

Research Article

Nonexpansive Mappings on New Premodular Special Space of Sequences

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For different premodular, which is a generalization of modular, defined by weighted Orlicz sequence space and its prequasi operator ideal, we have examined the existence of a fixed point for both Kannan contraction and nonexpansive mappings acting on these spaces. Some numerous numerical experiments and practical applications are presented to support our results.

1. Introduction

The spaces of all, bounded, r -absolutely summable, and null sequences of real numbers will be denoted throughout the article by $\mathbf{R}^{\mathcal{X}^+}$, ℓ_∞ , ℓ_r , and c_0 , respectively, where \mathcal{X}^+ is the set of nonnegative integers.

Definition 1. [1, 2] An Orlicz function is a function $M: [0, \infty) \rightarrow [0, \infty)$, which is continuous and strictly increasing with $M(0) = 0$, $M(v) > 0$ for $v > 0$, and $M(v) \rightarrow \infty$, as $v \rightarrow \infty$.

Definition 2. An Orlicz function M is said to satisfy Δ_2 -condition for every values of $v \geq 0$, if there is $k > 0$, such that $M(2v) \leq kM(v)$. The Δ_2 -condition is equivalent to $M(lv) \leq klM(v)$ for every values of $l > 1$ and v .

Lindentrauss and Tzafriri [3] utilized the idea of a convex Orlicz function to define Orlicz sequence space:

$$\begin{aligned} \ell_M &= \left\{ v \in \mathbf{R}^{\mathcal{X}^+} : \rho(\omega v) < \infty \text{ for some } \omega > 0 \right\}, \text{ where } \rho(v) \\ &= \sum_{y=0}^{\infty} M(|v_y|). \end{aligned} \quad (1)$$

$(\ell_M, \|\cdot\|)$ is a Banach space with the Luxemburg norm:

$$\|v\| = \inf \left\{ \omega > 0 : \rho\left(\frac{v}{\omega}\right) \leq 1 \right\}. \quad (2)$$

Every Orlicz sequence space contains a subspace that is isomorphic to c_0 or ℓ_r , for some $1 \leq r < \infty$ ([4], Theorem 4.a.9). The space of all bounded linear operators from a Banach space \mathfrak{X} into a Banach space \mathfrak{Y} will be denoted by $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and if $\mathfrak{X} = \mathfrak{Y}$, we write $\mathcal{B}(\mathfrak{X})$. $e_x = \{0, 0, \dots, 1, 0, 0, \dots\}$, while 1 lies in the x^{th} place, with $x \in \mathcal{X}^+$.

Definition 3. [5] An s -number function is a mapping from $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ into $[0, \infty)^{\mathcal{X}^+}$ which transforms every map $H \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ to $(s_x(H))_{x=0}^{\infty}$ satisfying the next conditions:

- (i) $\|H\| = s_0(H) \geq s_1(H) \geq s_2(H) \geq \dots \geq 0$, for every $H \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$,
- (ii) $s_{y+x-1}(H_1 + H_2) \leq s_y(H_1) + s_x(H_2)$, for every $H_1, H_2 \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and $y, x \in \mathcal{X}^+$,
- (iii) ideal property: $s_x(UTH) \leq \|U\| s_x(T) \|H\|$, for every $H \in \mathcal{B}(\mathfrak{X}_0, \mathfrak{X})$, $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and $U \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Y}_0)$, where \mathfrak{X}_0 and \mathfrak{Y}_0 are any two Banach spaces,

- (iv) for $H \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\omega \in \mathbf{R}$, we have $s_x(\omega H) = |\omega|s_x(H)$,
- (v) rank property: If $\text{rank}(H) \leq x$, then $s_x(H) = 0$, for all $H \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$,
- (vi) norming property: $s_{I_{\geq x}}(I_x) = 0$ or $s_{I_{< x}}(I_x) = 1$, where I_x explains the unit map on the x -dimensional Hilbert space ℓ_2^x .

The x th approximation number, $\alpha_x(W)$, is defined as

$$\alpha_x(H) = \inf\{\|H - Y\|: Y \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \text{ and } \text{rank}(Y) \leq x\}. \quad (3)$$

Notations 1. The sets $S_W, S_W(\mathcal{X}, \mathcal{Y}), S_W^{\text{app}}$, and $S_W^{\text{app}}(\mathcal{X}, \mathcal{Y})$ (cf. [6]) are defined as follows:

$$\begin{aligned} S_W &:= \{S_W(\mathcal{X}, \mathcal{Y})\}, \text{ where } S_W(\mathcal{X}, \mathcal{Y}) \\ &:= \{H \in \mathcal{B}(\mathcal{X}, \mathcal{Y}): ((s_x(H))_{x=0}^\infty \in W)\}. \text{ Also} \\ S_W^{\text{app}} &:= \{S_W^{\text{app}}(\mathcal{X}, \mathcal{Y})\}, \text{ where } S_W^{\text{app}}(\mathcal{X}, \mathcal{Y}) \\ &:= \{H \in \mathcal{B}(\mathcal{X}, \mathcal{Y}): ((\alpha_x(H))_{x=0}^\infty \in W)\}. \end{aligned} \quad (4)$$

Fixed point theory, Banach space geometry, normal series theory, ideal transformations, and approximation theory are all examples of ideal operator theorems and summability. Faried and Bakery [6] established the concept of a prequasi operator ideal that encapsulates the quasi operator ideal. Bakery and Abou Elmatty investigated the sufficient (but not necessary) conditions on $\ell(\gamma, r)$ that allowed $S_{\ell(\gamma, r)}$ to build a simple Banach prequasi operator ideal in [7]. For varied weights and powers, the prequasi operator ideal $S_{\ell(\gamma, r)}^{\text{app}}$ was once rigorously contained and small prequasi operator ideal. Several mathematicians were able to investigate many extensions for contraction maps defined on the space or on the space itself thanks to the Banach fixed point theorem [8]. Kannan [9] investigated an example of a class of operators that perform the same fixed point actions as contractions but are not continuous. Kannan operators in modular vector spaces have only been described by Ghoncheh [10]. He demonstrated the existence of a Kannan mapping fixed point in complete modular spaces with Fatou property. For more details on Kannan's fixed point theorems and modular vector spaces (see [11–14]). Bakery and Mohamed [15] introduced the concept of the prequasi norm on $\ell^{(r_a)}$ with variable exponent in $(0, 1]$. They looked at the Fatou property of different prequasi norms on $\ell^{(r_a)}$, as well as the sufficient requirements on $\ell^{(r_a)}$ with the definite prequasi norm to construct prequasi Banach and closed space. They also demonstrated the existence of a fixed point of Kannan prequasi norm contraction maps on $\ell^{(r_a)}$ and the prequasi Banach operator ideal constructed by $\ell^{(r_a)}$ and s -numbers. Recently, Reich and Zaslavski [16] showed the existence of a unique fixed point for nonlinear contractive self-mappings of a nonbounded closed subset of a Banach space. They extended this conclusion to contractive mappings, which map into a Banach space a closed subset of the space. For nonexpansive

mappings defined by an intersection of a finite number of closed bounded and convex nonempty subsets in Banach spaces, Dehici and Redjel [17] obtained certain fixed point results. According to Bendahmane and Bendoukha [18], a (p, q) -metric space is a generalization of the metric and S -metric spaces. They equipped them a Hausdorff topology and specified several fundamental features. Several well-known findings from fixed point theory are generalized to these new spaces. The paper is structured as follows: we present conditions on the weighted Orlicz sequence space $(\ell_M(\lambda))_\mu$, under definite prequasi norm of μ to construct prequasi Banach and closed sequence space in Section 3. The Fatou property of $\ell_M(\lambda)$ has been investigated for various prequasi norms. In Section 4, the existence of fixed point for Kannan μ -contraction mapping acting on $(\ell_M(\lambda))_\mu$ equipped with different prequasi norms are presented. Several numerical experiments are shown to demonstrate our results. In Section 5, the conditions for which the space $(\ell_M(\lambda))_\mu$ satisfies the property (R) and has the μ -normal structure property are presented. The existence of a fixed point of Kannan prequasi norm nonexpansive mapping on $(\ell_M(\lambda))_\mu$ has been given. In Section 6, we explain the existence of a fixed point of Kannan prequasi norm contraction mapping in the prequasi Banach operator ideal $S_{(\ell_M(\lambda))_\mu}$. In Section 7, we give some applications to the existence of solutions of summable equations.

2. Definitions and Preliminaries

Here and after, the space of all functions $\mu: Y \rightarrow [0, \infty)$ is $[0, \infty)^Y$, θ is the zero vector of Y , $[x/2]$ is the integral part of $x/2$, F is the space of finite sequences, and \mathcal{B} is the class of each bounded linear mapping between any two Banach spaces. Nakano [19] introduced the concept of modular vector spaces.

Definition 4. Let Y be a vector space. A function $\mu \in [0, \infty)^Y$ is called modular if the following conditions hold:

- (i) If $\beta \in Y$, $\beta = \theta \Leftrightarrow \mu(\beta) = 0$ and $\mu(\beta) \geq 0$,
- (ii) if $\beta \in Y$ and $|\omega| = 1$, then $\mu(\omega\beta) = \mu(\beta)$,
- (iii) assume $\beta, \eta \in Y$ and $\omega \in [0, 1]$, then $\mu(\omega\beta + (1 - \omega)\eta) \leq \mu(\beta) + \mu(\eta)$.

The concept of premodular vector spaces, which is more general than modular vector spaces.

Definition 5. [6] The linear space of sequences Y is said to be a special space of sequences (sss), if:

- (1) $\{e_x\}_{x \in \mathcal{Z}^+} \subseteq Y$,
- (2) Y is solid, i.e., for $\beta = (\beta_x) \in \mathbf{R}^{\mathcal{Z}^+}$, $\eta = (\eta_x) \in Y$ and $|\beta_x| \leq |\eta_x|$, for all $x \in \mathcal{Z}^+$, then $\beta \in Y$,
- (3) If $(\beta_x)_{x=0}^\infty \in Y$, then $(\beta_{[x/2]})_{x=0}^\infty \in Y$.

Definition 6. [6] A subclass Y_μ of Y is called a premodular (sss), if we have $\mu \in [0, \infty)^{Y_\mu}$ that satisfies the following conditions:

- (i) When $\beta \in Y$, $\beta = \theta \Leftrightarrow \mu(\beta) = 0$,
- (ii) For every $\beta \in Y$ and $\omega \in \mathbf{R}$, then there is $B \geq 1$ with $\mu(\omega\beta) \leq B|\omega|\mu(\beta)$,
- (iii) $\mu(\beta + \eta) \leq J(\mu(\beta) + \mu(\eta))$, for all $\beta, \eta \in Y$, holds for some $J \geq 1$,
- (iv) If $x \in \mathcal{X}^+$ and $|\beta_x| \leq |\eta_x|$, then $\mu((\beta_x)) \leq \mu((\eta_x))$,
- (v) For some $J_0 \geq 1$, we have $\mu((\beta_x)) \leq \mu((\beta_{\lfloor x/2 \rfloor})) \leq J_0 \mu((\beta_x))$,
- (vi) $\bar{F} = Y_\mu$,
- (vii) There exists $\zeta > 0$ such that $\mu(\omega, 0, 0, 0, \dots) \geq \zeta|\omega|\mu(1, 0, 0, 0, \dots)$, for all $\omega \in \mathbf{R}$.

Example 1. The function $\mu(\beta) = (\sum_{x \in \mathcal{X}^+} \sqrt[5]{|\beta_x|})^5$ is a premodular (not a modular) on the vector space $\ell_{1/5}$. As for every $\beta, \eta \in \ell_{1/5}$, one has

$$\mu\left(\frac{\beta + \eta}{2}\right) = \left(\sum_{x \in \mathcal{X}^+} \sqrt[5]{\left|\frac{\beta_x + \eta_x}{2}\right|}\right)^5 \leq 8(\mu(\beta) + \mu(\eta)). \quad (5)$$

Definition 7. [15] Suppose Y is a (sss). The function $\mu \in [0, \infty)^Y$ is said to be prequasi norm on Y , if it holds the settings (i), (ii), and (iii) of Definition 6.

Theorem 1. [15] Let Y be a premodular (sss), then it is prequasi normed (sss).

Theorem 2. [15] Y is a prequasi normed (sss), when it is quasi-normed (sss).

Definition 8. [20]

- (i) The prequasi norm μ on X_μ is said to be μ -convex, when $\mu(\omega\beta + (1 - \omega)\eta) \leq \omega\mu(\beta) + (1 - \omega)\mu(\eta)$, for all $\omega \in [0, 1]$ and $\beta, \eta \in X_\mu$.
- (ii) $\{\beta_x\}_{x \in \mathcal{X}^+} \subseteq (X)_\mu$ is μ -convergent to $\beta \in (X)_\mu$, if and only if, $\lim_{x \rightarrow \infty} \mu(\beta_x - \beta) = 0$. If the μ -limit exists, hence it is unique.
- (iii) $\{\beta_x\}_{x \in \mathcal{X}^+} \subseteq (X)_\mu$ is μ -Cauchy, if $\lim_{x, h \rightarrow \infty} \mu(\beta_x - \beta_h) = 0$.
- (iv) $\Phi \subset (X)_\mu$ is μ -closed, if for every μ -converging $\{\beta_x\}_{x \in \mathcal{X}^+} \subset \Phi$ to β , then $\beta \in \Phi$.
- (v) $\Phi \subset (X)_\mu$ is μ -bounded, if $v_\mu(\Phi) = \sup\{\mu(\beta - \eta) : \beta, \eta \in \Phi\} < \infty$.
- (vi) The μ -ball of radius $r \geq 0$ and center β , for every $\beta \in (X)_\mu$, is defined as

$$\mathcal{B}_\mu(\beta, r) = \{\eta \in (X)_\mu : \mu(\beta - \eta) \leq r\}. \quad (6)$$

- (vii) A prequasi norm μ on X satisfies the Fatou property, if for every sequence $\{\eta^x\} \subseteq (X)_\mu$ with $\lim_{x \rightarrow \infty} \mu(\eta^x - \eta) = 0$ and any $\beta \in (X)_\mu$, we have $\mu(\beta - \eta) \leq \sup_m \inf_{x \geq m} \mu(\beta - \eta^x)$.

Recall that the μ -balls are μ -closed under the Fatou property.

Definition 9. [21] A subclass \mathcal{G} of \mathcal{B} is called an operator ideal, if every vector $\mathcal{G}(\mathcal{X}, \mathcal{Y}) = \mathcal{G} \cap \mathcal{B}(\mathcal{X}, \mathcal{Y})$ holds the following conditions:

- (i) $I_\gamma \in \mathcal{G}$, where γ indicates Banach space of one dimension.
- (ii) The space $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ is linear over \mathbf{R} .
- (iii) If $H \in \mathcal{B}(\mathcal{X}_0, \mathcal{X})$, $T \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$, and $V \in \mathcal{B}(\mathcal{Y}, \mathcal{Y}_0)$, then $VTH \in \mathcal{G}(\mathcal{X}_0, \mathcal{Y}_0)$, where \mathcal{X}_0 and \mathcal{Y}_0 are normed spaces.

Recall that the quasi operator ideals are a special case of the prequasi operator ideals.

Definition 10. [6] A function $Y \in [0, \infty)^{\mathcal{G}}$ is said to be a prequasi norm on the ideal \mathcal{G} if the following conditions verify:

- (1) Suppose $H \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$, $Y(H) \geq 0$ and $Y(H) = 0$, if and only if, $H = 0$,
- (2) there exists $D \geq 1$ such that $Y(\omega H) \leq D|\omega|Y(H)$, for every $H \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$ and $\omega \in \mathbf{R}$,
- (3) we have $J \geq 1$ so that $Y(H_1 + H_2) \leq J[Y(H_1) + Y(H_2)]$, for all $H_1, H_2 \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$,
- (4) we get $\omega \geq 1$ so that if $H \in \mathcal{B}(\mathcal{X}_0, \mathcal{X})$, $T \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$, and $V \in \mathcal{B}(\mathcal{Y}, \mathcal{Y}_0)$, then $Y(VTH) \leq \omega \|V\| Y(T) \|H\|$.

Theorem 3. [15] The function $Y(H) = \mu(s_x(H))_{x=0}^\infty$ is a prequasi norm on S_{Y_μ} , when Y_μ is a premodular (sss).

Theorem 4. [6] If Y is a quasi norm on the ideal \mathcal{G} , then Y is a prequasi norm on the ideal \mathcal{G} .

Lemma 1. [22, 23] Assume $M: (0, \infty) \rightarrow [0, \infty)$ is a continuous function and strictly increasing with $\lim_{x \rightarrow 0} M(x) = 0$, and if the functions $M(x)$ and $\ln(M(e^x))$ are convex on $[0, \infty)$, then

$$M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\beta_x + \eta_x)\right) \leq M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\beta_x)\right) + M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\eta_x)\right). \quad (7)$$

$$\lambda_x, \beta_x, \eta_x \in [0, \infty), \text{ for all } x \in \mathcal{X}^+ \text{ and } \sum_{x=0}^\infty \lambda_x = 1.$$

3. Main Results

3.1. Properties of Different Prequasi Norms. In this section, we have studied some topological structures and the Fatou property of the weighted Orlicz sequence space, $\ell_M(\lambda)$, for various prequasi norms.

Lemma 2. If M is a concave Orlicz function, then $M(x + y) \leq M(x) + M(y)$, for all $x, y \in [0, \infty)$.

Proof. It is easy so omitted. □

Theorem 5. $(\ell_M(\lambda))_\mu$ where $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for each $\beta \in \ell_M(\lambda)$, is a premodular (sss), if M is a concave Orlicz function or convex Orlicz function satisfying Δ_2 -condition.

Proof. Suppose M is a convex Orlicz function satisfying Δ_2 -condition. First, we must demonstrate that $\ell_M(\lambda)$ is a (sss):

(1)

(i) Let $\beta, \eta \in \ell_M(\lambda)$. As M is a strictly increasing and convex function satisfying Δ_2 -condition, we get

$$\begin{aligned} \mu(\beta + \eta) &= \sum_{y=0}^\infty \lambda_y M(|\beta_y + \eta_y|) \\ &\leq \frac{k}{2} \left[\sum_{y=0}^\infty \lambda_y M(|\beta_y|) + \sum_{y=0}^\infty \lambda_y M(|\eta_y|) \right] \quad (8) \\ &= \frac{k}{2} (\mu(\beta) + \mu(\eta)) < \infty, \end{aligned}$$

this implies $\beta + \eta \in \ell_M(\lambda)$.

(ii) Suppose $\omega \in \mathbf{R}$ and $\beta \in \ell_M(\lambda)$. Since M satisfies Δ_2 -condition, we have

$$\begin{aligned} \mu(\omega\beta) &= \sum_{y=0}^\infty \lambda_y M(|\omega\beta_y|) \\ &\leq k|\omega| \sum_{y=0}^\infty \lambda_y M(|\beta_y|) \leq D|\omega|\mu(\beta) < \infty. \end{aligned} \quad (9)$$

So $\omega\beta \in \ell_M(\lambda)$. Therefore, from conditions 1 (i) and (ii), one has $\ell_M(\lambda)$ is linear. We have $e_y \in \ell_M(\lambda)$, for every $y \in \mathcal{Z}^+$, as

(2) Let $|\beta_y| \leq |\eta_y|$, for every $y \in \mathcal{Z}^+$ and $\eta \in \ell_M(\lambda)$. Since M is a nondecreasing function, then

$$\begin{aligned} \mu(\beta) &= \sum_{y=0}^\infty \lambda_y M(|\beta_y|) \\ &\leq \sum_{y=0}^\infty \lambda_y M(|\eta_y|) = \mu(\eta) < \infty, \end{aligned} \quad (10)$$

one has $\beta \in \ell_M(\lambda)$.

(3) Assume $(\beta_y) \in \ell_M(\lambda)$, we get

$$\begin{aligned} \mu((\beta_{[y/2]})) &= \sum_{y=0}^\infty \lambda_y M(|\beta_{[y/2]}|) \\ &\leq 2 \sum_{y=0}^\infty \lambda_y M(|\beta_y|) = 2\mu((\beta_y)) < \infty, \end{aligned} \quad (11)$$

then $(\beta_{[y/2]}) \in \ell_M(\lambda)$. Second, to prove that the functional μ on $\ell_M(\lambda)$ is a premodular:

- (i) Obviously, $\mu(\beta) \geq 0$ and $\mu(\beta) = 0 \Leftrightarrow \beta = \theta$.
- (ii) There are $D = \max\{1, k\} \geq 1$ with $\mu(\omega\beta) \leq D|\omega|\mu(\beta)$, for every $\beta \in \ell_M(\lambda)$ and $\omega \in \mathbf{R}$.

(iii) There exists $J = \max\{1, k/2\} \geq 1$ with $\mu(\beta + \eta) \leq J(\mu(\beta) + \mu(\eta))$, for every $\beta, \eta \in \ell_M(\lambda)$.

(iv) Follows the proof part (2).

(v) Follows from the proof part (3) that $J_0 = 2 \geq 1$.

(vi) Obviously, $\overline{F} = \ell_M(\lambda)$.

(vii) There exists $0 < \zeta \leq M_0(|\omega|)/|\omega|M_0(1)$, for $\omega \neq 0$ or $\zeta > 0$, for $\omega = 0$ so that $\mu(\omega, 0, 0, \dots) \geq \zeta|\omega|\mu(1, 0, 0, \dots)$.

If M is a concave Orlicz function. By applying Lemma 2 and the parallel proof follows. \square

Theorem 6. If M is a concave Orlicz function or convex Orlicz function satisfying Δ_2 -condition, then $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss), where $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for each $\beta \in \ell_M(\lambda)$.

Proof. Suppose M is a convex Orlicz function satisfying Δ_2 -condition. By using Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). From Theorem 1, the space $(\ell_M(\lambda))_\mu$ is a prequasi normed (sss). To prove that $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss), let $\beta^r = (\beta_y^r)_{y=0}^\infty$ be a Cauchy sequence in $(\ell_M(\lambda))_\mu$. Therefore, for all $\epsilon \in (0, 1)$, we have that for every $r, t \geq r_0$, we get

$$\mu(\beta^r - \beta^t) = \sum_{y=0}^\infty \lambda_y M(|\beta_y^r - \beta_y^t|) < \epsilon. \quad (12)$$

Hence, for $r, t \geq r_0$ and $y \in \mathcal{Z}^+$, one has $|\beta_y^r - \beta_y^t| < \epsilon$. Then (β_y^t) is a Cauchy sequence in \mathbf{R} , for fixed $y \in \mathcal{Z}^+$. This gives $\lim_{t \rightarrow \infty} \beta_y^t = \beta_y^0$, for constant $y \in \mathcal{Z}^+$. Therefore, $\mu(\beta^r - \beta^0) < \epsilon$, for all $r \geq r_0$. To investigate that $\beta^0 \in \ell_M(\lambda)$, one has $\mu(\beta^0) = \mu(\beta^0 - \beta^r + \beta^r) \leq J(\mu(\beta^r - \beta^0) + \mu(\beta^r)) < \infty$, so $\beta^0 \in \ell_M(\lambda)$. This implies that $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss). If M is a concave Orlicz function. By applying Lemma 2 and the parallel proof follows. \square

Theorem 7. If M is a concave Orlicz function or convex Orlicz function satisfying Δ_2 -condition, then $(\ell_M(\lambda))_\mu$ is a prequasi closed (sss), where $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for every $\beta \in \ell_M(\lambda)$.

Proof. Let M be a convex Orlicz function satisfying Δ_2 -condition. According to Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). From Theorem 1, the space $(\ell_M(\lambda))_\mu$ is a prequasi normed (sss). To prove that $(\ell_M(\lambda))_\mu$ is a prequasi closed (sss), suppose $\beta^r = (\beta_y^r)_{y=0}^\infty \in (\ell_M(\lambda))_\mu$ and $\lim_{r \rightarrow \infty} \mu(\beta^r - \beta^0) = 0$, hence for all $\epsilon \in (0, 1)$, one has $r_0 \in \mathcal{Z}^+$ so that for every $r \geq r_0$, we have

$$\mu(\beta^r - \beta^0) = \sum_{y=0}^\infty \lambda_y M(|\beta_y^r - \beta_y^0|) < \epsilon. \quad (13)$$

Therefore, for $r \geq r_0$ and $y \in \mathcal{Z}^+$, one has $|\beta_y^r - \beta_y^0| < \epsilon$. Hence, (β_y^r) is a convergent sequence in \mathbf{R} , for constant $y \in \mathcal{Z}^+$. So, $\lim_{r \rightarrow \infty} \beta_y^r = \beta_y^0$, for constant $y \in \mathcal{Z}^+$. Finally to show that $\beta^0 \in \ell_M(\lambda)$, one has

$$\mu(\beta^0) = \mu(\beta^0 - \beta^r + \beta^r) \leq J(\mu(\beta^r - \beta^0) + \mu(\beta^r)) < \infty. \quad (14)$$

Hence, HTML translation failed. This implies that $(\mathcal{L}_M(\lambda))_\mu$ is a prequasi closed (sss). If M is a concave Orlicz function, by applying Lemma 2 and the parallel proof follows. \square

Theorem 8. *If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex, then the function*

$$\begin{aligned} \mu(\beta - \eta) &= M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|) \right) \leq M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|) \right) + M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|) \right) \\ &\leq \sup_j \inf_{b \geq j} \mu(\beta - \eta^b). \end{aligned} \quad (15)$$

Hence, μ satisfies the Fatou property. \square

Theorem 9. *If M is a concave Orlicz function, then the function $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$ holds the Fatou property, for all $\beta \in \mathcal{L}_M(\lambda)$.*

Proof. Suppose $\{\eta^b\} \subseteq (\mathcal{L}_M(\lambda))_\mu$ so that $\lim_{b \rightarrow \infty} \mu(\eta^b - \eta) = 0$. As the space $(\mathcal{L}_M(\lambda))_\mu$ is a prequasi closed space; hence, $\eta \in (\mathcal{L}_M(\lambda))_\mu$. As M is continuous, concave and $M(0) = 0$. Therefore, for every $\beta \in (\mathcal{L}_M(\lambda))_\mu$, one has

$$\begin{aligned} \mu(\beta - \eta) &= \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|) \leq \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|) \\ &+ \sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|) \leq \sup_j \inf_{b \geq j} \mu(\beta - \eta^b). \end{aligned} \quad (16)$$

Therefore, μ does not hold the Fatou property. \square

Example 2. For every $\beta \in \mathcal{L}_M(\lambda)$, the function $\mu(\beta) = \ln(1 + \sum_{y=0}^{\infty} \lambda_y (e^{|\beta_y|} - 1))$ is a prequasi norm, not quasi, and not a norm.

Example 3. For all $\beta \in \mathcal{L}_M(\lambda)$, the function $\mu(\beta) = (\sum_{y=0}^{\infty} \lambda_y \sqrt{|\beta_y|})^2$ is a prequasi norm, quasi norm, and not a norm.

Example 4. The function $\mu(\beta) = \inf\{\kappa > 0: \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|/\kappa) \leq 1\}$ is a prequasi norm, a quasi norm, and a norm on $\mathcal{L}_M(\lambda)$.

$\mu(\beta) = M^{-1}(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y|))$ verifies the Fatou property, for all $\beta \in \mathcal{L}_M(\lambda)$.

Proof. Assume that $\{\eta^b\} \subseteq (\mathcal{L}_M(\lambda))_\mu$ such that $\lim_{b \rightarrow \infty} \mu(\eta^b - \eta) = 0$. As the space $(\mathcal{L}_M(\lambda))_\mu$ is a prequasi closed space, one has $\eta \in (\mathcal{L}_M(\lambda))_\mu$. Hence, for every $\beta \in (\mathcal{L}_M(\lambda))_\mu$, from Lemma 1, we have

Hence, μ satisfies the Fatou property. \square

Theorem 10. *The function $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$ does not satisfy the Fatou property, for all $\beta \in \mathcal{L}_M(\lambda)$, if M is a strictly convex Orlicz function satisfying Δ_2 -condition.*

Proof. Since M is a strictly convex Orlicz function satisfying Δ_2 -condition, then there exists $k > 2$ such that $2M(u) < M(2u) < kM(u)$, for all $u \geq 0$. Let the conditions be fulfilled and $\{\eta^b\} \subseteq (\mathcal{L}_M(\lambda))_\mu$ with $\lim_{b \rightarrow \infty} \mu(\eta^b - \eta) = 0$. As the space $(\mathcal{L}_M(\lambda))_\mu$ is a prequasi closed space; hence, $\eta \in (\mathcal{L}_M(\lambda))_\mu$. Since M is continuous, then for any $\beta \in (\mathcal{L}_M(\lambda))_\mu$, we have

$$\mu(\beta - \eta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|) \leq \frac{k}{2} \left[\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|) + \sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|) \right] \leq \frac{k}{2} \sup_j \inf_{b \geq j} \mu(\beta - \eta^b). \quad (17)$$

4. Kannan μ -Contraction Operator

We now define Kannan μ -Lipschitzian mapping acting on $(\mathcal{L}_M(\lambda))_\mu$. The sufficient conditions for a fixed point of Kannan contraction mapping on $(\mathcal{L}_M(\lambda))_\mu$ under various prequasi norms are investigated.

Definition 11. An operator $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ is called a Kannan μ -Lipschitzian, if there exists $\nu \geq 0$, so that

$$\mu(H\beta - H\eta) \leq \nu(\mu(H\beta - \beta) + \mu(H\eta - \eta)), \quad (18)$$

for every $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$.

- (1) The operator H is said to be Kannan μ -contraction, when $\nu \in [0, 1/2)$.

(2) The operator H is said to be Kannan μ -non-expansive, whenever $\nu = 1/2$.

A vector $\beta \in (\mathcal{L}_M(\lambda))_\mu$ is called a fixed point of H , when $H(\beta) = \beta$.

Theorem 11. *If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex, and*

$H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ is Kannan μ -contraction mapping, where $\mu(\beta) = M^{-1}(\sum_{y=0}^\infty \lambda_y M(|\beta_y|))$, for all $\beta \in \mathcal{L}_M(\lambda)$; hence, H has a unique fixed point.

Proof. Assume that $\beta \in \mathcal{L}_M(\lambda)$, one has $H^t \beta \in \mathcal{L}_M(\lambda)$. Since H is a Kannan μ -contraction mapping, we have

$$\begin{aligned} \mu(H^{t+1}\beta - H^t\beta) &\leq \nu(\mu(H^{t+1}\beta - H^t\beta) + \mu(H^t\beta - H^{t-1}\beta)) \Rightarrow \\ \mu(H^{t+1}\beta - H^t\beta) &\leq \frac{\nu}{1-\nu} \mu(H^t\beta - H^{t-1}\beta) \leq \left(\frac{\nu}{1-\nu}\right)^2 \mu(H^{t-1}\beta - H^{t-2}\beta) \leq \dots \leq \left(\frac{\nu}{1-\nu}\right)^t \mu(H\beta - \beta). \end{aligned} \tag{19}$$

Therefore, for every $t, \nu \in \mathcal{L}^+$ with $\nu > t$, then we get $\mu(H^t\beta - H^\nu\beta) \leq \nu(\mu(H^t\beta - H^{t-1}\beta) + \mu(H^\nu\beta - H^{\nu-1}\beta))$

$$\leq \nu \left(\left(\frac{\nu}{1-\nu}\right)^{t-1} + \left(\frac{\nu}{1-\nu}\right)^{\nu-1} \right) \mu(H\beta - \beta). \tag{20}$$

So, $\{H^t\beta\}$ is a Cauchy sequence in $(\mathcal{L}_M(\lambda))_\mu$. As the space $(\mathcal{L}_M(\lambda))_\mu$ is prequasi Banach space. Therefore, there is $\eta \in (\mathcal{L}_M(\lambda))_\mu$ such that $\lim_{t \rightarrow \infty} H^t\beta = \eta$. To prove that $H\eta = \eta$. As μ holds the Fatou property, we obtain

$$\begin{aligned} \mu(H\eta - \eta) &\leq \sup_p \inf_{t \geq p} \mu(H^{t+1}\beta - H^t\beta) \\ &\leq \sup_p \inf_{t \geq p} \left(\frac{\nu}{1-\nu}\right)^t \mu(H\beta - \beta) = 0, \end{aligned} \tag{21}$$

hence $H\eta = \eta$. Hence, η is a fixed point of H . To prove the uniqueness of the fixed point. For different fixed points $\zeta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ of H . We have that

$$\mu(\zeta - \eta) \leq \mu(H\zeta - H\eta) \leq \nu(\mu(H\zeta - \zeta) + \mu(H\eta - \eta)) = 0. \tag{22}$$

Therefore, $\zeta = \eta$. □

Corollary 1. *Let M be a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ be convex, and $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ be Kannan μ -contraction mapping, with $\mu(\beta) = M^{-1}(\sum_{y=0}^\infty \lambda_y M(|\beta_y|))$, for every $\beta \in \mathcal{L}_M(\lambda)$, then H has a unique fixed point ζ such that $\mu(H^t\beta - \zeta) \leq \nu(\nu/1 - \nu)^{t-1} \mu(H\beta - \beta)$.*

Proof. From Theorem 11, there is a unique fixed point ζ of H . Hence, one has

$$\mu(H^t\beta - \zeta) = \mu(H^t\beta - H\zeta) \leq \nu(\mu(H^t\beta - H^{t-1}\beta) + \mu(H\zeta - \zeta)) = \nu \left(\frac{\nu}{1-\nu}\right)^{t-1} \mu(H\beta - \beta). \tag{23}$$

Theorem 12. *Suppose M is a concave Orlicz function, and $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ is Kannan μ -contraction mapping, where $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for all $\beta \in \mathcal{L}_M(\lambda)$; hence, H has a unique fixed point.*

Proof. It is easy so omitted. □

Definition 13. *Assume $(\mathcal{L}_M(\lambda))_\mu$ is a pr-quasi normed (sss), $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ and $\zeta \in (\mathcal{L}_M(\lambda))_\mu$. The operator H is called μ -sequentially continuous at ζ , if and only if, when $\lim_{y \rightarrow \infty} \mu(\beta_y - \zeta) = 0$, then $\lim_{y \rightarrow \infty} \mu(H\beta_y - H\zeta) = 0$.*

Theorem 14. *Let M be a strictly convex Orlicz function satisfying Δ_2 -condition, and $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ where $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for every $\beta \in \mathcal{L}_M(\lambda)$. The element $\eta \in (\mathcal{L}_M(\lambda))_\mu$ is the unique fixed point of H , if the next conditions are satisfied:*

- (i) H is Kannan μ -contraction mapping,

- (ii) H is μ -sequentially continuous at a point $\eta \in (\mathcal{L}_M(\lambda))_\mu$
- (iii) There exists $\beta \in (\mathcal{L}_M(\lambda))_\mu$ such that the sequence of iterates $\{H^t\beta\}$ has a subsequence $\{H^{t_p}\beta\}$ converging to η .

Proof. Since M is a strictly convex Orlicz function satisfying Δ_2 -condition, then there exists $k > 2$ such that $2M(u) < M(2u) < kM(u)$, for all $u \geq 0$. Let the conditions be verified. If η is not a fixed point of H , then $H\eta \neq \eta$. By the conditions (ii) and (iii), we have

$$\begin{aligned} \lim_{t_p \rightarrow \infty} \mu(H^{t_p}\beta - \eta) &= 0, \\ \lim_{t_p \rightarrow \infty} \mu(H^{t_p+1}\beta - H\eta) &= 0. \end{aligned} \tag{24}$$

As the operator H is Kannan μ -contraction, one can see

$$\begin{aligned}
 0 < \mu(H\eta - \eta) &= \mu((H\eta - H^{t_p+1}\beta) + (H^{t_p}\beta - \eta) + (H^{t_p+1}\beta - H^{t_p}\beta)) \\
 &\leq \frac{k^2}{4}\mu(H^{t_p+1}\beta - H\eta) + \frac{k^2}{4}\mu(H^{t_p}\beta - \eta) + \frac{k}{2}\nu\left(\frac{\nu}{1-\nu}\right)^{t_p-1}\mu(H\beta - \beta).
 \end{aligned}
 \tag{25}$$

Since $t_p \rightarrow \infty$, this gives a contradiction. Hence, η is a fixed point of H . To prove that the uniqueness of the fixed point η . For different fixed points $\eta, \zeta \in (\mathcal{L}_M(\lambda))_\mu$ of H . Therefore, one has

$$\mu(\eta - \zeta) \leq \mu(H\eta - H\zeta) \leq \nu(\mu(H\eta - \eta) + \mu(H\zeta - \zeta)) = 0.
 \tag{26}$$

So, $\eta = \zeta$. □

Example 15. Assume $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$ where $M(t) = \sqrt[3]{t} + \sqrt[4]{t}$ and $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$, for all $\beta \in \mathcal{L}_M(\lambda)$ and

$$H(\beta) = \begin{cases} \frac{\beta}{18}, & \mu(\beta) \in [0, 1), \\ \frac{\beta}{20}, & \mu(\beta) \in [1, \infty). \end{cases}
 \tag{27}$$

As for each $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0, 1)$, one has

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{18}\right) \leq \frac{1}{\sqrt[4]{17}}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{17\beta_2}{18}\right)\right) \\
 &= \frac{1}{\sqrt[4]{17}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{28}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1, \infty)$, one has

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{20} - \frac{\beta_2}{20}\right) \leq \frac{1}{\sqrt[4]{19}}\left(\mu\left(\frac{19\beta_1}{20}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right) \\
 &= \frac{1}{\sqrt[4]{19}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{29}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0, 1)$ and $\mu(\beta_2) \in [1, \infty)$, we obtain

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{20}\right) \leq \frac{1}{\sqrt[4]{17}}\mu\left(\frac{17\beta_1}{18}\right) + \frac{1}{\sqrt[4]{19}}\mu\left(\frac{19\beta_2}{20}\right) \\
 &\leq \frac{1}{\sqrt[4]{17}}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right) \\
 &= \frac{1}{\sqrt[4]{17}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{30}$$

Hence, the operator H is Kannan μ -contraction. As μ verifies the Fatou property. From Theorem 11, the operator H has a unique fixed point $\theta \in (\mathcal{L}_M(\lambda))_\mu$.

Assume $\{\beta^{(y)}\} \subseteq (\mathcal{L}_M(\lambda))_\mu$ is such that $\lim_{y \rightarrow \infty} \mu(\beta^{(y)} - \beta^{(0)}) = 0$, where $\beta^{(0)} \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta^{(0)}) = 1$. As the prequasi norm μ is continuous, one can see

$$\begin{aligned}
 \lim_{t_p \rightarrow \infty} \mu(H\beta^{(y)} - H\beta^{(0)}) &= \lim_{t_p \rightarrow \infty} \mu\left(\frac{\beta^{(y)}}{18} - \frac{\beta^{(0)}}{20}\right) \\
 &= \mu\left(\frac{\beta^{(0)}}{180}\right) > 0.
 \end{aligned}
 \tag{31}$$

Therefore, H is not μ -sequentially continuous at $\beta^{(0)}$. Hence, the operator H is not continuous at $\beta^{(0)}$.

Let $\mu(\beta) = [\sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)]^4$, for all $\beta \in \mathcal{L}_M(\lambda)$.

As for all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0, 1)$, one has

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{18}\right) \leq \frac{8}{17}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{17\beta_2}{18}\right)\right) \\
 &= \frac{8}{17}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{32}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1, \infty)$, one has

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{20} - \frac{\beta_2}{20}\right) \leq \frac{8}{19}\left(\mu\left(\frac{19\beta_1}{20}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right) \\
 &= \frac{8}{19}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{33}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0, 1)$ and $\mu(\beta_2) \in [1, \infty)$, we get

$$\begin{aligned}
 \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{20}\right) \leq \frac{8}{17}\mu\left(\frac{17\beta_1}{18}\right) + \frac{8}{19}\mu\left(\frac{19\beta_2}{20}\right) \\
 &\leq \frac{8}{17}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right) \\
 &= \frac{8}{17}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).
 \end{aligned}
 \tag{34}$$

So, the operator H is Kannan μ -contraction and $H^t(\beta) =$

$$\begin{cases} \beta/18^t & \mu(\beta) \in [0, 1) \\ \beta/20^t & \mu(\beta) \in [1, \infty) \end{cases}$$

Clearly, H is μ -sequentially continuous at $\theta \in (\mathcal{L}_M(\lambda))_\mu$ and $\{H^t\beta\}$ contains a subsequence $\{H^{t_p}\beta\}$ converging to θ . From Theorem 14, then $\theta \in (\mathcal{L}_M(\lambda))_\mu$ is the unique fixed point of H .

Example 5. Assume $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$, where $M(t) = t^2$ and $\mu(\beta) = \sqrt{\sum_{y=0}^\infty \lambda_y M(|\beta_y|)}$, for all $\beta \in \mathcal{L}_M(\lambda)$ and

$$H(\beta) = \begin{cases} \frac{\beta}{4}, & \mu(\beta) \in [0, 1), \\ \frac{\beta}{5}, & \mu(\beta) \in [1, \infty). \end{cases} \tag{35}$$

As for each $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0, 1)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{4}\right) \leq \frac{1}{3} \left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{3\beta_2}{4}\right) \right) \\ &= \frac{1}{3} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{36}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1, \infty)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{5} - \frac{\beta_2}{5}\right) \leq \frac{1}{4} \left(\mu\left(\frac{4\beta_1}{5}\right) + \mu\left(\frac{4\beta_2}{5}\right) \right) \\ &= \frac{1}{4} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{37}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0, 1)$ and $\mu(\beta_2) \in [1, \infty)$, we get

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{5}\right) \leq \frac{1}{3} \mu\left(\frac{3\beta_1}{4}\right) + \frac{1}{4} \mu\left(\frac{4\beta_2}{5}\right) \\ &\leq \frac{1}{3} \left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{4\beta_2}{5}\right) \right) \\ &= \frac{1}{3} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{38}$$

Hence, the operator H is Kannan μ -contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H has one fixed point $\theta \in (\mathcal{L}_M(\lambda))_\mu$.

Suppose $\{\beta^{(y)}\} \subseteq (\mathcal{L}_M(\lambda))_\mu$ is so that $\lim_{y \rightarrow \infty} \mu(\beta^{(y)} - \beta^{(0)}) = 0$, where $\beta^{(0)} \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta^{(0)}) = 1$. As the prequasi norm μ is continuous, one can see

$$\begin{aligned} \lim_{y \rightarrow \infty} \mu(H\beta^{(y)} - H\beta^{(0)}) &= \lim_{y \rightarrow \infty} \mu\left(\frac{\beta^{(y)}}{4} - \frac{\beta^{(0)}}{5}\right) \\ &= \mu\left(\frac{\beta^{(0)}}{20}\right) > 0. \end{aligned} \tag{39}$$

Therefore, H is not μ -sequentially continuous at $\beta^{(0)}$. Hence, the map H is not continuous at $\beta^{(0)}$.

Let $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for every $\beta \in \mathcal{L}_M(\lambda)$.

As for each $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0, 1)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{4}\right) \leq \frac{2}{9} \left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{3\beta_2}{4}\right) \right) \\ &= \frac{2}{9} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{40}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1, \infty)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{5} - \frac{\beta_2}{5}\right) \leq \frac{1}{8} \left(\mu\left(\frac{4\beta_1}{5}\right) + \mu\left(\frac{4\beta_2}{5}\right) \right) \\ &= \frac{1}{8} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{41}$$

For all $\beta_1, \beta_2 \in (\mathcal{L}_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0, 1)$ and $\mu(\beta_2) \in [1, \infty)$, we obtain

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{5}\right) \leq \frac{2}{9} \mu\left(\frac{3\beta_1}{4}\right) + \frac{1}{8} \mu\left(\frac{4\beta_2}{5}\right) \\ &\leq \frac{2}{9} \left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{4\beta_2}{5}\right) \right) \\ &= \frac{2}{9} (\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)). \end{aligned} \tag{42}$$

So, the operator H is Kannan μ -contraction and.

$$H^t(\beta) = \begin{cases} \beta/4^t & \mu(\beta) \in [0, 1) \\ \beta/5^t & \mu(\beta) \in [1, \infty) \end{cases}$$

Obviously, H is μ -sequentially continuous at $\theta \in (\mathcal{L}_M(\lambda))_\mu$ and $\{H^t\beta\}$ has a subsequence $\{H^{t_p}\beta\}$ converging to θ . From Theorem 14, then $\theta \in (\mathcal{L}_M(\lambda))_\mu$ is the unique fixed point of H .

Example 16. Suppose $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$, where $M(t) = \sqrt[3]{t} + \sqrt[4]{t}$ and $\mu(\beta) = (\sum_{y=0}^\infty \lambda_y M(|\beta_y|))^4$, for every $\beta \in \mathcal{L}_M(\lambda)$ and

$$H(\beta) = \begin{cases} \frac{1}{18}(e_0 + \beta), & \beta_0 \in (-\infty, \frac{1}{17}), \\ \frac{1}{17}e_0, & \beta_0 = \frac{1}{17}, \\ \frac{1}{18}e_0, & \beta_0 \in \left(\frac{1}{17}, \infty\right). \end{cases} \tag{43}$$

As for each $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/17)$, one has

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{1}{18}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \dots)\right) \\ &\leq \frac{8}{17} \left(\mu\left(\frac{17\beta}{18}\right) + \mu\left(\frac{17\eta}{18}\right) \right) \\ &\leq \frac{8}{17} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \tag{44}$$

For every $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/17, \infty)$, then for all $\epsilon > 0$ one has

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (45)$$

For every $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ and $\eta_0 \in (1/17, \infty)$, we get

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{18}\right) \leq \frac{1}{17}\mu\left(\frac{17\beta}{18}\right) = \frac{1}{17}\mu(H\beta - \beta) \\ &\leq \frac{1}{17}(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (46)$$

Hence, the operator H is Kannan μ -contraction. Evidently, H is μ -sequentially continuous at $1/17e_0 \in (\mathcal{L}_M(\lambda))_\mu$ and we have $\beta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ under $\{H^t\beta\} = \{\sum_{n=1}^t 1/18^n e_0 + 1/18^t \beta\}$ contains a subsequence $\{H^{t_p}\beta\} = \{\sum_{n=1}^{t_p} 1/18^n e_0 + 1/18^{t_p} \beta\}$ converging to $1/17e_0$. From Theorem 14, the map H has a unique fixed point $1/17e_0 \in (\mathcal{L}_M(\lambda))_\mu$. Observe that H is not continuous at $1/17e_0 \in (\mathcal{L}_M(\lambda))_\mu$.

If $\mu(\beta) = \sum_{y \in \mathcal{X}^+} \lambda_y M(|\beta_y|)$, for every $\beta \in \mathcal{L}_M(\lambda)$. As for all $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/17)$, one has

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{1}{18}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \dots)\right) \\ &\leq \frac{1}{\sqrt[3]{17}} \left(\mu\left(\frac{17\beta}{18}\right) + \mu\left(\frac{17\eta}{18}\right) \right) \\ &\leq \frac{1}{\sqrt[3]{17}} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (47)$$

For each $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/17, \infty)$, then for all $\epsilon > 0$ we get

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (48)$$

For every $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ and $\eta_0 \in (1/17, \infty)$, this gives

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{18}\right) \leq \frac{1}{\sqrt[3]{17}}\mu\left(\frac{17\beta}{18}\right) = \frac{1}{\sqrt[3]{17}}\mu(H\beta - \beta) \\ &\leq \frac{1}{\sqrt[3]{17}} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (49)$$

So, the operator H is Kannan μ -contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H holds one fixed point $1/17e_0 \in (\mathcal{L}_M(\lambda))_\mu$.

Example 6. Assume $H: (\mathcal{L}_M(\lambda))_\mu \rightarrow (\mathcal{L}_M(\lambda))_\mu$, where $M(t) = t^2 + 2t$ and $\mu(\beta) = \sum_{y=0}^\infty \lambda_y M(|\beta_y|)$, for every $\beta \in \mathcal{L}_M(\lambda)$ and

$$H(\beta) = \begin{cases} \frac{1}{6}(e_1 + \beta), & \beta_0 \in (-\infty, \frac{1}{5}), \\ \frac{1}{5}e_1, & \beta_0 = \frac{1}{5}, \\ \frac{1}{6}e_1, & \beta_0 \in (\frac{1}{5}, \infty). \end{cases} \quad (50)$$

As for each $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/5)$, one has

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{1}{6}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \dots)\right) \\ &\leq \frac{2}{5} \left(\mu\left(\frac{5\beta}{6}\right) + \mu\left(\frac{5\eta}{6}\right) \right) \\ &\leq \frac{2}{5} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (51)$$

Suppose $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/5, \infty)$, then for any $\epsilon > 0$ we obtain

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (52)$$

Assume $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ and $\eta_0 \in (1/5, \infty)$, one can see

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{6}\right) \leq \frac{1}{5}\mu\left(\frac{5\beta}{6}\right) = \frac{1}{5}\mu(H\beta - \beta) \\ &\leq \frac{1}{5} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (53)$$

Hence, the operator H is Kannan μ -contraction. Clearly, H is μ -sequentially continuous at $1/5e_1 \in (\mathcal{L}_M(\lambda))_\mu$ and there exists $\beta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ under $\{H^t\beta\} = \{\sum_{n=1}^t 1/6^n e_1 + 1/6^t \beta\}$ contains a subsequence $\{H^{t_p}\beta\} = \{\sum_{n=1}^{t_p} 1/6^n e_1 + 1/6^{t_p} \beta\}$ converging to $1/5e_1$. From Theorem 14, the operator H holds a unique fixed point $1/5e_1 \in (\mathcal{L}_M(\lambda))_\mu$. Observe that H is not continuous at $1/5e_1 \in (\mathcal{L}_M(\lambda))_\mu$.

If $M(t) = t^2$ and $\mu(\beta) = \sqrt{\sum_{y=0}^\infty \lambda_y M(|\beta_y|)}$, for every $\beta \in \mathcal{L}_M(\lambda)$.

Since for all $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/5)$, one has

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{1}{6}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \dots)\right) \\ &\leq \frac{1}{5} \left(\mu\left(\frac{5\beta}{6}\right) + \mu\left(\frac{5\eta}{6}\right) \right) \\ &\leq \frac{1}{5} (\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \quad (54)$$

If $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/5, \infty)$, then for all $\epsilon > 0$ one has

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \tag{55}$$

Assume $\beta, \eta \in (\mathcal{L}_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ and $\eta_0 \in (1/5, \infty)$, we get

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{6}\right) \leq \frac{1}{5}\mu\left(\frac{5\beta}{6}\right) = \frac{1}{5}\mu(H\beta - \beta) \\ &\leq \frac{1}{5}(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \tag{56}$$

So, the operator H is Kannan μ -contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H contains one fixed point $1/5e_1 \in (\mathcal{L}_M(\lambda))_\mu$.

5. Kannan Nonexpansive Operator

We have presented in this section the uniform convexity of the space $(\mathcal{L}_M(\lambda))_\mu$ where

$$\mathcal{L}_M(\lambda) = \{u \in \mathbf{R}^{\mathcal{F}^+} : \varrho(\omega u) < \infty, \text{ for some } \omega > 0\}, \tag{57}$$

and $\varrho(u) = \sum_{y=0}^{\infty} \lambda_y M(|u_y|)$, under the Luxemburg norm

$$\mu(u) = \inf\left\{\omega > 0 : \varrho\left(\frac{u}{\omega}\right) \leq 1\right\}. \tag{58}$$

Definition 12.

(1) The continuous function M is called strictly convex (SC), if

$$M\left(\frac{v+t}{2}\right) < \frac{M(v) + M(t)}{2}, \tag{59}$$

for all $v, t \in [0, \infty)$ and $v \neq t$.

(2) [24] The following statements are equivalent:

- (i) M is a uniformly convex function on $[0, \infty)$.
- (ii) For any $\epsilon > 0$ and $u_0 > 0$, there exists a number $\delta \in (0, 1)$ such that for all u, v , and $|u - v| \geq \epsilon \max\{|u|, |v|\} \geq \epsilon u_0$ imply

$$M\left(\frac{u+v}{2}\right) \leq \frac{1-\delta}{2}(M(u) + M(v)), \tag{60}$$

if $u \geq u_0$.

(iii) For any $u_0 > 0$ and $a \in (0, 1)$, there exists a number $\delta \in (0, 1)$ such that

if $u \geq u_0$.

(3) [25] A normed space (X, μ) is said to be strictly convex if for any $u, v \in X$ and $b > 0$ satisfying $\mu(u) \leq b$, $\mu(v) \leq b$, and $\mu(u - v) > 0$ imply $\mu(u + v/2) < b$.

(4) [26] A normed space (X, μ) is said to be uniformly convex if for any $b > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $u, v \in X$ satisfying $\mu(u) \leq b$, $\mu(v) \leq b$ and $\mu(u - v) \geq \epsilon$ imply $\mu(u + v/2) \leq b - \delta$.

Theorem 17. If $\lim_{n \rightarrow \infty} \mu(x_n) = b$, $\lim_{n \rightarrow \infty} \mu(y_n) = b$ and $\lim_{n \rightarrow \infty} \mu(x_n + y_n/2) = b$ imply $\lim_{n \rightarrow \infty} \mu(x_n - y_n) = 0$, for all $\{x_n\}, \{y_n\} \subset \mathcal{L}_M(\lambda)$ and $b > 0$, then $\mathcal{L}_M(\lambda)$ is uniformly convex, where M is a convex Orlicz function satisfying Δ_2 -condition.

Proof. Let the conditions be satisfied and $\mathcal{L}_M(\lambda)$ is not uniformly convex, then there exists $\epsilon_0 > 0$ and $\{x_n\}, \{y_n\} \subset \mathcal{L}_M(\lambda)$ such that $\mu(x_n) \leq b$, $\mu(y_n) \leq b$, $\mu(x_n - y_n) \geq \epsilon_0$ we get $\mu(x_n + y_n/2) > b - 1/n$, for some $b > 0$. To prove that $\lim_{n \rightarrow \infty} \mu(x_n) = b$, let $\lim_{n \rightarrow \infty} \mu(x_n) = b_1 < b$ and $\lim_{n \rightarrow \infty} \mu(y_n) = b$. Since M is satisfying Δ_2 -condition, we have $\lim_{n \rightarrow \infty} \varrho(x_n/b_1) = 1$ and $\lim_{n \rightarrow \infty} \varrho(x_n/b) = 1$. Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varrho\left(\frac{x_n + y_n}{2b}\right) &\leq \frac{b_1}{2b} \lim_{n \rightarrow \infty} \varrho\left(\frac{x_n}{b_1}\right) \\ &+ \frac{1}{2} \lim_{n \rightarrow \infty} \varrho\left(\frac{y_n}{b}\right) < 1. \end{aligned} \tag{61}$$

This is equivalent to $\overline{\lim}_{n \rightarrow \infty} \mu(x_n + y_n/2) < b$. This contradicts $\lim_{n \rightarrow \infty} \mu(x_n + y_n/2) = b$, so $\lim_{n \rightarrow \infty} \varrho(x_n) = b$. Similarly, we can prove that $\lim_{n \rightarrow \infty} \varrho(y_n) = b$. Also since

$$1 < \liminf_{n \rightarrow \infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \leq \lim_{n \rightarrow \infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \leq \overline{\lim}_{n \rightarrow \infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \leq \frac{1}{2} \left(\overline{\lim}_{n \rightarrow \infty} \varrho\left(\frac{x_n}{b}\right) + \overline{\lim}_{n \rightarrow \infty} \varrho\left(\frac{y_n}{b}\right) \right) = 1. \tag{62}$$

Then $\lim_{n \rightarrow \infty} \varrho(x_n + y_n/2b) = 1$. This implies $\lim_{n \rightarrow \infty} \mu(x_n + y_n/2) = b$. But $\lim_{n \rightarrow \infty} \mu(x_n - y_n) \geq \epsilon_0 > 0$, this gives a contradiction. \square

Theorem 18. The space $\mathcal{L}_M(\lambda)$ is uniformly convex, if M is a uniformly convex Orlicz function satisfying Δ_2 -condition.

Proof. Assume the settings are satisfied, $\lim_{n \rightarrow \infty} \mu(x_n) = b$, $\lim_{n \rightarrow \infty} \mu(y_n) = b$, and $\lim_{n \rightarrow \infty} \mu(x_n + y_n/2) = b$, we will prove that $\lim_{n \rightarrow \infty} \mu(x_n - y_n) = 0$. For any $\epsilon \in (0, 1/2)$, let us choose $u_0 > 0$ such that $M(2u_0) < \epsilon$. Since M is uniformly convex, then there exists $\delta \in (0, 1)$ such that $|u - v| \geq \epsilon \max\{|u|, |v|\} \geq \epsilon u_0$ imply

$$M\left(\frac{u+v}{2}\right) \leq \frac{1-\delta}{2} (M(u) + M(v)). \tag{63}$$

For each $n \in \mathcal{Z}^+$, put

$$\begin{aligned} G_n &= \left\{ i \in \mathcal{Z}^+ : \left| \frac{x_n(i)}{b} \right|, \left| \frac{y_n(i)}{b} \right| < u_0 \right\}, \\ E_n &= \left\{ i \in \mathcal{Z}^+ : \left| \frac{x_n(i) - y_n(i)}{b} \right| < \epsilon \max \left\{ \left| \frac{x_n(i)}{b} \right|, \left| \frac{y_n(i)}{b} \right| \right\} < \epsilon u_0 \right\}, \\ F_n &= \left\{ i \in \mathcal{Z}^+ : \left| \frac{x_n(i) - y_n(i)}{b} \right| \geq \epsilon \max \left\{ \left| \frac{x_n(i)}{b} \right|, \left| \frac{y_n(i)}{b} \right| \right\} \geq \epsilon u_0 \right\}. \end{aligned} \tag{64}$$

Then we deduce

$$\sum_{i \in G_n} M\left(\left| \frac{x_n(i) - y_n(i)}{b} \right|\right) \leq M(2u_0), \tag{65}$$

and thus

$$\begin{aligned} \sum_{i \in E_n} M\left(\left| \frac{x_n(i) - y_n(i)}{b} \right|\right) &\leq 2\epsilon \sum_{i \in \mathcal{Z}^+} M\left(\frac{|x_n(i)| + |y_n(i)|}{2b}\right) \\ &\leq \epsilon \left(\sum_{i \in \mathcal{Z}^+} M\left(\left| \frac{x_n(i)}{b} \right|\right) + \sum_{i \in \mathcal{Z}^+} M\left(\left| \frac{y_n(i)}{b} \right|\right) \right) \leq 2\epsilon. \end{aligned} \tag{66}$$

Hence, we get

$$\begin{aligned} 0 &\leftarrow \frac{\mu(x_n) + \mu(y_n)}{2} - \mu\left(\frac{x_n + y_n}{2}\right) \\ &= \frac{\sum_{i \in \mathcal{Z}^+} M(|x_n(i)/b|) + \sum_{i \in \mathcal{Z}^+} M(y_n(i)/b)}{2} \\ &\quad - \sum_{i \in \mathcal{Z}^+} M\left(\frac{|x_n(i) + y_n(i)|}{2b}\right) \\ &\geq \frac{\sum_{i \in F_n} M(|x_n(i)/b|) + \sum_{i \in F_n} M(y_n(i)/b)}{2} \\ &\quad - \sum_{i \in F_n} M\left(\frac{|x_n(i) + y_n(i)|}{2b}\right) \\ &\geq \frac{\sum_{i \in F_n} M(|x_n(i)/b|) + \sum_{i \in F_n} M(y_n(i)/b)}{2} \\ &\quad - \frac{1-\delta}{2} \left[\sum_{i \in F_n} M(|x_n(i)/b|) + \sum_{i \in F_n} M(y_n(i)/b) \right] \\ &= \frac{\delta}{2} \left[\sum_{i \in F_n} M(|x_n(i)/b|) + \sum_{i \in F_n} M(y_n(i)/b) \right]. \end{aligned} \tag{67}$$

Since u_0 and ϵ are arbitrary, then $\lim_{n \rightarrow \infty} \varrho(x_n - y_n/2b) = 0$. As M verifies Δ_2 -condition. Therefore, $\lim_{n \rightarrow \infty} \mu(x_n - y_n) = 0$. From Theorem 17, the proof follows.

Here, we discuss the property (R) and the μ -normal structure property of the space $(\mathcal{L}_M(\lambda))_\mu$. \square

Definition 13. The space $(Y)_\mu$ holds the property (R), if for all decreasing sequence $\{\Phi_x\}_{x \in \mathcal{Z}^+}$ of μ -closed and μ -convex

nonempty subsets of $(Y)_\mu$ so that $\sup_{x \in \mathcal{Z}^+} d_\mu(\beta, \Phi_x) < \infty$, for some $\beta \in (Y)_\mu$; hence, we have $\bigcap_{x \in \mathcal{Z}^+} \Phi_x \neq \emptyset$.

Definition 14. The space $(Y)_\mu$ holds the μ -normal structure property if for all nonempty μ -bounded, μ -convex, and μ -closed subset Φ of $(Y)_\mu$ not decreased to one point, there exists $\beta \in \Phi$ with

$$\sup_{\eta \in \Phi} \mu(\beta - \eta) < v_\mu(\Phi) := \sup\{\mu(\beta - \eta) : \beta, \eta \in \Phi\} < \infty. \tag{68}$$

Theorem 19. If M is a uniformly convex Orlicz function satisfying Δ_2 -condition, then

- (1) Assume Φ is a nonempty μ -closed and μ -convex subset of $(\mathcal{L}_M(\lambda))_\mu$. For $\beta \in (\mathcal{L}_M(\lambda))_\mu$ with

$$d_\mu(\beta, \Phi) = \inf\{\mu(\beta - \eta) : \eta \in \Phi\} < \infty. \tag{69}$$

Therefore, we have one $\phi \in \Phi$ with $d_\mu(\beta, \Phi) = \mu(\beta - \phi)$.

- (2) $(\mathcal{L}_M(\lambda))_\mu$ satisfies the property (R).

Proof. For (1), assume $\beta \notin \Phi$ as Φ is μ -closed. So, one has $D := d_\mu(\beta, \Phi) > 0$. Therefore, there is $\eta_t \in \Phi$ so that $\lim_{t \rightarrow \infty} \mu(\beta - \eta_t) = D$. To prove that $\{\eta_t\}$ is a μ -Cauchy. For any two subsequences $\{\eta_{t_a}\}$ and $\{\eta_{t_b}\} \subset \{\eta_t\}$, we have $\mu(\eta_{t_a} + \eta_{t_b}/2 - \beta) \geq D$, as $\lim_{a \rightarrow \infty} \mu(\beta - \eta_{t_a}) = D$ and $\lim_{b \rightarrow \infty} \mu(\beta - \eta_{t_b}) = D$. Moreover,

$$\mu\left(\frac{\eta_{t_a} + \eta_{t_b}}{2} - \beta\right) = \mu\left(\frac{\eta_{t_a} - \beta}{2} + \frac{\eta_{t_b} - \beta}{2}\right) < \frac{1}{2} (D + D) = D. \tag{70}$$

Therefore, we have $\lim_{a,b \rightarrow \infty} \mu(\eta_{t_a} + \eta_{t_b}/2 - \beta) = D$. Since the space $(\mathcal{L}_M(\lambda))_\mu$ is uniformly convex, we get

$$\lim_{a,b \rightarrow \infty} \mu(\eta_{t_a} - \beta - (\eta_{t_b} - \beta)) = \lim_{a,b \rightarrow \infty} \mu(\eta_{t_a} - \eta_{t_b}) = 0. \tag{71}$$

Thus, $\{\eta_t\}$ is a μ -Cauchy in Φ . Since Φ is closed and the space $(\mathcal{L}_M(\lambda))_\mu$ is complete, then there exists $\phi \in \Phi$ with $\mu(\beta - \phi) = d_\mu(\beta, \Phi)$. Since the space $(\mathcal{L}_M(\lambda))_\mu$ is uniformly convex, then it is (SC), which implies the uniqueness of ϕ . To show (2), for some $t_0 \in \mathcal{Z}^+$, suppose $\beta \notin \Phi_{t_0}$. Since $(d_\mu(\beta, \Phi_t))_{t \in \mathcal{Z}^+} \in \mathcal{L}_\infty$ is increasing. Set $\lim_{t \rightarrow \infty} d_\mu(\beta, \Phi_t) = D$, when $D > 0$. Otherwise, $\beta \in \Phi_t$, for each $t \in \mathcal{Z}^+$. From (1), we have a unique $\eta_t \in \Phi_t$ with $d_\mu(\beta, \Phi_t) = \mu(\beta - \eta_t)$, for all $t \in \mathcal{Z}^+$. A consistent proof will show that $\{\eta_t/2\}$ μ -converges to some $\eta \in (\mathcal{L}_M(\lambda))_\mu$. Since $\{\Phi_t\}$ are μ -convex, decreasing and μ -closed, we get $2\eta \in \cap_{t \in \mathcal{Z}^+} \Phi_t$. \square

Theorem 20. *If M is a uniformly convex Orlicz function satisfying Δ_2 -condition, then $(\mathcal{L}_M(\lambda))_\mu$ has the μ -normal structure property.*

Proof. Let the conditions are satisfied. Theorem 18 gives that $(\mathcal{L}_M(\lambda))_\mu$ is uniformly convex. Assume Φ is a μ -bounded, μ -convex, and μ -closed subset of $(\mathcal{L}_M(\lambda))_\mu$ not decreased to one point. Hence, $v_\mu(\Phi) > 0$. Set $D = v_\mu(\Phi)$. Let $\beta, \eta \in \Phi$ with $\beta \neq \eta$. Hence, $\mu(\beta - \eta/2) > 0$. For every $\phi \in \Phi$, one has $\mu(\beta - \phi) \leq D$ and $\mu(\eta - \phi) \leq D$. As Φ is μ -convex, then $\beta + \eta/2 \in \Phi$. Hence,

$$\mu\left(\frac{\beta + \eta}{2} - \phi\right) = \mu\left(\frac{(\beta - \phi) + (\eta - \phi)}{2}\right) < D, \tag{72}$$

for every $\phi \in \Phi$. So

$$\sup_{\phi \in \Phi} \mu\left(\frac{\beta + \eta}{2} - \phi\right) < D = v_\mu(\Phi). \tag{73}$$

\square

Lemma 3. *Let the space $(\mathcal{L}_M(\lambda))_\mu$ verify the (R) property and the μ -quasi-normal property. Assume Φ is a nonempty μ -bounded, μ -convex, and μ -closed subset of $(\mathcal{L}_M(\lambda))_\mu$. Suppose $H: \Phi \rightarrow \Phi$ is a Kannan μ -nonexpansive mapping. For $x > 0$. If $W_x = \{\beta \in \Phi: \mu(\beta - H(\beta)) \leq x\} \neq \emptyset$. Set*

$$\Phi_x = \cap \left\{ \mathcal{B}_\mu(t, v): H(W_x) \subset \mathcal{B}_\mu(t, v) \right\} \cap \Phi. \tag{74}$$

Then Φ_x is a nonempty, μ -convex, μ -closed subset of Φ with $H(\Phi_x) \subset \Phi_x \subset W_x$ and $v_\mu(\Phi_x) \leq x$.

Proof. As $H(W_x) \subset \Phi_x$, this gives $\Phi_x \neq \emptyset$. Since the μ -balls are μ -convex, and μ -closed, then Φ_x is a μ -closed and μ -convex subset of Φ . To prove that $\Phi_x \subset W_x$. Assume $\beta \in \Phi_x$. If $\mu(\beta - H(\beta)) = 0$, we have $\beta \in W_x$. Otherwise, suppose $\mu(\beta - H(\beta)) > 0$. Set

$$t = \sup\{\mu(H(\zeta) - H(\beta)): \zeta \in W_x\}. \tag{75}$$

From the definition of t , then $H(W_x) \subset \mathcal{B}_\mu(H(\beta), t)$. Hence, $\Phi_x \subset \mathcal{B}_\mu(H(\beta), t)$, which implies $\mu(\beta - H(\beta)) \leq t$.

Assume $d > 0$. Hence, there is $\zeta \in W_x$ so that $t - d \leq \mu(H(\zeta) - H(\beta))$. Then

$$\begin{aligned} \mu(\beta - H(\beta)) - d &\leq t - d \leq \mu(H(\zeta) - H(\beta)) \\ &\leq \frac{1}{2}(\mu(\beta - H(\beta)) + \mu(\zeta - H(\zeta))) \\ &\leq \frac{1}{2}(\mu(\beta - H(\beta)) + x). \end{aligned} \tag{76}$$

Since d is arbitrarily positive, we have $\mu(\beta - H(\beta)) \leq x$, then we have $\beta \in W_x$. For $H(W_x) \subset \Phi_x$, we get $H(\Phi_x) \subset H(W_x) \subset \Phi_x$, this indicates Φ_x is H -invariant. Consequent to prove that $v_\mu(\Phi_x) \leq x$. As

$$\mu(H(\beta) - H(\eta)) \leq \frac{1}{2}(\mu(\beta - H(\beta)) + \mu(\eta - H(\eta))), \tag{77}$$

For every $\beta, \eta \in W_x$. Let $\beta \in W_x$. So $H(W_x) \subset \mathcal{B}_\mu(H(\beta), x)$. From the definition of Φ_x , one has $\Phi_x \subset \mathcal{B}_\mu(H(\beta), x)$. Hence, $H(\beta) \in \cap_{\eta \in \Phi_x} \mathcal{B}_\mu(\eta, x)$. Therefore, we have $\mu(\eta - \zeta) \leq x$, for every $\eta, \zeta \in \Phi_x$, which implies $v_\mu(\Phi_x) \leq x$. This finishes the proof.

In this part, we give enough settings on $(\mathcal{L}_M(\lambda))_\mu$ so that the Kannan μ -nonexpansive mapping defined on it contains a fixed point. \square

Theorem 21. *Let $(\mathcal{L}_M(\lambda))_\mu$ hold the μ -quasinormal property and the (R) property. Assume Φ is a nonempty, μ -convex, μ -closed, and μ -bounded subset of $(\mathcal{L}_M(\lambda))_\mu$. If $H: \Phi \rightarrow \Phi$ is a Kannan μ -nonexpansive mapping, then H has a fixed point.*

Proof. Let $x_t = x_0 + 1/t$, for all $t \geq 1$, where $x_0 = \inf\{\mu(\beta - H(\beta)): \beta \in \Phi\}$. We have for each $t \geq 1$ that $W_{x_t} = \{\beta \in \Phi: \mu(\beta - H(\beta)) \leq x_t\} \neq \emptyset$. Suppose Φ_{x_t} explained as in Lemma 3. Clearly, $\{\Phi_{x_t}\}$ is a decreasing sequence of nonempty μ -bounded, μ -closed, and μ -convex subsets of Φ . The property (R) gives that $\Phi_\infty = \cap_{t \geq 1} \Phi_{x_t} \neq \emptyset$. Let $\beta \in \Phi_\infty$, we have $\mu(\beta - H(\beta)) \leq x_t$, for every $t \geq 1$. If $t \rightarrow \infty$, one has $\mu(\beta - H(\beta)) \leq x_0$, which implies $\mu(\beta - H(\beta)) = x_0$. Hence, $W_{x_0} \neq \emptyset$. So $x_0 = 0$. Otherwise, $x_0 > 0$ which investigates that H has no fixed point. Assume Φ_{x_0} as defined in Lemma 3. Since H has no fixed point and Φ_{x_0} is H -invariant, hence Φ_{x_0} holds more than one point, which gives, $v_\mu(\Phi_{x_0}) > 0$. By the μ -quasinormal property, one has $\beta \in \Phi_{x_0}$ with

$$\mu(\beta - \eta) < v_\mu(\Phi_{x_0}) \leq x_0, \tag{78}$$

for every $\eta \in \Phi_{x_0}$. By Lemma 3, we have $\Phi_{x_0} \subset W_{x_0}$. By definition of Φ_{x_0} , then $H(\beta) \in W_{x_0} \subset \Phi_{x_0}$. Obviously, one has

$$\mu(\beta - H(\beta)) < v_\mu(\Phi_{x_0}) \leq x_0, \tag{79}$$

which contradicts the definition of x_0 . So $x_0 = 0$ this implies that any point in W_{x_0} is a fixed point of H , i.e., H has a fixed point in Φ .

According to Theorem 19, Theorem 20, and Theorem 21, we obtain the next corollary: \square

Corollary 2. *If M is a uniformly convex Orlicz function satisfying Δ_2 -condition. Assume Φ is a nonempty, μ -convex, μ -closed, and μ -bounded subset of $(\ell_M(\lambda))_\mu$. Suppose $H: \Phi \rightarrow \Phi$ is a Kannan μ -nonexpansive operator. Then H holds a fixed point.*

Example 7. Let $H: \Phi \rightarrow \Phi$ with $H(\beta) = \begin{cases} \beta/4, & \mu(\beta) \in [0, 1), \\ \beta/5, & \mu(\beta) \in [1, \infty), \end{cases}$

where $\Phi = \{\beta \in (\ell_M(\lambda))_\mu : \beta_0 = \beta_1 = 0\}$, where $\varrho(\beta) = \sum_{x=0}^\infty \lambda_x |\beta_x|^2$, for every $\beta \in (\ell_M(\lambda))_\mu$. As Example 5, the operator H is Kannan μ -contraction mapping. So it is Kannan μ -nonexpansive operator. Clearly, Φ is a nonempty, μ -convex, μ -closed and μ -bounded subset of $(\ell_M(\lambda))_\mu$. By Corollary 2, the operator H has a fixed point in Φ .

6. Kannan Y – Contraction Mapping on $S_{(\ell_M(\lambda))_\mu}$

For any two Banach spaces \mathfrak{X} and \mathfrak{Y} , we examine in this section the existence of a fixed point of Kannan Y - contraction mapping on $S_{(\ell_M(\lambda))_\mu}$, where $Y(Q) = M^{-1}(\sum_{y=0}^\infty \lambda_y M(|s_y(Q)|))$, for all $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$.

$$Y(Q) = \mu\left((s_y(Q))_{y=0}^\infty\right) = \mu\left((s_y(Q - Q_r + Q_r))_{y=0}^\infty\right) \leq \mu\left((s_{[y/2]}(Q - Q_r))_{y=0}^\infty\right) + \mu\left((s_{[y/2]}(Q_r))_{y=0}^\infty\right) \leq \mu\left((\|Q_r - Q\|)_{y=0}^\infty\right) + 2\mu\left((s_y(Q_r))_{y=0}^\infty\right) < \epsilon. \tag{81}$$

Therefore, $(s_y(Q))_{y=0}^\infty \in (\ell_M(\lambda))_\mu$, this implies $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. \square

Theorem 23. *If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex, then $(S_{(\ell_M(\lambda))_\mu}, Y)$ is a prequasi closed operator ideal, where $Y(Q) = \mu((s_y(Q))_{y=0}^\infty)$.*

Proof. As Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). Therefore, from Theorem 3, one has $Y(Q) = \mu((s_y(Q))_{y=0}^\infty)$ is a prequasi norm on $S_{(\ell_M(\lambda))_\mu}$. Assume $Q_r \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, for every $r \in \mathcal{Z}^+$ and $\lim_{r \rightarrow \infty} Y(Q_r - Q) = 0$. Hence, there is $\zeta > 0$ and since $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, we get

$$Y(Q_r - Q) = \mu\left((s_y(Q_r - Q))_{y=0}^\infty\right) \geq \mu(s_0(Q_r - Q), 0, 0, 0, \dots) = M^{-1}(\lambda_0 \|Q_r - Q\|). \tag{82}$$

Hence $(Q_r)_{r \in \mathcal{Z}^+}$ is convergent in $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. i.e., $\lim_{r \rightarrow \infty} \|Q_r - Q\| = 0$ and while $(s_y(Q_r))_{y=0}^\infty \in (\ell_M(\lambda))_\mu$ for all $r \in \mathcal{Z}^+$ and $(\ell_M(\lambda))_\mu$ is a premodular (sss). Hence, we have

Theorem 22. *If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex, then $(S_{(\ell_M(\lambda))_\mu}, Y)$ is a prequasi Banach operator ideal, where $Y(Q) = \mu((s_y(Q))_{y=0}^\infty)$.*

Proof. As Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). Therefore, from Theorem 3, one has $Y(Q) = \mu((s_y(Q))_{y=0}^\infty)$ is a prequasi norm on $S_{(\ell_M(\lambda))_\mu}$. Suppose $Q_r \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is a Cauchy sequence. As $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, one obtains

$$Y(Q_r - Q_t) = \mu\left((s_y(Q_r - Q_t))_{y=0}^\infty\right) \geq \mu(s_0(Q_r - Q_t), 0, 0, 0, \dots) = M^{-1}(\lambda_0 \|Q_r - Q_t\|). \tag{80}$$

Hence $(Q_r)_{r \in \mathcal{Z}^+}$ is a Cauchy sequence in $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Since $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a Banach space, so there is $Q \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ with $\lim_{r \rightarrow \infty} \|Q_r - Q\| = 0$. Since $(s_y(Q_r))_{y=0}^\infty \in (\ell_M(\lambda))_\mu$ for every $r \in \mathcal{Z}^+$. We have

$$Y(Q) = \mu\left((s_y(Q))_{y=0}^\infty\right) = \mu\left((s_y(Q - Q_r + Q_r))_{y=0}^\infty\right) \leq \mu\left((s_{[y/2]}(Q - Q_r))_{y=0}^\infty\right) + \mu\left((s_{[y/2]}(Q_r))_{y=0}^\infty\right) \leq \mu\left((\|Q_r - Q\|)_{y=0}^\infty\right) + 2\mu\left((s_y(Q_r))_{y=0}^\infty\right) < \epsilon, \tag{83}$$

we have $(s_y(Q))_{y=0}^\infty \in (\ell_M(\lambda))_\mu$, then $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. \square

Definition 15. A prequasi norm Y on the ideal $S_{(\ell_M(\lambda))_\mu}$, where $Y(Q) = \mu((s_y(Q))_{y=0}^\infty)$, satisfies the Fatou property if for any sequence $\{Q_y\}_{y \in \mathcal{Z}^+} \subseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ with $\lim_{y \rightarrow \infty} Y(Q_y - Q) = 0$ and any $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, then $Y(V - Q) \leq \sup_{j \geq y} \inf_{j \geq y} \mu(V - Q_j)$.

Theorem 25. *The prequasinorm $Y(Q) = M^{-1}(\sum_{y=0}^\infty \lambda_y M(s_y(Q)))$, for all $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ does not satisfy the Fatou property, if M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex.*

Proof. Assume the settings are satisfied and $\{Q_t\}_{t \in \mathcal{Z}^+} \subseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ with $\lim_{t \rightarrow \infty} Y(Q_t - Q) = 0$. Since the space $S_{(\ell_M(\lambda))_\mu}$ is a prequasi closed ideal, then,

$Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. Hence, for any $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\begin{aligned} Y(V - Q) &= M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(s_y(V - Q)) \right) \leq M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(s_{\lfloor y/2 \rfloor}(V - Q_j)) \right) + M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(s_{\lfloor \lceil y/2 \rceil}(Q_j - Q)) \right) \\ &\leq 2 \sup_t \inf_{j \geq t} M^{-1} \left(\sum_{y=0}^{\infty} \lambda_y M(s_y(V - Q_j)) \right). \end{aligned} \tag{84}$$

Hence, Y does not satisfy the Fatou property. \square

Definition 16. An operator $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \rightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is called a Kannan Y -Lipschitzian, if there exists $\nu \geq 0$, so that for every $Q, T \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, we have

$$Y(PQ - PT) \leq \nu(Y(PQ - Q) + Y(PT - T)). \tag{85}$$

- (1) If $\nu \in [0, 1/2)$, the operator P is said to be Kannan Y -contraction.
- (2) If $\nu = 1/2$, the operator P is said to be Kannan Y -nonexpansive.

Definition 17. An operator $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \rightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is said to be Y -sequentially continuous at V , if and only if, when $\lim_{t \rightarrow \infty} Y(Q_t - V) = 0$, then $\lim_{t \rightarrow \infty} Y(PQ_t - PV) = 0$.

Theorem 26. If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is conve, and

$$\begin{aligned} 0 < Y(PT - T) &= Y((PT - P^{t_j+1}V) + (P^{t_j}V - T) + (P^{t_j+1}V - P^{t_j}V)) \\ &\leq 2Y(P^{t_j+1}V - PT) + 4Y(P^{t_j}V - T) + 4\nu \left(\frac{\nu}{1-\nu} \right)^{t_j-1} Y(PV - V). \end{aligned} \tag{87}$$

Since $t_j \rightarrow \infty$, we have a contradiction. Hence, T is a fixed point of P . To prove the uniqueness of the fixed point T . Let we have two different fixed points $T, U \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ of P . Therefore, one has

$$\begin{aligned} Y(T - U) &\leq Y(PT - PU) \\ &\leq \nu(Y(PT - T) + Y(PU - U)) = 0. \end{aligned} \tag{88}$$

Hence, $T = U$. \square

Example 8. Assume $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \rightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, where $Y(Q) = \sum_{y=0}^{\infty} \lambda_y \sqrt{s_y(Q)}$, for every $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ and

$P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \rightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. The point $T \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is the unique fixed point of P , when the following conditions are satisfied:

- (i) P is Kannan Y -contraction mapping,
- (ii) P is Y -sequentially continuous at a point $T \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$,
- (iii) There exists $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ so that the sequence of iterates $\{P^t V\}$ has a subsequence $\{P^{t_j} V\}$ converging to T .

Proof. Let the conditions be verified. If T is not a fixed point of P , then $PT \neq T$. From the conditions (ii) and (iii), we have

$$\begin{aligned} \lim_{t_j \rightarrow \infty} Y(P^{t_j} V - T) &= 0, \\ \lim_{t_j \rightarrow \infty} Y(P^{t_j+1} V - PT) &= 0. \end{aligned} \tag{86}$$

Since P is Kannan Y -contraction mapping, one can see

$$P(Q) = \begin{cases} \frac{Q}{26}, & Y(Q) \in [0, 1), \\ \frac{Q}{37}, & Y(Q) \in [1, \infty). \end{cases} \tag{89}$$

As for every $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1), Y(Q_2) \in [0, 1)$, one has

$$\begin{aligned} Y(PQ_1 - PQ_2) &= Y\left(\frac{Q_1}{26} - \frac{Q_2}{26}\right) \leq \frac{2}{5} \left(Y\left(\frac{25Q_1}{26}\right) + Y\left(\frac{25Q_2}{26}\right) \right) \\ &= \frac{2}{5} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \end{aligned} \tag{90}$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1), Y(Q_2) \in [1, \infty)$, we get

$$Y(PQ_1 - PQ_2) = Y\left(\frac{Q_1}{37} - \frac{Q_2}{37}\right) \leq \frac{1}{3} \left(Y\left(\frac{36Q_1}{37}\right) + Y\left(\frac{36Q_2}{37}\right) \right) = \frac{1}{3} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \tag{91}$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1) \in [0, 1)$ and $Y(Q_2) \in [1, \infty)$, one can see

$$\begin{aligned} Y(PQ_1 - PQ_2) &= Y\left(\frac{Q_1}{26} - \frac{Q_2}{37}\right) \leq \frac{2}{5} Y\left(\frac{25Q_1}{26}\right) + \frac{1}{3} Y\left(\frac{36Q_2}{37}\right) \leq \frac{2}{5} \left(Y\left(\frac{25Q_1}{26}\right) + Y\left(\frac{36Q_2}{37}\right) \right) \\ &= \frac{2}{5} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \end{aligned} \tag{92}$$

So, the operator Q is Kannan Y -contraction and $P^t(Q) = \begin{cases} Q/26^t, & Y(Q) \in [0, 1), \\ Q/37^t, & Y(Q) \in [1, \infty). \end{cases}$

Clearly, P is Y -sequentially continuous at the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ and $\{P^t Q\}$ has a subsequence $\{P^{t_j} Q\}$ converging to Θ . From Theorem 27, the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ is the unique fixed point of P . Suppose with $\lim_{t \rightarrow \infty} Y(Q^{(t)} - Q^{(0)}) = 0 - b \pm \sqrt{b^2 - 4ac}/2a$, where $Q^{(0)} \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q^{(0)}) = 1$. From the continuously of the prequasi norm Y , one has

$$\begin{aligned} \lim_{t \rightarrow \infty} Y(PQ^{(t)} - PQ^{(0)}) &= \lim_{t \rightarrow \infty} Y\left(\frac{Q^{(t)}}{26} - \frac{Q^{(0)}}{37}\right) \\ &= Y\left(\frac{11Q^{(0)}}{962}\right) > 0. \end{aligned} \tag{93}$$

So P is not Y -sequentially continuous at $Q^{(0)}$. This implies the operator P is not continuous at $Q^{(0)}$.

Example 9. Suppose $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \rightarrow S_{(\ell_M(\lambda))_\gamma}(\mathfrak{X}, \mathfrak{Y})$, where $Y(Q) = \sqrt{\sum_{y=0}^\infty \lambda_y (s_y(Q))^2}$, for every $Q \in S_{(\ell_M(\lambda))_\gamma}(\mathfrak{X}, \mathfrak{Y})$ and

$$P(Q) = \begin{cases} \frac{Q}{5}, & Y(Q) \in [0, 1), \\ \frac{Q}{6}, & Y(Q) \in [1, \infty). \end{cases} \tag{94}$$

As for each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1), Y(Q_2) \in [0, 1)$, one can see

$$\begin{aligned} Y(PQ_1 - PQ_2) &= Y\left(\frac{Q_1}{5} - \frac{Q_2}{5}\right) \leq \frac{\sqrt{2}}{4} \left(Y\left(\frac{4Q_1}{5}\right) + Y\left(\frac{4Q_2}{5}\right) \right) \\ &= \frac{\sqrt{2}}{4} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \end{aligned} \tag{95}$$

For every $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1), Y(Q_2) \in [1, \infty)$, this implies

$$\begin{aligned} Y(PQ_1 - PQ_2) &= Y\left(\frac{Q_1}{6} - \frac{Q_2}{6}\right) \leq \frac{\sqrt{2}}{5} \left(Y\left(\frac{5Q_1}{6}\right) + Y\left(\frac{5Q_2}{6}\right) \right) \\ &= \frac{\sqrt{2}}{5} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \end{aligned} \tag{96}$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q_1) \in [0, 1)$ and $Y(Q_2) \in [1, \infty)$, one obtains

$$\begin{aligned} Y(PQ_1 - PQ_2) &= Y\left(\frac{Q_1}{5} - \frac{Q_2}{6}\right) \leq \frac{\sqrt{2}}{4} Y\left(\frac{4Q_1}{5}\right) + \frac{\sqrt{2}}{5} Y\left(\frac{5Q_2}{6}\right) \\ &\leq \frac{\sqrt{2}}{4} \left(Y\left(\frac{4Q_1}{5}\right) + Y\left(\frac{5Q_2}{6}\right) \right) \\ &= \frac{\sqrt{2}}{4} (Y(PQ_1 - Q_1) + Y(PQ_2 - Q_2)). \end{aligned} \tag{97}$$

So, the operator Q is Kannan Y -contraction and. $P^t(Q) = \begin{cases} Q/5^t, & Y(Q) \in [0, 1), \\ Q/6^t, & Y(Q) \in [1, \infty). \end{cases}$

Evidently, P is Y -sequentially continuous at the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ and $\{P^t Q\}$ has a subsequence $\{P^{t_j} Q\}$ converging to Θ . From Theorem 27, the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ is the unique fixed point of P . Suppose

$\{Q^{(t)}\} \subseteq S_{(\ell_M(\lambda))_\mu}$ with $\lim_{t \rightarrow \infty} Y(Q^{(t)} - Q^{(0)}) = 0$, where $Q^{(0)} \in S_{(\ell_M(\lambda))_\mu}$ with $Y(Q^{(0)}) = 1$. From the continuity of the prequasi norm Y , one has

$$\begin{aligned} \lim_{t \rightarrow \infty} Y(PQ^{(t)} - PQ^{(0)}) &= \lim_{t \rightarrow \infty} Y\left(\frac{Q^{(t)}}{5} - \frac{Q^{(0)}}{6}\right) \\ &= Y\left(\frac{Q^{(0)}}{30}\right) > 0. \end{aligned} \tag{98}$$

So P is not Y -sequentially continuous at $Q^{(0)}$. Hence, the operator P is not continuous at $Q^{(0)}$.

7. Applications on Summable Equations

We investigate here a solution to (101), which studied by many authors (see [27–29]), in $(\ell_M(\lambda))_\mu$.

$$\begin{aligned} &M\left(\left|\sum_{y \in \mathcal{Z}^+} D(x, y)(h(y, \beta_y) - h(y, \eta_y))\right|\right) \\ &\leq M(\nu) \left[M\left(\left|r_x - \beta_x + \sum_{y=0}^{\infty} D(x, y)f(y, \beta_y)\right|\right) + M\left(\left|r_x - \eta_x + \sum_{y=0}^{\infty} D(x, y)f(y, \eta_y)\right|\right) \right], \end{aligned} \tag{101}$$

then equation (101) hold a solution in $(\ell_M(\lambda))_\mu$ where $\mu(\beta) = M^{-1}\left(\sum_{x=0}^{\infty} M(|\beta_x|)\right)$, for every $\beta \in \ell_M(\lambda)$.

$$\beta_x = r_x + \sum_{y=0}^{\infty} D(x, y)h(y, \beta_y). \tag{99}$$

Suppose $H: (\ell_M(\lambda))_\mu \rightarrow (\ell_M(\lambda))_\mu$ constructed by

$$H(\beta_x)_{x \in \mathcal{Z}^+} = \left(r_x + \sum_{y=0}^{\infty} D(x, y)h(y, \beta_y) \right)_{x \in \mathcal{Z}^+}. \tag{100}$$

Theorem 27. If M is a convex Orlicz function satisfying Δ_2 -condition and $\ln(M(e^x))$ is convex, $D: \mathcal{Z}^{+2} \rightarrow \mathbf{R}$, $h: \mathcal{Z}^+ \times \mathbf{R} \rightarrow \mathbf{R}$, $r: \mathcal{Z}^+ \rightarrow \mathbf{R}$, and for all $x \in \mathcal{Z}^+$, there exists $\nu \in [0, 1/2)$, with

Proof. Suppose the setups are verified. We have

$$\begin{aligned} \mu(H\beta - H\eta) &= M^{-1}\left(\sum_{x \in \mathcal{Z}^+} \lambda_x M(|H\beta_x - H\eta_x|)\right) = M^{-1}\left(\sum_{x \in \mathcal{Z}^+} \lambda_x M\left(\left|\sum_{m \in \mathcal{Z}^+} D(x, y)[h(y, \beta_y) - h(y, \eta_y)]\right|\right)\right) \\ &\leq \nu M^{-1}\left(\sum_{x \in \mathcal{Z}^+} \lambda_x M\left(\left|r_x - \beta_x + \sum_{m=0}^{\infty} D(x, y)h(y, \beta_y)\right|\right)\right) \\ &\quad + \nu M^{-1}\left(\sum_{x \in \mathcal{Z}^+} \lambda_x M\left(\left|r_x - \eta_x + \sum_{m=0}^{\infty} D(x, y)h(y, \eta_y)\right|\right)\right) \\ &= \nu(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \tag{102}$$

In view of Theorem 11, there exists a unique solution of equation (101) in $(\ell_M(\lambda))_\mu$. \square

Example 10. For the space $(\ell_M(\lambda))_\mu$, where $\mu(\beta) = \sqrt[4]{\sum_{x \in \mathcal{Z}^+} |\beta_x|^4}$, for all $\beta \in \ell_M(\lambda)$. Assume the summable equations are defined as

$$\beta_x = e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1}\right)^v \sin y, \tag{103}$$

where $\nu > 2$ and let $H: (\ell_M(\lambda))_\mu \rightarrow (\ell_M(\lambda))_\mu$ is defined by

$$H(\beta_x)_{x \in \mathcal{Z}^+} = \left(e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1}\right)^v \sin y \right)_{x \in \mathcal{Z}^+}. \tag{104}$$

We have

$$\begin{aligned} & \left| \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1} \right)^v (\sin y - \sin y) \right|^4 \\ & \leq \frac{1}{81} \left[\left| e^{-(3x+6)} - \beta_x + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1} \right)^v \sin y \right|^4 + \left| e^{-(3x+6)} - \eta_x + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\eta_x}{x^2 + y^2 + 1} \right)^v \sin y \right|^4 \right]. \end{aligned} \tag{105}$$

By Theorem 27, the summable equations (105) have one solution in $(\ell_M(\lambda))_\mu$.

Theorem 30. *If M is a concave Orlicz function, $D: \mathcal{Z}^{+2} \rightarrow \mathbf{R}$, $h: \mathcal{Z}^+ \times \mathbf{R} \rightarrow \mathbf{R}$, $r: \mathcal{Z}^+ \rightarrow \mathbf{R}$, and for every $x \in \mathcal{Z}^+$, there exists $v \in [0, 1/2]$, with*

$$\begin{aligned} & M \left(\left| \sum_{y \in \mathcal{Z}^+} D(x, y) (h(y, \beta_y) - h(y, \eta_y)) \right| \right) \\ & \leq v \left[M \left(\left| r_x - \beta_x + \sum_{y=0}^{\infty} D(x, y) f(y, \beta_y) \right| \right) + M \left(\left| r_x - \eta_x + \sum_{y=0}^{\infty} D(x, y) f(y, \eta_y) \right| \right) \right], \end{aligned} \tag{106}$$

then equation (101) contains one solution in $(\ell_M(\lambda))_\mu$ where $\mu(\beta) = \sum_{x=0}^{\infty} M(|\beta_x|)$, for each $\beta \in \ell_M(\lambda)$.

Proof. Suppose the setups are verified. One has

$$\begin{aligned} \mu(H\beta - H\eta) &= \sum_{x \in \mathcal{Z}^+} \lambda_x M(|H\beta_x - H\eta_x|) = \sum_{x \in \mathcal{Z}^+} \lambda_x M \left(\left| \sum_{y \in \mathcal{Z}^+} D(x, y) (h(y, \beta_y) - h(y, \eta_y)) \right| \right) \\ & \leq v \left[\sum_{x \in \mathcal{Z}^+} \lambda_x M \left(\left| r_x - \beta_x + \sum_{y=0}^{\infty} D(x, y) h(y, \beta_y) \right| \right) + \sum_{x \in \mathcal{Z}^+} \lambda_x M \left(\left| r_x - \eta_x + \sum_{y=0}^{\infty} D(x, y) h(y, \eta_y) \right| \right) \right] \\ & = v(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \end{aligned} \tag{107}$$

In view of Theorem 12, there exists a unique solution of equation (101) in $(\ell_M(\lambda))_\mu$. \square

Example 11. For the space $(\ell_M(\lambda))_\mu$ where $\mu(\beta) = \sum_{x \in \mathcal{Z}^+} \sqrt[3]{|\beta_x|}$, for every $\beta \in \ell_M(\lambda)$. Assume the summable equations

$$\beta_x = e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v, \tag{108}$$

where $v > 2$ and let $H: (\ell_M(\lambda))_\mu \rightarrow (\ell_M(\lambda))_\mu$ is defined by

$$H(\beta_x)_{x \in \mathcal{Z}^+} = \left(e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \right)_{x \in \mathcal{Z}^+}. \tag{109}$$

It is easy to see that

$$\begin{aligned} & \left| \sum_{y=0}^{\infty} (-1)^x \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v ((-1)^y - (-1)^y) \right|^{1/3} \\ & \leq \frac{1}{3} \left[\left| e^{-(3x+6)} - \beta_x + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \right|^{1/3} + \left| e^{-(3x+6)} - \eta_x + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \right|^{1/3} \right]. \end{aligned} \tag{110}$$

By Theorem 30, the summable equation (105) has an unique solution in $(\mathcal{L}_M(\lambda))_\mu$.

Example 12. Given the sequence space $(\mathcal{L}_M(\lambda))_\mu$ where $\mu(\beta) = \sqrt{\sum_{x \in \mathbb{Z}^+} |\beta_x|^2}$, for all $\beta \in \mathcal{L}_M(\lambda)$. Consider the summable equations (110), with $x \geq 2$ and $v > 2$ and let $H: \Phi \rightarrow \Phi$, where $\Phi = \{\beta \in (\mathcal{L}_M(\lambda))_\mu; \beta_0 = \beta_1 = 0\}$, defined by

$$\left| \sum_{y=0}^{\infty} (-1)^x \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \left((-1)^y - (-1)^y \right) \right|^2 \leq \frac{1}{9} \left[\left| e^{-(3x+6)} - \beta_x + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \right|^2 + \left| e^{-(3x+6)} - \eta_x + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\eta_x|}}{x^2 + y^2 + 1} \right)^v \right|^2 \right]. \quad (112)$$

By Theorem 27 and Corollary 2, the summable equation (110) have a solution in Φ .

8. Conclusion

We explored the presence of a fixed point for both Kannan contraction and nonexpansive mappings working on various premodular, which is a generalization of modular, defined by weighted Orlicz sequence space and its pre-quasi operator ideal. Numerous numerical experiments and practical applications are used to substantiate our findings.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally to the writing of this paper. All the authors read and approved the final manuscript.

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$$H(\beta_x)_{x \geq 2} = \left(e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^{x+y} \left(\frac{e^{|\beta_x|}}{x^2 + y^2 + 1} \right)^v \right)_{x \geq 2}. \quad (111)$$

Obviously, Φ is a nonempty, μ -convex, μ -closed, and μ -bounded subset of $(\mathcal{L}_M(\lambda))_\mu$. It is easy to see that

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