma\right)}{2}\right). \tag{36}$$

By applying limit and after simple calculation, we get contradiction; hence, $\vartheta^* = r^*$. Similarly, taking the liminf gives $\vartheta^* = r^*$. Since $\vartheta_\gamma \in T_i(r^*)$ for all $\gamma \geq 1$ and $T_i(r^*)$ is closed set, it follows that $r^* T_i(r^*)$. Now, from (i), we get $r^* \in T_i(r^*)$ and hence $F_x(T_1) = F_x(T_2)$.

To prove (iv), assume the set $F_x(T_1) \cap F_x(T_2)$ is a clique of $\widetilde{\mathbf{G}}$. We have shown that $F_x(T_1) \cap F_x(T_2)$ is singleton set. Assume on contrary that there exist ϑ and v such that $\vartheta, v \in F_x(T_1) \cap F_x(T_2)$ but $\vartheta \neq v$. As $(\vartheta, v) \in \mathbf{e}^*(\mathbf{G})$ and T_1 and T_2 form a graphic F_1- contraction, so for $(\vartheta_s, \vartheta_t) \in \mathbf{e}^*(\mathbf{G})$ which implies

$$\psi(d(\vartheta, v)) \leq \psi\left(\Lambda_{\psi_1}(\vartheta, v; \vartheta, v)\right) - \phi\left(\Lambda_{\psi_1}(\vartheta, v; \vartheta, v)\right), \tag{37}$$

where

$$\Lambda_{\psi_1}(\vartheta, v; \vartheta, v) = \psi_1\left(d(\vartheta, v), d(\vartheta, \vartheta), d(v, v), \frac{d\left(r_{2\gamma}, \vartheta_\gamma\right) + d\left(r_2, r_2\right)}{2}, \frac{d\left(r_2, r_2\right) + d\left(r_{2\gamma}, \vartheta_\gamma\right)}{2}\right), \tag{38}$$

which is a contradiction. Hence, $\vartheta = v$. Conversely, if $F_x(T_1) \cap F_x(T_2)$ is singleton, then it follows $F_x(T_1) \cap F_x(T_2)$ is a clique of $\widetilde{\mathbf{G}}$.

By the same technique, it is essay to prove the following result. □

Theorem 2. *Consider a complete (Q, d) metric space endowed with a directed graph $\mathbf{G}$ such that $\mathbf{v}(\mathbf{G}) = Q$ and $\mathbf{e}(\mathbf{G}) \supseteq \Delta$. If mapping $T_1, T_2: Q \longrightarrow CL(Q)$ make a graphic $(\psi, \phi)_2$ contraction, then the following statement holds:*

(i) $F_x(T_1) \neq \varnothing$ or $F_x(T_2) \neq \varnothing \Leftrightarrow F_x(T_1) = F_x(T_1) \neq \varnothing$;

(ii) $Q_{\{T_1, T_2\}} \neq \varnothing$ provided that $F_x(T_1) \cap F_x(T_2) \neq \varnothing$;

(iii) $\mathbf{G}$ is weakly connected and $Q_{\{T_1, T_2\}} \neq \varnothing$, then $F_x(T_1) = F_x(T_2) \neq \varnothing$ provided that either

(a) T_1 or T_2 is upper semicontinuous(USC) or

(b) $\mathbf{G}$ has property (P) and either T_1 or T_2 is bounded;

(iv) $F_x(T_1) \cap F_x(T_2)$ is a clique of $\widetilde{\mathbf{G}} \Leftrightarrow F_x(T_1) \cap F_x(T_2)$ which is singleton.

Now, we give example which satisfying Theorem 1.

Example 1. Let $Q = \left\{r_\gamma = (\gamma(\gamma+1)/2)\colon \gamma \in N\right\} = \mathbf{v}(\mathbf{G})$

$$\begin{aligned}\mathbf{e}(\mathbf{G}) &= \left\{\frac{(r,t)}{t} = 2r, \forall r, t \in \mathbf{v}(\mathbf{G})\right\},\\ \mathbf{e}^*(\mathbf{G}) &= \left\{\frac{(r,t)}{t} \neq 2r, \forall r, t \in \mathbf{v}(\mathbf{G})\right\}.\end{aligned} \tag{39}$$

Let $\mathbf{v}(\mathbf{G})$ be endowed with usual metric space. Let $T_1, T_2: Q \times Q \longrightarrow P(Q)$, $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, and $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ define by

$$\begin{aligned}T_1(r) &= \{r_2\},\\ T_2(r) &= \begin{cases}\{r_2\}, & \text{if } r = r_2,\\ \left\{r_1, r_\gamma\right\}, & \text{if } r = r_\gamma,\end{cases}\\ \phi(t) &= \frac{t}{4},\\ \psi(t) &= \frac{t}{2},\\ \psi_1(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5) &= \max\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}.\end{aligned} \tag{40}$$

For $(\vartheta_r, \vartheta_t) \in \mathbf{e}(\mathbf{G})$, we discuss the following cases:

Case 1.1. If $r = r_1$, $t = r_\delta$ for $\delta > 2$, then for $\vartheta_r = r_2 \in T_1(r)$, there exist $\vartheta_t = r_{\delta-1} \in T_2(t)$ such that

$$\begin{aligned}\psi(d(\vartheta_r, \vartheta_t)) &= \frac{d(\vartheta_r, \vartheta_t)}{2} = \frac{d(r_2, r_{\delta-1})}{2}\\ &= \frac{\delta^2 - \delta - 2}{4} \leq \frac{\delta^2 + \delta - 2}{4}\\ &\leq \frac{\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t)}{2}\\ &\leq \psi\left(\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r, t; \vartheta_r, \vartheta_t)\right).\end{aligned} \tag{41}$$

Case 1.2. If $r = r_\gamma$ and $t = r_{\gamma+1}$ for $\gamma > 1$, then for $\vartheta_r = r_2 \in T_1(r)$, there exist $\vartheta_t = r_{\gamma-1} \in T_2(t)$ such that

$$\begin{aligned}\psi(d(\vartheta_s,\vartheta_t)) &= \frac{d(\vartheta_r,\vartheta_t)}{2} = \frac{d(r_2,r_{\gamma-1})}{2}\\ &= \frac{\delta^2-\delta-2}{4} \le \frac{\delta^2+\delta-2}{4}\\ &\le \frac{\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)}{2}\\ &\le \psi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right).\end{aligned} \tag{42}$$

Case 1.3. If $r = r_\gamma$ and $t = r_\delta$ for $\delta > \gamma > 1$, then for $\vartheta_r = r_2 \in T_1(r)$, there exist $\vartheta_t = r_{\gamma-1} \in T_2(t)$ such that

$$\begin{aligned}\psi(d(\vartheta_r,\vartheta_t)) &= \frac{d(\vartheta_r,\vartheta_t)}{2} = \frac{d(r_2,r_{\gamma-1})}{2}\\ &= \frac{\gamma^2-\gamma-2}{4} \le \frac{\gamma^2+\gamma-2}{4}\\ &\le \frac{\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)}{2}\\ &\le \psi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right).\end{aligned} \tag{43}$$

Now, for $r,t \in Q, \vartheta_r \in T_2(r)$, there exist $\vartheta_t \in T_1(r)$ such that $(\vartheta_r,\vartheta_t) \in \mathbf{e}^*(\mathbf{G})$, we consider the following cases.

Case 1.4. If $r = r_\gamma$ and $t = r_1$ for $\gamma > 1$, then for $\vartheta_r = r_{\gamma-1} \in T_2(r)$, there exist $\vartheta_t = r_2 \in T_1(t)$ such that

$$\begin{aligned}\psi(d(\vartheta_r,\vartheta_t)) &= \frac{d(\vartheta_r,\vartheta_t)}{2} = \frac{d(r_{\gamma-1},r_2)}{2}\\ &= \frac{\gamma^2-\gamma-2}{4} \le \frac{\gamma^2+\gamma-2}{4}\\ &\le \frac{\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)}{2}\\ &\le \psi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right).\end{aligned} \tag{44}$$

Case 1.5. If $r = r_\gamma$ and $t = r_\delta$ for $\delta > \gamma > 1$, then for $\vartheta_r = r_{\gamma-1} \in T_2(r)$, there exist $\vartheta_t = r_2 \in T_1(t)$ such that

$$\begin{aligned}\psi(d(\vartheta_r,\vartheta_t)) &= \frac{d(\vartheta_r,\vartheta_t)}{2} = \frac{d(r_{\gamma-1},r_2)}{2}\\ &= \frac{\gamma^2-\gamma-2}{4} \le \frac{\gamma^2+\gamma-2}{4}\\ &\le \frac{\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)}{2}\\ &\le \psi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right) - \phi\left(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\right).\end{aligned} \tag{45}$$

3. Application to Integral Inclusion

Set-valued (SV) F.P results are explored extensively and have interesting application in integral and differential inclusion [16–19]. In this section, we derive sufficient conditions for the solutions of Fredholm-type integral inclusion. Set of all continuous function defined on $[\varsigma,\xi]$ is denoted by $W = C[\varsigma,\xi]$. Define $d\colon W \times W \longrightarrow \mathbf{R}^+$, by $d(r,t) = \sup_{t\in[\varsigma,\xi]}|r-t|$, which is a complete metric space on W. Assume that the metric is endowed with a graph $\mathbf{G}$. Consider the integral inclusion

$$r(\zeta) \in \phi(\zeta) + \int_\varsigma^\xi K(\zeta,\eta,r(\eta))\mathrm{d}\eta, \quad \zeta \in [\varsigma,\xi], \tag{46}$$

where $K\colon [\varsigma,\xi]\times[\varsigma,\xi]\times\mathbf{R} \longrightarrow P_{cv}(\mathbf{R})$, where $P_{cv}(\mathbf{R})$ is the family (collection) of nonempty compact and convex subsets of $\mathbf{R}$. For each $r \in C([\varsigma,\xi],\mathbf{R})$, the operator $K(.,.r)$ is lower semicontinuous. Further, the function $\phi\colon [\varsigma,\xi] \longrightarrow \mathbf{R}$ is continuous.

Define $T\colon C([\varsigma,\xi],\mathbf{R}) \longrightarrow CL(C[\varsigma,\xi],\mathbf{R})$ by,

$$\mathrm{Tr}(\zeta) = \left\{\vartheta \in C([\varsigma,\xi],\mathbf{R})\colon\, \vartheta \in \phi(\zeta) + \int_\varsigma^\xi K(\zeta,\eta,r(\eta))\mathrm{d}\eta\right\}, \tag{47}$$

$r \in C([\varsigma,\xi],\mathbf{R})$, and set $K_r\colon K(\zeta,\eta,r(\eta)), \zeta,\eta \in [\varsigma,\xi]$. Now, for $K\colon [\varsigma,\xi]\times[\varsigma,\xi]\times\mathbf{R} \longrightarrow P_{cv}(\mathbf{R})$, by Michael's selection theorem, there exist a continuous operator $m_s\colon [\varsigma,\xi]\times[\varsigma,\xi] \longrightarrow \mathbf{R}$ with $m_r \in K_r$

$$\phi(\zeta) + \int_\varsigma^\xi K(\zeta,\eta,r(\eta))\mathrm{d}\eta \in \mathrm{Tr}(\zeta). \tag{48}$$

This implies that $\mathrm{Tr} \ne \varnothing$. Therefore, Tr is closed [18].

Theorem 3. *Assume that the following assumptions holds:*

(A_1) *There exist a continuous function* $G\colon [\varsigma,\xi]\times[\varsigma,\xi] \longrightarrow [0,\infty)$ *such that*

$$H(K(\zeta,\eta,r),K(\zeta,\eta,t)) \le G(\zeta,\eta)\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t), \tag{49}$$

for each $\zeta,\eta \in [\varsigma,\xi]$.

(A_2) $sup_{\zeta\in[\varsigma,\xi]}\int_\varsigma^\xi |G(\zeta,\eta)| \le (1/2)$.

(A_3) *Assume that for every vertex* r *in* $\mathbf{G}$ *and for every* $\vartheta_r \in T_i(r), i = 1,2,$ *we have* $(r,\vartheta_r) \in \mathbf{e}(\mathbf{G})$.

(A_4) *If any* $r,t \in Q$ *with* $(r,t) \in \mathbf{e}(\mathbf{G})$ *and* $\vartheta_r \in T_i(r)$, *there exists* $\vartheta_t \in T_j(t)$ *for* $i, j = 1,2$ *and* $i \ne j$ *such that* $(\vartheta_r,\vartheta_t) \in \mathbf{e}^*(\mathbf{G})$.

Then, the integral inclusion (46) has a solution.

Proof. Let $r,t \in W$ be such that $t \in Ts$. Then, we have $m_r(\zeta,\eta) \in K_r(\zeta,\eta)$ for $\zeta,\eta \in [\varsigma,\xi]$ such that $u(\zeta) = \phi(\zeta) + \int_\varsigma^\xi m_r(\zeta,\eta,s(\eta))\mathrm{d}\eta$. On other side, hypothesis ensures that there exist $v(\zeta,\eta) \in \Lambda_t(\zeta,\eta)$ such that

$$\left|m_r(\zeta,\eta) - v(\zeta,\eta)\right| \le G(\zeta,\eta)\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t). \tag{50}$$

Consider the set valued operator S defined by

$$S(\zeta,\eta) = \Lambda_t(\zeta,\eta) \cap \{w \in \mathbf{R}\colon |m_r(\zeta,\eta) - \nu(\zeta,\eta)| \leq G(\zeta,\eta)\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\}. \quad (51)$$

Since the operator T is lower semicontinuous, there exists $m_t\colon [\varsigma,\xi] \times [\varsigma,\xi] \longrightarrow \mathbf{R}$ such that $m_t(\zeta,r) \in S(\zeta,\eta)$ for each $\zeta,\eta \in [\varsigma,\xi]$; thus,

$$\vartheta_r(\zeta) = \phi(\zeta) + \int_\varsigma^\xi m_r(\zeta,\eta,r(\eta))\mathrm{d}\eta \in \phi(\zeta) + \int_\varsigma^\xi K(\zeta,\eta,r(\eta))\mathrm{d}\eta, \quad \zeta \in [\varsigma,\xi]. \quad (52)$$

Now, we have

$$\begin{aligned} d(\vartheta_r(\zeta),\vartheta_t(\zeta)) &= \sup_{\zeta\in[\varsigma,\xi]}|\vartheta_r(\zeta) - \vartheta_t(\zeta)| \\ &\leq \sup_{\zeta\in[\varsigma,\xi]}\left(\int_\varsigma^\xi |m_r(\zeta,\eta,r(\eta)) - m(\zeta,\eta,t(\eta))|\mathrm{d}\eta\right) \\ &\leq \sup_{\zeta\in[\varsigma,\xi]}\left(\int_\varsigma^\xi G(\zeta,\eta)\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\mathrm{d}\eta\right. \\ &\leq \Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)\sup_{\zeta\in[\varsigma,\xi]}\int_\varsigma^\xi G(\zeta,\eta)\mathrm{d}\eta \\ &\leq \frac{\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)}{2}, \end{aligned} \quad (53)$$

and we have

$$\psi(d(\vartheta_r(\zeta),\vartheta_t(\zeta))) \leq \psi(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)) - \phi(\Lambda_{\psi_1}(r,t;\vartheta_r,\vartheta_t)). \quad (54)$$

By taking $\psi(\zeta) = \zeta, \phi(\zeta) = (\zeta/2)$. From Theorem 1, the integral inclusion (46) has a solution. □

4. Conclusion

Common fixed point results for single-valued mappings are used to solve nonlinear integral, functional, and matrix equations, etc. However, the studies of the set-valued mapping generalize the concept of single valued mappings. Such studies have been applied to prove the existence of solution for integral and differential inclusion and existence of equilibria in game theory. The result of Abbas et al. [12] generalized many results for set-valued mapping related with F-contraction while our results generalized many results related with weak-contraction; for detail, Alber and Guerre-Delabriere [20] established results for single valued mapping and relax contraction condition in form of weak contraction. Then, Doric [21] proved common fixed point for generalized (ψ,ϕ)-weak contractions. Dutta and Choudhury [22] generalized the result in metric space. Loung and Thuan [23] and Rhoades [24] also used weak contraction in this line and established fixed point results in metric spaces. Similarly Zhang and Song [25] further explored the results for generalized ϕ-weak contractions. Our results generalized the mention results and many more in this direction for set-valued mapping induced with graph without using Hausdorff metric.

Data Availability

No data were used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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