Hindawi
Journal of Mathematics
Volume 2022, Article ID 2138165, 21 pages
https://doi.org/10.1155/2022/2138165

Research Article

Dynamics of COVID-19 Using SEIQR Epidemic Model

N. Avinash[1], **G. Britto Antony Xavier**[1], **Ammar Alsinai**[2], **Hanan Ahmed**[3], **V. Rexma Sherine**[1], and **P. Chellamani**[1]

[1] *Department of Mathematics, Sacred Heart College, Tirupattur 635601, Tamilnadu, India*
[2] *Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570005, Karnataka, India*
[3] *Department of Mathematics, Ibb University, Ibb, Yemen*

Correspondence should be addressed to Hanan Ahmed; hananahmed1a@gmail.com

Received 19 May 2022; Revised 9 August 2022; Accepted 20 August 2022; Published 26 September 2022

Academic Editor: Ali Sajid

The major goal of this study is to create an optimal technique for managing COVID-19 spread by transforming the SEIQR model into a dynamic (multistage) programming problem with continuous and discrete time-varying transmission rates as optimizing variables. We have developed an optimal control problem for a discrete-time, deterministic susceptible class (S), exposed class (E), infected class (I), quarantined class (Q), and recovered class (R) epidemic with a finite time horizon. The problem involves finding the minimum objective function of a controlled process subject to the constraints of limited resources. For our model, we present a new technique based on dynamic programming problem solutions that can be used to minimize infection rate and maximize recovery rate. We developed suitable conditions for obtaining monotonic solutions and proposed a dynamic programming model to obtain optimal transmission rate sequences. We explored the positivity and unique solvability nature of these implicit and explicit time-discrete models. According to our findings, isolating the affected humans can limit the danger of COVID-19 spreading in the future.

1. Introduction

Mathematical models are useful for determining how an infection behaves when it enters a population and determining whether it will be eradicated or continue under different settings. COVID-19 is currently causing tremendous concern among researchers, governments, and the general public due to its rapid spread and a high number of deaths [1]. The transmission of this disease is caused by the tiny particles or droplets called aerosols that carry the virus into the atmosphere caused by a contaminated person while sneezing, coughing, or exhaling. Many researchers and scientists are continuously working to reduce the transmission of this vicious disease throughout the world. Infectious diseases are the disciplines that focus on the study of the dynamics of infectious diseases as well as the relationship between these diseases and the various factors involved in their appearance and evolution, in order to implement a fight against this spread. Despite its youth, mathematical modeling is a valuable tool for understanding disease transmission mechanisms, is playing an increasingly important role in epidemiology, and has already contributed to significant successes. The most influential work in the field of mathematics epidemiology was first introduced by Kermack and McKendrick as the SIR model in the year 1927 [2]. Cao et al.[3] discussed a modified model of the SIR (susceptible, infected, recovered) epidemic introduced in order to detect the confirmed number of infected cases and consecutive burdens on isolation wards and ICUs. Also, Nesteruk [4] developed the variables used in the proposed model by introducing a SIR epidemic model and explaining how to dominate the spread of the disease. To restore the pandemic with the involvement of social distancing and lockdown, Gerberry and Milner presented a data-driven susceptible, exposed, infected, quarantined, and recovered (SEIQR) model in [5]. From the publication of Zeb et al. [6], epidemiological's SEIQR model with isolation class in 2020 and their mathematical epidemiology has expanded in numerous directions, involving biology and computer science by Zima et al.[7], Zhou et al. [8], and Kermack and McKendrick

[9, 10]. Some recent studies have focused on this area of research by He et al. [11], Rahimi et al. [12], Hussain et al. [13], Youssef et al. [14], Prabakaran et al. [15], and Youssef et al. [16]. In this current paper, we implement the discrete type of SEIQR model and discuss the solvability of both continuous and discrete type SEIQR model. We examined the behaviour of the time-continuous model. We have developed two time-discrete models: time-implicit and time-explicit. We looked at the theory and methods for solving the time-implicit model. Then, to control and anticipate the dynamics of COVID dissemination, we establish appropriate transmission rate limits. To do this, we devised a dynamic programming problem to optimize transmission rate sequences under arbitrary beginning conditions. We propose safety guidelines and essential precautionary measures based on the optimized rate sequences to control COVID spread. The article gives the technique for optimizing the transmission rate sequences. The epidemic models and their time-discrete variations have been studied by Allen [17] and Ghosh et al. [18]. Several approaches towards fractional-order mathematical models of COVID-19 were studied by the authors Alqhtani et al. [19], Valliammal and Ravichandran [20], Nisar et al. [21], Vijayakumar et al. [22], and Alderremy et al. [23]. However, the aforementioned studies and references mostly contain explicit approaches with respect to time-discrete epidemic models.

1.1. Review Literature. In 2020, COVID-19 is a worldwide emergency. The first cases occurred in December 2019, and as of 6 : 34 pm CEST, 28 July 2022, there have been 571,198,904 confirmed cases of COVID-19, including 6,387,863 deaths, reported to WHO. As of 25 July 2022, a total of 12,248,795,623 vaccine doses have been administered. The rapid spread of COVID-19 has already caused great public attention and many heated discussions, and the Chinese mass media have been reporting relevant information about the virus and the outbreak.

Ming et al. [24] show that effective public health measures are required to be implemented in time to avoid the breakdown of the health system, and the media can certainly play a crucial role in conveying updated policies and regulations from authorities to the citizens. The finding that SARS-2-S exploits ACE2 for entry, which was also reported by Kermack and McKendrick [9] while the present manuscript was in revision, suggests that the virus might target a similar spectrum of cells as SARS-CoV. However, upon its outbreak, various research, including but not limited to Okhuese [25], began to predict the scale that the virus would hit the world; the ratio of the death to recovery rate has seemingly been a positive proportion. Allen [17] studied about time-discrete SI, SIR, and SIS epidemic models, and its properties. Kermack and McKendrick [10] analyzed an outbreak such as the one in Hubei is captured by SIR dynamics where the population is divided into three compartments that differentiate the state of individuals with respect to the contagion process: infected (I), susceptible (S) to infection, and removed (R) (i.e., not taking part in the transmission process). Mathematical modeling has been influential in providing a deeper understanding on the transmission mechanisms and burden of the ongoing COVID-19 pandemic, contributing to the development of public health policy and understanding. Most mathematical models of the COVID-19 pandemic can broadly be divided into either population-based, SIR (Kermack-McKendrick)-type models, driven by (potentially stochastic) differential equations proposed by Nesteruk [4] in which individuals typically interact on a network structure and exchange infection stochastically. This point emerges also clearly from a number of recent model-based contributions that have extended the basic SIR model to account for key insights from economic theory, namely by allowing for peoples' (rational) adjustment of work, consumption, and leisure activities in the face of infection risk. More generally, the idea is to model explicitly the exposure to the virus (of those people who are susceptible), as in the susceptible-exposed-infectious-recovered (SEIR) model which has been analyzed extensively by He et al. [11] in the context of the COVID-19 pandemic. The Jacobian method used for the SEIR model yields a biologically reasonable R_0, but for more complex compartmental models, especially those with more infected compartments, the method is hard to apply as it relies on the algebraic Routh Hurwitz conditions for stability of the Jacobian matrix. An alternative method proposed by the authors Van den Driessche and Watmough [26] gives a way of determining R_0 for an ODE compartmental model by using the next generation matrix. Batista [27] applies logistic growth regression model to predict the final size of the Covid-19 epidemic. Basically, NSFD is an iterative method in which we get closer to solution through iteration was given by Mickens [28]. The authors Vijayakumar et al. [22] discussed about approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$. Finding the variants that predict severe disease, we developed a collaboration of four international computational centers (Iran, Italy, Malaysia, and Greece). In [29], the authors Bairagi et al. have introduced a mathematical model for controlling the outbreak of COVID-19 by augmenting isolation and social distancing features of individuals and also solved the utility maximization problem by using a nonco-operative game. In 2021, the multidisciplinary approach was necessary to address the multidimensional aspects of COVID-19 infection by established collaborations discussed by the authors Rahimi et al. [12]. The study by the authors Prabakaran et al. [15] looked into the evolving geographic diversity of the SARS-CoV-2. We then consider how positive factors like social distancing measures and detrimental factors such as delays in testing onset affect optimal testing strategies and outbreak controllability. Throughout, Youssef et al. [14] focus their analyses on empirically supported parameter values including realistic testing rates. While many existing COVID-19 SIR-like compartmental models explore the effects of testing with forms of isolation like quarantine or hospitalization, the majority of these studies assume simple linear equations for the rates at which tests are administered and individuals are isolated. The authors Hussain et al. [13]

discussed about the complex systems and network science approaches, along with technological advances and data availability, are becoming instrumental for the design of effective containment strategies. In a nonsense region, Hilfer's neutral fractional derivative provided controllability results using Monch's method, Banach's contraction principle, fractional calculus, and semigroup property was studied by the authors Nisar et al. [21]. Some recent updates regarding the modeling of the coronavirus, the authors Alderremy et al. [23] constructed a mathematical model based on the fuzzy fractional derivative and obtained the results. The authors Valliammal and Ravichandran [20] are discussed in detail the fractional integro-differential equation with different conditions in various spaces. In [30], the authors Awal et al. proposed a framework that uses Bayesian optimization to optimize the hyperparameters of the classifier and adaptive synthetic (ADASYN) algorithm to balance the COVID and non-COVID classes of the dataset. In 2022, the authors Youssef et al. considered a modified model to analyze the disease dynamics of the coronavirus infection by taking the real cases from Saudi Arabia [16]. Alqhtani et al. [19] analyzed about spatiotemporal dynamical patterns arising from subdiffusion reaction-diffusion systems of predator-prey interaction are modeled in the sense of the Caputo fractional operator. Ghosh et al. [18] have studied about discrete-time epidemic model for the analysis of transmission of COVID-19 based upon data of epidemiological parameters.In this article, we have considered the epidemic model published in [6, 30]. Then, we have extended the idea of the article [1] to the considered model. As a result, we recap and extend certain conclusions on the features of the time-continuous classical SEIQR model, and we suggest an implicit time-discrete version of this classical SEIQR model, proving that it retains many of the qualities of the time-continuous version. As a result, the goal of this research is to propose a nonautonomous SEIQR model, investigate the properties of its time-continuous formulation, and design an implicit numerical solution approach that preserves the time-continuous variant's primary properties. The goal of this article is to propose, analyze, and optimize COVID-19 using the SEIQR epidemic model. According to our investigations, COVID-19 outbreaks might be caused by human-to-human interaction. As a result, isolation of the infected humans can reduce the COVID-19 spread in the future. Literature review and comparison of various of these models are presented in Table 1.

More precisely, our main contributions can be summarized as follows:

(i) First, we suggest a time-continuous SEIQR model modification with time-varying transmission and recovery rates.

(ii) Second, we draw the conclusion that the formulation of our time-continuous problem is well-posed. This comprises continuous reliance on initial conditions and time-varying rates, global existence in time, and global uniqueness in time, all of which are based on an inductive application of Banach's fixed point theorem.

(iii) In the case of the time-discrete implicit model, we provide unique solvability, monotonicity properties, and an upper error bound between the solution of the implicit time-discrete problem formulation and the solution of the time-continuous problem formulation.

(iv) In order to maximize transmission rate sequences under arbitrary beginning conditions, we have developed a dynamic programming problem. Based on the optimal rate sequences, we suggest safety guidelines and important safety precautions to control COVID spread.

The paper is arranged as follows: Section 1 is dedicated to the introduction. In section 2, we present the time-continuous and time-discrete SEIQR model. In section 3, we give the monotonicity properties and long-time behaviour. An error analysis is given in section 4. The conclusion of our research work is implemented in the last section 5.

2. Time-Continuous SEIQR Model

The time-continuous SEIQR model is formulated, and its behaviours are described using the Lipchitz condition and Grownwall and Bellman's inequality in this section.

2.1. Mathematical Background Material. Here, we revisit the Lipschitz continuity of a function on Euclidean spaces, the local Lipchitz condition, Banach's fixed point theorem, and the method of variation of the parameter, which will be used in the subsequent sections.

Definition 1 (see [46]). Let q_1 and q_2 be two positive integers and $D \subset \mathbb{R}^{q_1}$. A function $H\colon D \longrightarrow \mathbb{R}^{q_2}$ is said to be Lipchitz continuous on D if there exists a nonnegative constant $L \geq 0$ such that $\|H(x) - H(y)\|_{\mathbb{R}^{q_2}} \leq L \cdot \|x - y\|_{\mathbb{R}^{q_1}}$ holds for $x, y \in D$.

(i) Let $U \subset \mathbb{R}^{q_1}$ be an open set and $H\colon U \longrightarrow \mathbb{R}^{q_2}$. Then, H is called as locally Lipchitz continuous if for every element $y_0 \in U$ there exists a neighborhood V of y_0 such that the restrictions of H to V are Lipchitz continuous on V.In a more general framework, we consider a nonlinear initial value problem (IVP) $z\prime(t) = H(t, z(t))$; $z(0) = z_0$, where $z(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$ is solution vector, and $H(t, z(t)) = (h_1(t, z(t)), h_2(t, z(t)), \ldots, h_n(t, z(t)))^T$ is vectorial function with initial point $z_0 \in \mathbb{R}^n$. The following theorem, which is a direct consequence of Gronwall's lemma, can be used to prove global existence in time.

Theorem 1 (see [17]). *Let $G\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be locally Lipchitz conditions. If there exist nonnegative real constants B and k such that $\|G(t, z(t))\| \leq k\|z(t)\|_{\infty} + B$ holds for all $z(t) \in \mathbb{R}^n$, then the solution of the initial value problem* $\begin{cases} z\prime(t) = G(t, z(t)), \\ z(0) = z_0, \end{cases}$ *exists for all time $t \in \mathbb{R}$, and moreover, $z(t)$ satisfies $\|z(t)\|_{\mathbb{R}^n} \leq \|z_0\|_{\mathbb{R}}{}^{n} \cdot \exp(k \cdot |t|) + (B/k)(\exp(k \cdot |t|) - 1), \forall t \in \mathbb{R}$.*

We give the following: Banach's fixed point theorem, which will be used to preserve the global uniqueness in time [47, 47].

Table 1: Overview of models studied.

Models	Compartments	Model type	Study area
SEIR/SLIR	Susceptible (S), exposed/latent (E/L), infectious (I), removed (R)	Deterministic	Europe and North America, New York, Mexico, Zhejiang, Guangdong, Japan, India [31–36]
SEIQR	Susceptible (S), exposed (E), hospitalized infected (I), quarantine (Q), recovered or removed (R)	Deterministic	India [37]
SIR-X	Infected (I), susceptible (S), removed (R), quarantined (X)	Deterministic	China [38]
SIRD	Susceptible (S), infected (I), recovered (R), dead (D)	Deterministic	China, Italy, and France [39]
SEIHARD	Susceptible (S), exposed (E), symptomatic infectious (I), hospitalized (H), asymptomatic infectious (A), recovered (R), deaths (D)	Deterministic	Washington, New York [40]
SIRU	Susceptible (S), asymptomatic infectious (I), reported symptomatic infectious (R), unreported symptomatic infectious (U)	Deterministic	China, Hubei, Wuhan [41]
SEIPAHRF	Susceptible (S), exposed (E), symptomatic (I), super-spreaders class (P), asymptomatic infectious (A), hospitalized (H), recovery (R), fatality (F)	Deterministic	Wuhan [42]
SEIRU	Susceptible (S), asymptomatic noninfectious (E), asymptomatic infectious (I), reported symptomatic infectious (R), unreported symptomatic infectious (U)	Deterministic	China [43]
SEIHR	Susceptible (S), exposed (E), symptomatic infectious (I), hospitalized (H), recovered or death (R)	Deterministic	South Korea [44]
SEIRP	Susceptible (S), exposed (E), infectious (I), removed (R), pathogens (P)	Deterministic	Pakistan [45]

Theorem 2 (see [48]). *Let (X, μ) be a complete metric space. Let $T\colon X \longrightarrow X$ be a strict contraction, that is, there exists a constant $K \in [0, 1)$ such that $\mu(T(x), T(y)) \leq K \cdot \mu(x, y)$ holds for all $x, y \in X$. Then, the mapping T has a unique fixed point.*

In the following theorem, we present the Grownwall and Bellman inequality, which will be used in the subsequence theorems related to the continuous functions.

Theorem 3 (see [49]). *Let $I\colon = [a, b]$, $u, f\colon I \longrightarrow [0, \infty)$ be two continuous and nonnegative functions and $g\colon I \longrightarrow (0, \infty)$ be a continuous, positive, and nondecreasing function. If the inequality*

$$u(t) \leq g(t) + \int_a^t f(s) \cdot u(s) \mathrm{d}s, \tag{1}$$

holds for all $t \in I$, we have

$$u(t) \leq g(t) \cdot \exp\left(\int_a^t f(s) \cdot u(s) \mathrm{d}s\right). \tag{2}$$

Theorem 4. *(Method of variation of parameter) For a first-order nonhomogeneous linear differential equation, $y\prime(t) + p(t)y(t) = f(t)$ has the general solution $y(t) = v(t)e^{Pt} + Ae^{P(t)}$, where $v'(t) = e^{-P(t)} f(t)$, $P(t) = -\int p(t)dt$ and A is an arbitrary constant.*

2.2. Continuous Problem Formulation. At first, let us assume the following assumptions [50, 51] for the upcoming calculations.

(i) Let the population size varies over time be N is varying over time (i.e., population size $= \mu N(t)$ for all $t \in [0, \infty)$).

(ii) We divide the population into five homogeneous subgroups, namely susceptible people (S), exposed (E), infectious (I), quarantined (Q), and recovered (R). We can clearly assign every individual to exactly one subgroup. Hence, we obtain S, E, I, Q, R model satisfying the condition $\mu N = S(t) + E(t) + I(t) + Q(t) + R(t)$ for all $t \in [0, \infty)$.

(iii) Each time-varying transmission rate $x\colon [0, \infty) \longrightarrow [0, \infty)$ is Lipchitz continuous and continuously differentiable, and there exist constants $x_{\min}$ and $x_{\max}$ such that $0 < x_{\min} \leq x(t) \leq x_{\max}$ for all $t \geq 0$ and $x \in \{\pi, \beta, \gamma, \sigma, \theta, \mu\}$.

The choice of time-dependent transmission rates is possible because the countermeasures such as lockdowns, social distancing, or other political actions like curfews and different medical treatments reduce possible contact between susceptible and infectious people.Our equations of the time-continuous SEIQR model read as follows:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \mu N - \mu S(t) - \beta(N)S(t)[E(t) + I(t)], \\[2ex] \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta(N)S(t)[E(t) + I(t)] - \pi E(t) - (\mu + \gamma)E(t) \\[2ex] \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \pi E(t) - \sigma I(t) - \mu I(t), \\[2ex] \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t), \\[2ex] \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \theta Q(t) - \mu R(t), \end{cases} \tag{3}$$

with initial conditions $S(0) = S_0 > 0, E(0) = E_0 > 0,\ I(0) = I_0 > 0, Q(0) = Q_0 > 0$ and $R(0) = R_0 \geq 0$. The detailed parameters and description are given in Table 2

2.3. Nonnegativity and Boundedness of Solutions. Now, we prove the boundedness of the solution to (3). For this purpose, we modify ideas given in [51, 52] deriving the following lemmas, so consider the bounded, time-varying transmission rates given above.

Lemma 1. *Each solution of system (3) is bounded below by zero.*

Proof. Consider, the first relation of (3),

$$\frac{dS(t)}{dt} = \mu N - \mu S(t) - \beta(N)S(t)[E(t) + I(t)]. \tag{4}$$

By taking $(dS(t)/dt) = S'(t)$, equation (4) can be expressed as a first-order nonhomogeneous linear differential equation in $S(t)$ as

$$S'(t) + [\mu + \beta(N)E(t) + \beta(N)I(t)]S(t) = \mu N. \tag{5}$$

Applying Theorem 4, and by applying the same procedure to the first-order nonhomogenous linear equation in $E(t)$,

$$\frac{dE(t)}{dt} - [\beta(N)S(t) - \pi - \mu - \gamma]E(t) = \beta(N)S(t)I(t). \tag{6}$$

We can easily show that $E(t) \geq 0$ for all $t \in [0, \infty)$. Proceeding like this, we can show that $I(t), Q(t)$, and $R(t) \geq 0$ for all $t \in [0, \infty)$.

Since $\mu N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$, and total population is finite, $S(t), E(t), I(t), Q(t)$, and $R(t)$ are bounded above, and hence, the proof is complete. □

Theorem 5. *For all solution functions of (3), we have* $0 \leq X \leq \mu N$, *where* $X \in \{S(t), E(t), I(t), Q(t), R(t)\}$.

Proof. The proof follows from $\mu N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$ and Lemma 1. □

2.4. Global Existence in Time. We arrive at a theorem regarding global existence in the time of (3) based on Theorem 1. For abbreviation, we use the supremum norm $\|f(t)\|_\infty := \sup_{t\in[a,b]}|f(t)|$ for an arbitrary continuous function $f\colon [a,b] \longrightarrow \mathbb{R}$. A similar definition holds for vector-valued bounded functions. In our case, using the boundedness of S, E, I, Q, R on $[0, \infty)$, obtain the following global existence theorem.

Table 2: Parameters and description.

Parameters	Description
$S(t)$	At time t, the number of susceptible people
$E(t)$	At time t, the number of exposed people
$I(t)$	At time t, the number of infected people
$Q(t)$	At time t, the number of quarantined people
$R(t)$	At time t, the number of recovered people
β	The rate at which susceptible populations migrate to exposed and infected populations
π	The rate at which an exposed population moves to an infected population
γ	Transmission rate at which exposed people take outside as isolated
σ	Transmission rate at which infected people were added to isolated individual
θ	Transmission rate at which isolated persons recovered
μ	Natural death rate and disease-related death rate

Theorem 6. *The system of nonlinear first-order ODE (3) has at least one solution which exists for all* $t \geq 0$.

Proof. By denoting $z(t) = (S(t), E(t), I(t), Q(t), R(t))$, we can set

$G\colon [0, \infty) \times \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ by

$$(t, z(t)) \longrightarrow \begin{pmatrix} A - \mu S(t) - \beta(N)S(t)(E(t) + I(t)) \\ \beta(N)S(t)(E(t) + I(t)) - \pi E(t) - (\mu + \gamma)E(t) \\ \pi E(t) - \sigma I(t) - \mu I(t) \\ \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t) \\ \theta Q(t) - \mu R(t) \end{pmatrix}^T. \tag{7}$$

Clearly, G is Lipchitz continuous, due to the continuity of each components.

Assuming the supremum norm on our Euclidean space, and with the help of triangle inequality, we arrive

$$\begin{aligned}
\|G[t,z(t))\|_\infty &= \sup_{t\in[0,\infty)}\{|A-\mu S(t)-\beta S(t)[E(t)+I(t)]|, |\beta(N)S(t)[E(t)+I(t)]-\beta E(t)-(\mu+\gamma)E(t)|,\\
&\qquad |\pi E(t)-\sigma I(t)-\mu I(t)|, |\gamma E(t)+\sigma I(t)-\theta Q(t)-\mu Q(t)|, |\theta Q(t)-\mu R(t)|\},\\
\|G(t,z(t))\|_\infty &\le \sup_{t\in[0,\infty)}\{|A-\mu S(t)|, |\beta(N)S(t)[E(t)+I(t))|, |\pi E(t)|, |\gamma E(t)+\sigma I(t)|, |\theta Q(t)-\mu R(t)|\},\\
&\le \sup_{t\in[0,\infty)}\Big\{\left|\mu_{\max}N+\mu_{\max}S(t)\right|, \left|\beta_{\max}(N)S(t)[E(t)+I(t)]\right|, \left|\pi_{\max}E(t)\right|, \left|\gamma_{\max}E(t)+\sigma_{\max}I(t)\right|,\\
&\qquad \left|\theta_{\max}Q(t)-\mu_{\max}R(t)\right|\Big\},\\
&\le \sup_{t\in[0,\infty)}(\mu_{\max}+\beta_{\max}+\pi_{\max}+\gamma_{\max}+\sigma_{\max}+\theta_{\max})\{|S(t)|, |E(t)|, |I(t)|, |Q(t)|, |R(t)|\},\\
&\le \sup_{t\in[0,\infty)}(\mu_{\max}+\beta_{\max}+\pi_{\max}+\gamma_{\max}+\sigma_{\max}+\theta_{\max})\|z(t)\|_\infty,\\
\|G(t,z(t))\|_\infty &\le k\|z(t)\|_\infty,
\end{aligned} \tag{8}$$

where $k=\mu_{\max}+\beta_{\max}+\pi_{\max}+\gamma_{\max}+\sigma_{\max}+\theta_{\max}$.

From the boundedness of our solution functions and the boundedness of our time-varying transmission rates, all requirements of Theorem 1 are fulfilled, and our proof is complete. □

2.5. Global Uniqueness in Time. We present the global uniqueness theorem for (3) by utilizing the inductive application of Banach's fixed point theorem.

Theorem 7. *The nonlinear ODE system (3) has a unique solution that exists for all $t \ge 0$.*

Proof. Consider the system of equations given in (3):

(1) Consider the time interval $[0,\tau]$ is applicable to Banach's fixed point theorem

(2) For $x_1, x_2, y_1, y_2 \in \mathbb{R}$, by triangle inequality, we have $|x_1\cdot y_1 - x_2\cdot y_2| \le |x_1\|y_1-y_2| + |y_2\|x_1-x_2|$

(3) We assume that $S,E,I,Q,R,\overline{S},\overline{E},\overline{I},\overline{Q},\overline{R}$: $[0,\infty)\longrightarrow[0,\infty)$ are two solutions of (3)

Beginning the proof by letting

$$\begin{aligned}
\sup_{t\in[0,\tau]}|S(t)-\overline{S}(t)| &= \sup\left|S_1e^{P(t)}-\overline{S}_1e^{P(t)}\right|\text{where}P(t)\\
&= -\int P(t)\mathrm{d}t,\\
\sup_{t\in[0,\tau]}|S(t)-\overline{S}(t)| &= \sup\left|S_1-\overline{S}_1\|e^{P(t)}\right|,\\
\sup_{t\in[0,\tau]}|S(t)-\overline{S}(t)| &\le \left|S_1-\overline{S}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty.
\end{aligned} \tag{9}$$

Then, the second equation in (3) becomes

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}-[\beta(N)S(t)-\pi-\mu-\gamma]E(t)=\beta(N)S(t)I(t). \tag{10}$$

Since it is a first-order nonhomogenous linear equation in $E(t)$, we take $p_1(t)=-[\beta(N)S(t)-\pi-\mu-\gamma]$ and $f(t)=\beta(N)S(t)I(t)$. As $P_1(t)=-\int p_1(t)\mathrm{d}t$ and $v_1(t)=\int e^{-P_1(t)}\beta(N)S(t)I(t)\mathrm{d}t$. Hence, $E(t)=v_1(t)\mathrm{d}te^{P_1(t)}+E_1e^{P(t)}$ and

$$\sup_{t\in[0,\tau]}|E(t)-\overline{E}(t)| \le \left|E_1-\overline{E}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty. \tag{11}$$

Similarly, we can easily show the following inequalities:

$$\begin{aligned}
\sup_{t\in[0,\tau]}|Q(t)-\overline{Q}(t)| &\le \left|Q_1-\overline{Q}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty,\\
\sup_{t\in[0,\tau]}|I(t)-\overline{I}(t)| &\le \left|I_1-\overline{I}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty,\\
\sup_{t\in[0,\tau]}|R(t)-\overline{R}(t)| &\le \left|R_1-\overline{R}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty.
\end{aligned} \tag{12}$$

Summing implies

$$\|z(t)-\overline{z(t)}\|_\infty \le \sum_{S,E,I,Q,R}\left|S_1-\overline{S}_1\right|\tau\|z(t)-\overline{z(t)}\|_\infty. \tag{13}$$

By choosing $\tau<(1/2)(1/\sum|S_1-\overline{S}_1|)$. As a result, we conclude that the solution is unique in $[0,\tau]$.

2.6. Time-Discrete Implicit SEIQR Model. Assume that our time interval $[0,\tau]$ can be divided by strictly increasing sequence $\{t_j\}_{j=1}^{M}$ for $M\in\mathbb{N}$ with $t_1=0$ and $t_M=T$. We write $f(t_j)=f_j$, $j\in\{1,2,3,\ldots,M\}$ for an arbitrary time-dependent function f.

2.7. Discussion and Formulations. Here, we are transforming the continuous system (3) to the fully explicit discrete scheme (14) as given as follows:

$$\begin{cases} \dfrac{\Delta S_j}{\Delta t_j} = \mu N - \mu_{j+1} S_j(t) - \beta_{j+1}(N) S_j(t)\left[E_j(t) + I_j(t)\right], \\ \dfrac{\Delta E_j}{\Delta t_j} = \beta_{j+1}(N) S_j(t)\left[E_j + I_j\right] - \pi_{j+1} E_j - \left(\mu_{j+1} + \gamma_{j+1}\right) E_j, \\ \dfrac{\Delta I_j}{\Delta t_j} = \pi_{j+1} E_j - \sigma_{j+1} I_j - \mu_{j+1} I_j, \\ \dfrac{\Delta Q_j}{\Delta t_j} = \gamma_{j+1} E_j + \sigma_{j+1} I_j - \theta_{j+1} Q_j - \mu_{j+1} Q_j, \\ \dfrac{\Delta R_j}{\Delta t_j} = \theta_{j+1} Q_j - \mu_{j+1} R_j, \end{cases} \tag{14}$$

and a fully implicit scheme (15) as

$$\begin{cases} \dfrac{\Delta S_j}{\Delta t_j} = \mu_{j+1} N - \mu_{j+1} S_{j+1} - \beta_{j+1}(N) S_{j+1}\left[E_{j+1} + I_{j+1}(t)\right], \\ \dfrac{\Delta E_j}{\Delta t_j} = \beta_{j+1} N S_{j+1}\left[E_{j+1} + I_{j+1}\right] - \left[\pi_{j+1} + \mu_{j+1} + \gamma_{j+1}\right] E_{j+1}, \\ \dfrac{\Delta I_j}{\Delta t_j} = \pi_{j+1} E_{j+1} - \sigma_{j+1} I_{j+1} - \mu_{j+1} I_{j+1}, \\ \dfrac{\Delta Q_j}{\Delta t_j} = \gamma_{j+1} E_{j+1} + \sigma_{j+1} I_{j+1} - \theta_{j+1} Q_{j+1} - \mu_{j+1} Q_{j+1}, \\ \dfrac{\Delta R_j}{\Delta t_j} = \theta_{j+1} Q_{j+1} - \mu_{j+1} R_{j+1}, \end{cases} \tag{15}$$

where $(\Delta X_j/\Delta t_j) = (X_{j+1} - X_j)/(t_{j+1} - t_j)$ for $j \in \{1, 2, 3, \ldots, M-1\}$.

Observe that $\mu_{j+1} N = S_{j+1} + E_{j+1} + I_{j+1} + Q_{j+1} + R_{j+1}$ for all $j \in \{1, 2, 3, \ldots, M-1\}$.Since the fully explicit scheme (14) simply reduces to a linear system, our main interest is in a fully implicit discrete scheme because it preserves the nonlinear structure of the continuous problem.

2.8. Implicit Time-Discrete Problem Formulation. In this section and subsequent sections, we derive the recurrence type of solutions for the implicit scheme (15). For that, we assume that $0 < x_{\min} \le x_j \le x_{\max}$, $x \in \{\mu, \beta, \pi, \gamma, \sigma, \theta,\}$ for $j \in \{1, 2, 3, \ldots, M\}$ and that $0 < t_{j+1} - t_j \le 1$ for $j \in \{1, 2, 3, \ldots, M\}$ and that $S_0 > 0, E_0 > 0, I_0 > 0, Q_0 > 0, R_0 \ge 0$. Now, (15) can be expressed as

$$\begin{cases} S_{j+1} = \dfrac{S_j + \mu_{j+1} N\left(t_{j+1} - t_j\right)}{1 + \left[\mu_{j+1} + \beta_{j+1} N E_{j+1} + \beta_{j+1} N I_{j+1}\right]\left(t_{j+1} - t_j\right)}, \\ E_{j+1} = \dfrac{E_j + \beta_{j+1} N S_{j+1} I_{j+1}\left(t_{j+1} - t_j\right)}{1 + \left[\pi_{j+1} + \mu_{j+1} + \gamma_{j+1} - \beta N S_{j+1}\right]\left(t_{j+1} - t_j\right)}, \\ I_{j+1} = \dfrac{I_j + \pi_{j+1} E_{j+1}\left(t_{j+1} - t_j\right)}{1 + \left[\sigma_{j+1} + \mu_{j+1}\right]\left(t_{j+1} - t_j\right)}, \\ Q_{j+1} = \dfrac{Q_j + \left[\gamma_{j+1} E_{j+1} + \sigma_{j+1} I_{j+1}\right]\left(t_{j+1} - t_j\right)}{1 + \left[\theta_{j+1} + \mu_{j+1}\right]\left(t_{j+1} - t_j\right)}, \\ R_{j+1} = \dfrac{R_j + \theta_{j+1} Q_{j+1}\left(t_{j+1} - t_j\right)}{1 + \mu_{j+1}\left(t_{j+1} - t_j\right)}. \end{cases} \tag{16}$$

2.9. Unique Solvability. In this section, we provide the method for finding the solution of (16). Letting $t_{j+1} - t_j = \Delta t_j$ in S_{j+1}, we obtain

$$S_{j+1} = \frac{S_j + \mu_{j+1} N \Delta t_j}{1 + \left[\mu_{j+1} + \beta_{j+1} N E_{j+1} + \beta_{j+1} N I_{j+1}\right]\Delta t_j}, \tag{17}$$

when $j = 0$, $S_1 = (S_0 + \mu_1 N \Delta t_0)/1 + [\mu_1 + \beta_1 N E_1 + \beta_1 N I_1]\Delta t_0$.

when $j = 1$, $S_2 = (S_1 + \mu_2 N \Delta t_1)/1 + [\mu_2 + \beta_2 N E_2 + \beta_2 N I_2]\Delta t_1$.

Now, substituting S_1 value in S_2, we obtain

$$S_2 = \frac{N\mu_2 \Delta t_1\left[1 + (\mu_1 + N\beta_1 E_1 + N\beta_1 I_1)\Delta t_0\right] + N\mu_1 \Delta t_0 + S_0}{\prod_{m=1}^{2}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)\Delta t_{m-1}\right]}. \tag{18}$$

Similarly, we can find the value of S_3 as

$$S_3 = \frac{\sum_{z=1}^{2} N\mu_{z+1}(\Delta t_z)\prod_{m=1}^{z}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)(\Delta t_{m-1})\right]}{\prod_{m=1}^{3}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)(\Delta t_{m-1})\right]} + \frac{N\mu_1(\Delta t_0) + S_0}{\prod_{m=1}^{3}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)\Delta t_{m-1}\right]}. \tag{19}$$

Finally, we obtain the general solution as

$$S_k = \frac{S_0}{\prod_{m=1}^{k}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)\Delta t_{m-1}\right]} + \frac{\sum_{z=1}^{k} N\mu_z(\Delta t_{z-1})\prod_{m=0}^{z-1}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)\Delta t_{m-1}\right]}{\prod_{m=1}^{k}\left[1 + (\mu_m + N\beta_m E_m + N\beta_m I_m)\Delta t_{m-1}\right]}, \tag{20}$$

where $I_x = E_x = \mu_x = \beta_x = t_x = 0$ for $x \le 0$.

In a similar way, we can easily find the other parameters as

$$
\begin{aligned}
E_k &= \frac{E_0}{\prod_{m=1}^{k}\left[1+(\pi_m+\mu_m+\gamma_m-N\beta_m S_m)\Delta t_{m-1}\right]} + \frac{\sum_{z=1}^{k} N\beta_z S_z I_z \Delta t_{z-1}\prod_{m=0}^{z-1}\left[1+(\pi_m+\mu_m+\gamma_m-N\beta_m S_m)\Delta t_{m-1}\right]}{\prod_{m=1}^{k}\left[1+(\pi_m+\mu_m+\gamma_m-N\beta_m S_m)\Delta t_{m-1}\right]},\\
I_k &= \frac{I_0+\sum_{z=1}^{k}\pi_z E_z \Delta t_{z-1}\prod_{m=0}^{z-1}\left[1+(\sigma_m+\mu_m)\Delta t_{m-1}\right]}{\prod_{m=1}^{k}\left[1+(\sigma_m+\mu_m)\Delta t_{m-1}\right]},\\
Q_k &= \frac{Q_0+\sum_{z=1}^{k}(\gamma_z E_z+\sigma_z I_z)\Delta t_{z-1}\prod_{m=0}^{z-1}\left[1+(\theta_m+\mu_m)\Delta t_{m-1}\right]}{\prod_{m=1}^{k}\left[1+(\theta_m+\mu_m)\Delta t_{m-1}\right]},\\
R_k &= \frac{R_0+\sum_{z=1}^{k}\theta_z Q_z \Delta t_{z-1}\prod_{m=0}^{z-1}\left[1+\mu_m \Delta t_{m-1}\right]}{\prod_{m=1}^{k}\left[1+\mu_m \Delta t_{m-1}\right]},
\end{aligned}
\tag{21}
$$

where $S_x = E_x = I_x = Q_x = R_x = 0$ and $\mu_x = \beta_x = \pi_x = \gamma_x = \sigma_x = \theta_x = t_x = 0$ for $x \le 0$.

Theorem 8. *Assume* $0<\mu_{\min}\le\mu_j\le\mu_{\max}<1$, $0<\beta_{\min}\le\beta_j\le\beta_{\max}<1$, $0<\pi_{\min}\le\pi_j\le\pi_{\max}<1$, $0<\gamma_{\min}\le\gamma_j\le\gamma_{\max}<1$, $0<\sigma_{\min}\le\sigma_j\le\sigma_{\max}<1$, *and* $0<\theta_{\min}\le\theta_j\le\theta_{\max}<1$ *and* $0<t_{j+1}-t_j\le 1$ *holds for all* $j\in\{1,2,\ldots,M-1\}$, *and* $S_1>0$, $E_1>0$, $I_1>0$, $Q_1>0$ *and* $R_1>0$. *The implicit solution scheme (16) uniquely solvable for all* $j\in\{1,2,\ldots,M-1\}$, *and we have*

$$
E_{j+1} = \frac{-B\pm\sqrt{B^2-4AC}}{2A},\quad j\in\{1,2,\ldots,M-1\}, \tag{22}
$$

where A, B, and C are given in upcoming equations in (27)–(29).

Proof. Substituting S_{j+1} in E_{j+1} and taking $S_{j+1}=\overline{S}$, $E_{j+1}=\overline{E}$, $I_{j+1}=\overline{I}$, $Q_{j+1}=\overline{Q}$ and $R_{j+1}=\overline{R}$, we get

$$
\overline{E} = \frac{E_j+E_jXT+\overline{I}\left[E_jN\beta T+\beta NT\left(N\mu T+S_j\right)\right]}{1+XT+\phi T+\phi T^2X-N\beta\left[N\mu T+S_j\right]T+\overline{I}\left[N\beta T+\phi T^2 N\beta\right]}, \tag{23}
$$

where $\pi+\mu+\gamma=\phi$, $\mu+N\beta\overline{E}=X$, and $\Delta t_j=T$.

Again, substituting the $\overline{I}$ value above, we arrive at

$$
\overline{E} = \frac{\text{Numerator}\,(\widetilde{N})}{\text{Denominator}\,(\widetilde{D})}, \tag{24}
$$

$$
\begin{aligned}
\widetilde{N} &= \left(E_j+E_j\mu T\right)\delta+\left(E_jN\beta T+\beta NT\left(N\mu T+S_j\right)\right)I_j\\
&\quad+\overline{E}\left[N\beta E_jT\delta+\left(E_jN\beta T+\beta NT\left(N\mu T+S_j\right)\right)\pi T\right],\\
\widetilde{D} &= \left(1+\mu T+\phi T+\phi T^2\mu-N\beta\left(N\mu T+S_j\right)T\right)\delta\\
&\quad+I_j\left(N\beta T+\phi T^2N\beta\right)\\
&\quad+\overline{E}\left[\left(N\beta T+\phi T^2N\beta\right)\delta+\pi T\left(N\beta T+\phi T^2N\beta\right)\right],
\end{aligned}
\tag{25}
$$

where $\Delta t_j=T, 1+(\sigma+\mu)T=\delta$.

We get the quadratic equation in the form when we solve equation (24)

$$
A\overline{E}^2+B\overline{E}+C=0\cdot(i.e)AE_{j+1}^2+BE_{j+1}+C=0, \tag{26}
$$

where

$$
\begin{aligned}
A &= \left(N\beta\left(\Delta t_j\right)+(\pi+\mu+\gamma)\left(\Delta t_j\right)^2N\beta\right)\left(1+(\sigma+\mu)\left(\Delta t_j\right)\right)\\
&\quad+\pi\left(\Delta t_j\right)\left(N\beta\left(\Delta t_j\right)+(\pi+\mu+\gamma)\left(\Delta t_j\right)^2N\beta\right),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
B &= \left(1+\mu\left(\Delta t_j\right)+(\pi+\mu+\gamma)\left(\Delta t_j\right)+(\pi+\mu+\gamma)\mu\left(\Delta t_j\right)^2\right.\\
&\quad\left.-N\beta\left(N\mu\left(\Delta t_j\right)+S_j\right)\left(\Delta t_j\right)\right)\left(1+(\sigma+\mu)\left(\Delta t_j\right)\right)\\
&\quad+I_j\left(N\beta\left(\Delta t_j\right)+(\pi+\mu+\gamma)\left(\Delta t_j\right)^2N\beta\right)\\
&\quad-N\beta E_j\left(\Delta t_j\right)\left(1+(\sigma+\mu)\left(\Delta t_j\right)\right)\\
&\quad-\left(E_jN\beta\left(\Delta t_j\right)+\beta N\left(\Delta t_j\right)\left(N\mu\left(\Delta t_j\right)+S_j\right)\right)\pi\left(\Delta t_j\right),
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
C &= \left(E_j+E_j\mu\left(\Delta t_j\right)\right)\left(1+(\sigma+\mu)\left(\Delta t_j\right)\right)\\
&\quad+\left(E_jN\beta\left(\Delta t_j\right)N\beta\left(\Delta t_j\right)\left(N\mu\left(\Delta t_j\right)+S_j\right)\right)I_j.
\end{aligned}
\tag{29}
$$

The proof is completed by taking the roots of equation (26).

Similarly, by substituting I_{j+1} in Q_{j+1} in the system, the equation Q_{j+1} is in the function of E_{j+1}. Then, replacing E_{j+1}

by equation (24), we will get an explicit formula for computing $\overline{Q}$.

Since $S_{j+1} + E_{j+1} + I_{j+1} + Q_{j+1} + R_{j+1} = \mu_{j+1}N$, we can easily get an explicit formula for R_{j+1} for $j \in \{1, 2, \ldots, M-1\}$. □

3. Monotonicity Properties and Long-Time Behaviour

In this section, we develop a suitable atmosphere in which our implicit scheme obeys the monotonic properties as in the continuous case. For this, we give the following lemmas and finally provide a nonlinear programming problem to optimize the transmission sequences.

Lemma 2. *If $S_j[\beta_j\mu_j^{-1}(E_j + I_j) + N^{-1}] \geq 1$, then $S_{j+1} \leq S_j$, i.e., $(S_n)_{n=1}^{\infty}$ becomes a decreasing sequence.*

Proof. Taking $y = \mu_{j+1}N\Delta t_j$ and $x = 1 + [\mu_{j+1} + \beta_{j+1}N(E_{j+1} + I_{j+1})]$ in S_{j+1} of (16). We arrive the relation

$$S_{j+1} = \frac{y + S_j}{1 + x}. \tag{30}$$

From (30), we obtain $(1 + x)S_{j+1} = y + S_j$ and which implies $S_{j+1} - S_j = y - xS_{j+1}$.

Clearly, $y - xS_{j+1} \leq 0$ if $S_{j+1}[\beta_{j+1}\mu_{j+1}^{-1}(E_{j+1} + I_{j+1}) + N^{-1}] \geq 1$. Thus,

$$S_{j+1} - S_j \leq 0, \quad \text{for}\, j \in \{1, 2, \ldots, m\}, \tag{31}$$

and the proof is complete.

Lemma 3. *If $\beta_jNS_j[1 + I_jE_j^{-1}] \leq \pi_j + \mu_j + \gamma_j$, then $(E_n)_{n=1}^{\infty}$ is a decreasing sequence.*

Proof. Taking $x = \beta_{j+1}NS_{j+1}I_{j+1}\Delta t_j$ and $y = (\pi_{j+1} + \mu_{j+1} + \gamma_{j+1} - \beta_{j+1}NS_{j+1})\Delta t_j$ in E_{j+1} of (16), we derive

$$E_{j+1} - E_j = x - yE_{j+1}. \tag{32}$$

Here, $x - yE_{j+1} \leq 0$ follows from $\beta_{j+1}NS_{j+1}[1 + I_{j+1}E_{j+1}^{-1}] \leq \pi_{j+1} + \mu_{j+1} + \gamma_{j+1}$, and hence, we obtain $E_{j+1} \leq E_j$ for all j, and the proof is complete. □

Lemma 4. *If $\pi_j(\sigma_j + \mu_j)^{-1} \leq I_jE_j^{-1}$, then $(I_n)_{n=1}^{\infty}$ be a decreasing sequence.*

Proof. Taking $I_{j+1} = ((I_j + x)/(1 + y))$, where $x = \pi_{j+1}E_{j+1}\Delta t_j$ and $y = (\sigma_{j+1} + \mu_{j+1})\Delta t_j$ in (16), then we find

$$I_{j+1} - I_j = x - yI_{j+1}. \tag{33}$$

Since $\pi_{j+1}(\sigma_{j+1} + \mu_{j+1})^{-1} \leq I_{j+1}E_{j+1}^{-1}$, we get $x - yI_{j+1} \leq 0$ and $I_{j+1} \leq I_j$ for all j. □

Lemma 5. *If $(\gamma_jE_j + \sigma_jI_j)(\theta_j + \mu_j)^{-1} \geq Q_j$, then $(Q_j)_{n=1}^{\infty}$ becomes an increasing sequence.*

Proof. Taking $Q_{j+1} = (Q_j + x)/(1 + y)$, where $x = (\gamma_{j+1}E_{j+1} + \sigma_{j+1}I_{j+1})\Delta t_j$ and $y = (\theta_{j+1} + \mu_{j+1})\Delta t_j$ in (16), then it is easy to arrive

$$Q_{j+1}Q_j = x - yQ_{j+1}. \tag{34}$$

From our assumption $(\gamma_{j+1}E_{j+1} + \sigma_{j+1}I_{j+1})(\theta_{j+1} + \mu_{j+1})^{-1} \geq Q_{j+1}$, we derive

$$x - yQ_{j+1} \geq 0, \tag{35}$$

and the proof is complete. □

Lemma 6. *If $\theta_jQ_j\mu_j^{-1} \geq R_j$ for all j, then $R_{j+1} \geq R_j$ for all j.Expressing $R_{j+1} = (R_j + x)/(1 + y)$ in (16), we can easily find*

$$R_{j+1} - R_j = x - yR_{j+1}, \tag{36}$$

from the given condition, $\theta_{j+1}Q_{j+1}\mu_{j+1}^{-1} \geq R_{j+1}$ and straightforward calculations, we obtain

$$x - yR_{j+1} \geq 0, \tag{37}$$

and the proof is complete.

Remark 1. Since R_j is monotonic and bounded above by total population, then it will converge and $\lim_{j\longrightarrow\infty}R_j = R^*$ exists, and (E_j) and (S_j) are decreasing sequence, and we easily observe that $\lim_{j\longrightarrow\infty}I_j = 0$.

3.1. Formulation and Discussion. Due to the ongoing nature of the COVID-19 pandemic, it was impossible to fully comprehend the short- or long-term implications of this global disruption. In our study, when there is no quarantine, a single infected individual can spread the infection to about two other people; however, when quarantine is imposed, there is a chance of preventing further transmission of infection. However, some of the exposed individuals may avoid quarantine due to fear of stigma and death. In other words, this does not achieve zero infection in the population, implying that additional interventions are required to eradicate the virus. If there are no adequate interventions in place, the virus will remain in the population for a long time, but it will eventually drop over time. But, still, there will be a small number of sick people who have the ability to start another outbreak even after measures like quarantine and public health education/awareness raise the number of exposed and infected people dramatically but not to zero. According to this, COVID-19 will not be completely eradicated even with prompt development of measures.In order to study the effects of isolation, quarantine, and the percentage of exposed people who will be quarantined, we did numerical simulations. Many authors have developed numerous mathematical models to limit the spread of viruses. Here, we have developed the optimization technique to control the spread of the virus. This approach enables us to control a viruses future spread and predict how it will spread in the future.After summarizing all of the prior principles, the problem is transformed into a dynamic programming problem model with constraints imposed by the previous lemmas by keeping S_k, E_k, I_k, Q_k, and R_k are constants and $\beta_k, \pi_k, \mu_k, \gamma_k, \sigma_k$, and θ_k are variables at each level of the optimization. Note that the following dynamic programming problem preserves the monotonic properties.

The dynamic programming problem is given by

$\mathrm{Min}\,(\beta_k + \pi_k + \mu_k) - (\gamma_k + \sigma_k + \theta_k)$ and subject to the constraints:

$$\begin{aligned}
S_k\left[\beta_k (E_k + I_k) + \mu_k N^{-1}\right] - \mu_k &\geq 0,\\
(\pi_k + \mu_k + \gamma_k)E_k - \beta_k N S_k (E_k + I_k) &\geq 0,\\
I_k (\sigma_k + \mu_k) - \pi_k E_k &\geq 0,\\
\gamma_k E_k + \sigma_k I_k - Q_k (\theta_k + \mu_k) &\geq 0,\\
\theta_k Q_k - R_k \mu_k &\geq 0,\\
S_k + E_k + I_k + Q_k + R_k - \mu_k N &= 0,\\
S_k, E_k, I_k, Q_k, R_k &\geq 0,\\
\beta_k, \pi_k, \mu_k, \gamma_k, \sigma_k, \theta_k &\geq 0.
\end{aligned} \tag{38}$$

Since S_0, E_0, I_0, Q_0, and R_0 are known initial conditions, we get a dynamic programming problem for level-0 time if we keep $k = 0$ in the above model. We shall obtain an optimal (feasible) solution $(\beta_0^*, \pi_0^*, \mu_0^*, \gamma_0^*, \sigma_0^*, \theta_0^*)$ by employing the optimization technique in operation research. Then, these values are assigned as $\beta_1 = \beta_0^*$, $\pi_1 = \pi_0^*$, $\mu_1 = \mu_0^*$, $\gamma_1 = \gamma_0^*$, $\sigma_1 = \sigma_0^*$, $\theta_1 = \theta_0^*$. We will receive the values for finding S_1, E_1, I_1, Q_1, and R_1. We will receive a dynamic programming problem for level-1 time if we keep $k = 1$ in this model. We will achieve the ideal solution as $(\beta_1^*, \pi_1^*, \mu_1^*, \gamma_1^*, \sigma_1^*, \theta_1^*)$ using the optimization technique. These values are assigned as $\beta_2 = \beta_1^*, \pi_2 = \pi_1^*, \mu_2 = \mu_1^*, \gamma_2 = \gamma_1^*$, $\sigma_2 = \sigma_1^*, \theta_2 = \theta_1^*$. We will receive the values for S_2, E_2, I_2, Q_2, and R_2. If we keep going in this direction, we will end up with transmission rate sequences $(\beta_k)_{k=1}^{\infty}$, $(\pi_k)_{k=1}^{\infty}$, $(\mu_k)_{k=1}^{\infty}$, $(\gamma_k)_{k=1}^{\infty}$, $(\sigma_k)_{k=1}^{\infty}$, and $(\theta_k)_{k=1}^{\infty}$ which provide a sufficiently viable stable solution for the situation, correspond to S_0, E_0, I_0, Q_0, and R_0. After a certain stage, each transmission rate becomes constant. We must follow the appropriate standard operating procedures and safety precautions in order to acquire these sequences in practice. The isolation class, it appears, plays a significant role in achieving this possible solution.

4. Error Analysis

Now, we will set an upper limit for error propagation. We need to construct certain assumptions for our convergence analysis before proving the required statements. The following is a list of them:

(i) Let $[0, T]$ be the time interval under consideration with $t_1 = 0 < t_2 < \cdots < t_{M-1} < t_M = T$

(ii) Allow the time-continuous and time-discrete models beginning circumstances to coincide

(iii) Let $S, E, I, Q, R\colon\ [0, T] \longrightarrow [0, \infty)$ be twice continuously differentiable solution functions

(iv) Allow the time-varying transmission rates $\mu, \beta, \pi, \gamma, \sigma, \theta\colon\ [0 \cdot T] \longrightarrow [0, \infty)$ be continuously differentiable just once

(v) Allow the time-varying transmission and recovery rates to be bounded by 0 and 1

(vi) Choose $\Delta_p < \min\{1/12\ \ (\mu_{\max} + \beta_{\max}) + (4\pi_{\max} + \gamma_{\max} + \sigma_{\max} + \theta_{\max})\} \leq 1$ for all $p \in N$ and $\Delta{:} = \max_{p\in N} \Delta_p$

We get the following theorem under these conditions, in which we adopt notions from the error analysis of an explicit-implicit solution algorithm.

Theorem 9. *The difference between the solutions of the time-continuous system formulation (3) and the time-discrete system (16) fulfills if the aforementioned assumptions are met, then*

$$\left\| z_{p+1} - z\left(t_{p+1}\right)\right\| \leq C_{loc} \cdot \Delta \left\{ \left(\frac{1}{1 - \left[6\left(\mu_{\max} + \beta_{\max}\right) + 2\left(\pi_{\max} + \theta_{\max} + \gamma_{\max} + \sigma_{\max}\right)\right]\Delta} \right)^{p} - 1 \right\}. \tag{39}$$

Proof. Since this is technical proof, we will start with a brief description of our technique. The first step is the estimation of local errors between time-continuous and time-discrete solutions. After that, we look at error propagation over time. Finally, we look into the accumulation of these errors. At the same time, time-discrete solutions are expressed as S_p at time t_p, whereas time-continuous solutions are written as $S(t_p)$.

1) For the purpose of examining local errors, we assume that

$$\begin{aligned}
\left(t_p, S_p\right)^T &= \left(t_p, S\left(t_p\right)\right)^T,\\
\left(t_p, E_p\right)^T &= \left(t_p, E\left(t_p\right)\right)^T,\\
\left(t_p, I_p\right)^T &= \left(t_p, I\left(t_p\right)\right)^T,\\
\left(t_p, Q_p\right)^T &= \left(t_p, Q\left(t_p\right)\right)^T,\\
\left(t_p, R_p\right)^T &= \left(t_p, R\left(t_p\right)\right)^T,
\end{aligned} \tag{40}$$

hold for arbitrary $p \in \{1, 2, \ldots, M-1\}$ on the time interval $[t_p, t_{p+1}]$. Here, we consider one time step and denote corresponding time-discrete solutions by $\widetilde{S_{p+1}}$, $\widetilde{E_{p+1}}$, $\widetilde{I_{p+1}}$, $\widetilde{Q_{p+1}}$, and $\widetilde{R_{p+1}}$.

a) For the time-discrete solution S_p at time t_p, we take

$$\widetilde{S_{p+1}} = \frac{S_p + \mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}, \tag{41}$$

and in similar way for the time-continuous solutions $S(t_p)$, we have

$$\widetilde{S_{p+1}} = S(t_p) - \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1} S(t_p)}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}} + \frac{\mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}, \tag{42}$$

which yields

$$\left|S(t_{p+1}) - \widetilde{S_{p+1}}\right| = \left|S(t_{p+1}) - S(t_p) + \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1} S(t_p) - \mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}\right|, \tag{43}$$

which implies

$$\left|\int_{t_p}^{t_{p+1}} S\prime(\tau)\mathrm{d}\tau + \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1} S(t_p) - \mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}\right|. \tag{44}$$

Now, add and subtract $\Delta_{p+1} \cdot S\prime(t_p)$. And then, applying the triangle inequality in the above relation, we get

$$\begin{aligned}
\left|S(t_{p+1}) - \widetilde{S_{p+1}}\right| &\leq \left|\int_{t_p}^{t_{p+1}} S\prime(\tau)\mathrm{d}\tau - \Delta_{p+1} \cdot S\prime(t_p)\right| + \left|\Delta_{p+1} \cdot S\prime(t_p) + \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1} S(t_p) - \mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}\right| \\
I_{S,1} &:= = \left|\int_{t_p}^{t_{p+1}} \left\{S\prime(\tau) - S\prime(t_p)\right\} d\tau\right| = \left|\int_{t_p}^{t_{p+1}} (\tau - t_p) \frac{S\prime(\tau) - S\prime(t_p)}{\tau - t_p} \mathrm{d}\tau\right|.
\end{aligned} \tag{45}$$

By applying the mean value theorem, there exists $\xi_{S,1} \in (t_p, t_{p+1})$ such that

$$\left|S''(\xi_{S,1})\right| = \left|\frac{S\prime(\tau) - S\prime(t_p)}{\tau - t_p}\right| \leq \|S''(t)\|_\infty. \tag{46}$$

This yields

$$I_{S,1} \leq \|S\prime\prime(t)\|_\infty \cdot \int_{t_p}^{t_{p+1}} (\tau - t_p)\mathrm{d}\tau = \frac{\Delta_{p+1}^2}{2} \cdot \|S\prime\prime(t)\|_\infty. \tag{47}$$

Also,

$$\begin{aligned}
II_{S,1} &:= = \left|\Delta_{p+1}.S\prime(t_p) + \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1} S(t_p) - \mu_{p+1} N \Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)\Delta t_{p+1}}\right|, \\
II_{S,1} &= \Delta t_{p+1} \cdot \left|S\prime(t_p) + \frac{\left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right) S(t_p) - \mu_{p+1} N}{1 + \left(\mu_{p+1} + \beta_{p+1} N \widetilde{E_{p+1}} + \beta_{p+1} N \widetilde{I_{p+1}}\right)}\right|.
\end{aligned} \tag{48}$$

Substituting equation (42) in the above and then solving the above relation, we get $II_{S,1} \leq \Delta t_{p+1} \cdot III_{S,1} + IV_{S,1}$, where

$$\begin{aligned}
III_{S,1} &:= = \left|N(\mu_p - \mu_{p+1}) - S(t_p)\left\{(\mu_p - \mu_{p+1}) + N\left[\beta_p\left(E(t_p) + I(t_p)\right) - \beta_{p+1}\left(\widetilde{E_{p+1}} + \widetilde{I_{p+1}}\right)\right]\right\}\right|, \\
IV_{S,1} &:= = \left|\left(\mu_p N - S(t_p)\left[\mu_p + \beta_p N\left(E(t_p) + I(t_p)\right)\right]\right)\left(\mu_{p+1} + \beta_{p+1} N\left(\widetilde{E_{p+1}} + \widetilde{I_{p+1}}\right)\right)\Delta t_{p+1}\right|.
\end{aligned} \tag{49}$$

Now, applying the triangle inequality in $III_{S,1}$, we obtain $\leq N|\mu_{p+1} - \mu_p| + |S(t_p)| \cdot V_{S,1}$, where

$$V_{S,1} := = \Big|(\mu_p - \mu_{p+1}) + N\big[\beta_p(E(t_p) + I(t_p)) - \beta_{p+1}(\widetilde{E_{p+1}} + \widetilde{I_{p+1}})\big]\Big|,$$

$$\text{let} N\left|\mu_{p+1} - \mu_p\right| = N(t_{p+1} - t_p)\left|\frac{\mu_{p+1} - \mu_p}{t_{p+1} - t_p}\right|,$$

$$N\left|\mu_{p+1} - \mu_p\right| = N\Delta t_{p+1}\left|\frac{\mu_{p+1} - \mu_p}{t_{p+1} - t_p}\right|. \tag{50}$$

By the mean value theorem, there exists $\xi_{\mu,1} \in [t_p, t_{p+1}]$ such that

$$\left|\mu\prime(\xi_{\mu,1})\right| = \left|\frac{\mu_{p+1} - \mu_p}{t_{p+1} - t_p}\right| \leq \|\mu\prime(t)\|_\infty. \tag{51}$$

Hence,

$$N \cdot \left|\mu_{p+1} - \mu_p\right| \leq N \cdot \Delta t_{p+1} \cdot \|\mu\prime(t)\|_\infty. \tag{52}$$

Using the triangle inequality again in $V_{S,1}$ yields

$$V_{S,1} \leq \left|\mu_{p+1} - \mu_p\right| + N\left|\beta_p E(t_p) - \beta_{p+1}(\widetilde{E_{p+1}})\right| + N\left|\beta_p I(t_p) - \beta_{p+1}(\widetilde{I_{p+1}})\right|. \tag{53}$$

From equation (52), we can easily find that

$$\left|\mu_{p+1} - \mu_p\right| \leq \Delta t_{p+1} \cdot \|\mu\prime(t)\|_\infty. \tag{54}$$

Now, substituting $\widetilde{E_{p+1}}$ value (53), we arrive

$$N\left|\beta_p E(t_p) - \beta_{p+1}(\widetilde{E_{p+1}})\right| = \left|N\beta_p E(t_p) - \beta_{p+1}\left(\frac{E(t_p) + \beta_{p+1}N\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta t_{p+1}}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1}}\right)\right|, \leq N \cdot I_{S,2} + N \cdot II_{S,2}, \tag{55}$$

where

$$I_{S,2} := = \left|\beta_p E(t_p) - \frac{\beta_{p+1}E(t_p)}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1}}\right|,$$
$$II_{S,2} := = \left|\frac{\beta_{p+1}^2 N\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta t_{p+1}}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - N\beta_{p+1}\widetilde{S_{p+1}})\Delta t_{p+1}}\right|. \tag{56}$$

Now, $I_{S,2} \leq |E(t_p)|\cdot$

$$\left|\frac{\beta_p + \beta_p(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1} - \beta_{p+1}}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1}}\right| \leq \frac{N}{1 - \beta_{\max}}\left\{\left|\beta_{p+1} - \beta_p\right| + \left|\beta_p(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1}\right|\right\},$$

$$I_{S,2} \leq \frac{N \cdot \Delta t_{p+1}}{1 - \beta_{\max}}\left\{\|\beta'(t)\|_\infty + \beta_{\max}(\pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max})\right\}. \tag{57}$$

Similarly, for $II_{S,2}$, it is easy to arrive

$$II_{S,2} = \left|\frac{N\beta_{p+1}^2\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta t_{p+1}}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}})\Delta t_{p+1}}\right| \leq \left|\frac{N\beta_{p+1}^2\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta t_{p+1}}{1 - \beta_{\max}}\right|, \tag{58}$$

$$II_{S,2} \leq \left|\frac{N \cdot \beta_{\max^2} \cdot \Delta t_{p+1}}{1 - \beta_{\max}}\right|.$$

Applying equation (57) and (58) in (55), we get

$$N\left|\beta_p E(t_p) - \beta_{p+1}(\widetilde{E_{p+1}})\right| \leq \frac{N^2\Delta t_{p+1}}{1 - \beta_{\max}}\left\{\|\beta'(t)\|_\infty + \beta_{\max}(\pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max}) + \beta_{\max}^2\right\}. \tag{59}$$

Proceeding through the similar steps, we can find

$$N\left|\beta_p I\left(t_p\right)-\beta_{p+1}\left(\widetilde{I_{p+1}}\right)\right| \leq \frac{N^2 \Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\sigma_{\max}+\mu_{\max}\right)\right\}+\frac{N \Delta t_{p+1}}{1-\beta_{\max}} \pi_{\max}. \tag{60}$$

Applying equations (54), (59), and (60) in $V_{S,1}$, we obtain

$$\begin{aligned} V_{S,1} \leq {} & \Delta t_{p+1}\|\mu\prime(t)\|_{\infty}+\frac{N^2 \Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\pi_{\max}+\mu_{\max}+\gamma_{\max}+\beta_{\max}\right)+\beta_{\max}^2\right\} \\ & +\frac{N^2 \Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\sigma_{\max}+\mu_{\max}\right)\right\}+\frac{N \Delta t_{p+1}}{1-\beta_{\max}} \pi_{\max}. \end{aligned} \tag{61}$$

Using equations (52) and (61), $III_{S,1}$ will be

$$\begin{aligned} III_{S,1} \leq {} & 2N\Delta t_{p+1} \cdot \|\mu\prime(t)\|_{\infty}+\frac{N^3 \Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\frac{N^3 \Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\sigma_{\max}+\mu_{\max}\right)\right\}\right. \\ & \left.+\beta_{\max}\left(\pi_{\max}+\mu_{\max}+\gamma_{\max}+\beta_{\max}\right)+\beta_{\max}^2\right\}+\frac{N \Delta t_{p+1}}{1-\beta_{\max}} \pi_{\max}. \end{aligned} \tag{62}$$

Now, the term $IV_{S,1}$ satisfies the inequality

$$IV_{S,1} \leq I_{S,3}+\left|S\left(t_p\right)\right| \cdot II_{S,3}, \tag{63}$$

where

$$\begin{aligned} I_{S,3} := {} & =\left|\mu_p N\left[\mu_{p+1}+\beta_{p+1} N\left(\widetilde{E_{p+1}}+\widetilde{I_{p+1}}\right)\right] \Delta t_{p+1}\right|, \\ II_{S,3} := {} & =\left|\mu_p+\beta_p N\left(E\left(t_p\right)+I\left(t_p\right)\right)\right. \\ & \left.\cdot\left(\mu_{p+1}+\beta_{p+1} N\left(\widetilde{E_{p+1}}+t\widetilde{I_{p+1}}\right)\right) \Delta t_{p+1}\right|, \end{aligned} \tag{64}$$

which yields

$$\begin{aligned} I_{S,3} & = N\Delta t_{p+1}\left|\mu_p\left[\mu_{p+1}+\beta_{p+1} N\left(\widetilde{E_{p+1}}+\widetilde{I_{p+1}}\right)\right]\right|, \\ & \leq N\Delta t_{p+1}\left\{\mu_{\max}\left(\mu_{\max}+\beta_{\max}\right)\right\}, \end{aligned} \tag{65}$$

and

$$\begin{aligned} II_{S,3} & \leq N\Delta t_{p+1}\left\{\left(\mu_{\max}+\beta_{\max}\right)\left(\mu_{\max}+\beta_{\max}\right)\right\}, \\ II_{S,3} & = N\Delta t_{p+1}\left(\mu_{\max}+\beta_{\max}\right)^2. \end{aligned} \tag{66}$$

From (65) and (66), we get

$$\begin{aligned} IV_{S,1} \leq {} & N\Delta t_{p+1}\left\{\mu_{\max}\left(\mu_{\max}+\beta_{\max}\right)\right\} \\ & +N^2 \Delta t_{p+1}\left(\mu_{\max}+\beta_{\max}\right)^2. \end{aligned} \tag{67}$$

Similarly, from (62) and (66),

$$\begin{aligned} II_{S,1} \leq {} & 2N\left(\Delta t_{p+1}\right)^2 \cdot \|\mu\prime(t)\|_{\infty}+\frac{N^3\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\pi_{\max}+\mu_{\max}+\gamma_{\max}+\beta_{\max}\right)+\beta_{\max}^2\right\} \\ & +\frac{N^3\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\sigma_{\max}+\mu_{\max}\right)\right\}+\frac{N\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}} \pi_{\max}+N\left(\Delta t_{p+1}\right)^2\left\{\mu_{\max}\left(\mu_{\max}+\beta_{\max}\right)\right\} \\ & +N^2\left(\Delta t_{p+1}\right)^2\left(\mu_{\max}+\beta_{\max}\right)^2. \end{aligned} \tag{68}$$

Hence,

$$\begin{aligned} \left|S\left(t_{p+1}\right)-\widetilde{S_{p+1}}\right| \leq {} & \frac{\left(\Delta_{p+1}\right)^2}{2} \cdot \|S\prime\prime(t)\|_{\infty}+2N\left(\Delta t_{p+1}\right)^2 \cdot \|\mu\prime(t)\|_{\infty} \\ & +\frac{N^3\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\pi_{\max}+\mu_{\max}+\gamma_{\max}+\beta_{\max}\right)+\beta_{\max}^2\right\} \\ & +\frac{N^3\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_{\infty}+\beta_{\max}\left(\sigma_{\max}+\mu_{\max}\right)\right\} \\ & +\frac{N\left(\Delta t_{p+1}\right)^2}{1-\beta_{\max}} \pi_{\max}+N\left(\Delta t_{p+1}\right)^2\left\{\mu_{\max}\left(\mu_{\max}+\beta_{\max}\right)\right\}+N^2\left(\Delta t_{p+1}\right)^2\left(\mu_{\max}+\beta_{\max}\right)^2. \end{aligned} \tag{69}$$

b) For the time-discrete solution for E_p at time t_p, we take

$$\widetilde{E_{p+1}} = \frac{E_p + \beta_{p+1} N\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta t_{p+1}}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}}, \tag{70}$$

as well as for time-continuous solutions $E(t_p)$, we find

$$\widetilde{E_{p+1}} = E(t_p) - \frac{\left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1} E(t_p)}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}} + \frac{\beta_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} N\Delta t_{p+1}}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}}. \tag{71}$$

Now, applying the procedure similar to the case (a), we can easily find

$$\begin{aligned} \left|E(t_{p+1}) - \widetilde{E_{p+1}}\right| \leq& \left|\int_{t_p}^{t_{p+1}} E'(\tau)\mathrm{d}\tau - \Delta t_{p+1} E'(t_p)\right| \\ &+ \left|\Delta t_{p+1} E'(t_p) + \frac{\left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1} E(t_p)}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}} - \frac{\beta_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} N\Delta t_{p+1}}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}}\right|, \end{aligned} \tag{72}$$

which becomes

$$\left|E(t_{p+1}) - \widetilde{E_{p+1}}\right| \leq \frac{\left(\Delta t_{p+1}\right)^2}{2}\|E''(t)\|_\infty + \Delta t_{p+1} \cdot I_{E,1}, \tag{73}$$

where

$$I_{E,1} := = \left|E'(t_p) + \frac{\left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)E(t_p) - \beta_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} N}{1 + \left(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}}\right|, \tag{74}$$

which can be written as

$$\left|\frac{E'(t_p)\left[1 + \left(\phi_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}\right]}{1 + \left(\phi_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}} + \frac{\left(\phi_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)E(t_p) - \beta_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} N}{1 + \left(\phi_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta t_{p+1}}\right|, \tag{75}$$

and $\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} = \phi_{p+1}$.Thus,

$$\begin{aligned} I_{E,1} \leq& \frac{1}{1 - \beta_{\max}} \cdot \left|\left(\beta_p NS(t_p)I(t_p) - \beta_{p+1} N\widetilde{S_{p+1}}\widetilde{I_{p+1}}\right)\right| \\ &+ \left|E(t_p)\left\{\left(\beta_p NS(t_p) - \beta_{p+1} N\widetilde{S_{p+1}}\right) + \left(\pi_{p+1} - \pi_p\right) + \left(\mu_{p+1} - \mu_p\right) + \left(\gamma_{p+1} - \gamma_p\right)\right\}\right| \\ &+ \left|\left[\beta_p NS(t_p)I(t_p) + E(t_p)\left(\beta_p NS(t_p) - \phi_{p+1}\right)\right]\left(\phi_{p+1} - \beta_{p+1} N\widetilde{S_{p+1}}\right)\Delta_{p+1}\right|, \end{aligned} \tag{76}$$

where

$$
\begin{aligned}
II_{E,2} &:= = \left|\left(\beta_p NS(t_p)I(t_p) - \beta_{p+1}N\widetilde{S_{p+1}}\widetilde{I_{p+1}}\right)\right|,\\
III_{E,2} &:= = E(t_p)\left\{\left(\beta_p NS(t_p) - \beta_{p+1}N\widetilde{S_{p+1}}\right) + \left(\pi_{p+1} - \pi_p\right) + \left(\mu_{p+1} - \mu_p\right) + \left(\gamma_{p+1} - \gamma_p\right)\right\},\\
IV_{E,2} &:= = \left[\beta_p NS(t_p)I(t_p) + E(t_p)\left(\beta_p NS(t_p) - \phi_{p+1}\right)\right]\left(\phi_{p+1} - \beta_{p+1}N\widetilde{S_{p+1}}\right)\Delta_{p+1},\\
II_{E,2} &= N\left|\beta_p S(t_p)I(t_p) - \beta_{p+1}\widetilde{I_{p+1}}\left(\frac{S(t_p) + \mu_{p+1}N\Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{E_{p+1}}\right)\Delta t_{p+1}}\right)\right|,\\
&\leq N\left|\beta_p S(t_p)I(t_p) - \frac{\beta_{p+1}\widetilde{I_{p+1}}S(t_p)}{1 + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{E_{p+1}}\right)\Delta t_{p+1}}\right| + N\left|\frac{\beta_{p+1}\widetilde{I_{p+1}}\mu_{p+1}N\Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{E_{p+1}}\right)\Delta t_{p+1}}\right|,
\end{aligned}
\tag{77}
$$

where

$$
\begin{aligned}
V_{E,2} &:= = \left|\beta_p S(t_p)I(t_p) - \frac{\beta_{p+1}\widetilde{I_{p+1}}S(t_p)}{1 + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{E_{p+1}}\right)\Delta t_{p+1}}\right|,\\
VI_{E,2} &:= = \left|\frac{\beta_{p+1}\widetilde{I_{p+1}}\mu_{p+1}N\Delta t_{p+1}}{1 + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{E_{p+1}}\right)\Delta t_{p+1}}\right|.
\end{aligned}
\tag{78}
$$

Similarly, substituting the $\widetilde{I_{p+1}}$ value in $V_{E,2}$ and then solving the $V_{E,2}$, we get

$$
\begin{aligned}
V_{E,2(a)} &= \frac{1}{1-\beta_{\max}}\left|\beta_p I(t_p) - \beta_{p+1}I(t_p)\right|\\
&\quad + \frac{1}{1-\beta_{\max}}\left|\beta_p I(t_p)\left[\left(\sigma_{p+1} + \mu_{p+1}\right)\Delta t_{p+1} + \left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{I_{p+1}}\right)\right.\right.\\
&\quad \left.\left. + \left(\sigma_{p+1} + \mu_{p+1}\right)\left(\mu_{p+1} + \beta_{p+1}N\widetilde{E_{p+1}} + \beta_{p+1}N\widetilde{I_{p+1}}\right)\Delta^2 t_{p+1}\right]\right|,\\
V_{E,2(a)} &\leq \frac{N\Delta t_{p+1}}{1-\beta_{\max}}\left\|\beta'(t)\right\|_\infty + \frac{N\Delta t_{p+1}}{1-\beta_{\max}}\left\{\beta_{\max}\left(\sigma_{\max} + 2\left(\mu_{\max} + \beta_{\max}\right) + \left(\mu_{\max+2\beta_{\max}}\right)\left(\sigma_{\max} + \mu_{\max}\right)\Delta t_{p+1}\right)\right\}.
\end{aligned}
\tag{79}
$$

We can also easily find a

$$
V_{E,2(b)} \leq \frac{\pi_{\max}\Delta t_{p+1}}{1-\beta_{\max}}. \tag{80}
$$

Finally,

$$
\begin{aligned}
V_{E,2} &\leq \frac{N^2\Delta t_{p+1}}{1-\beta_{\max}}\left\|\beta'(t)\right\|_\infty + \frac{N\Delta t_{p+1}}{1-\beta_{\max}}\pi_{\max}\\
&\quad + \frac{N^2\Delta t_{p+1}}{1-\beta_{\max}}\left\{\beta_{\max}\left(\sigma_{\max} + 2\left(\mu_{\max} + \beta_{\max}\right)\right.\right.\\
&\quad \left.\left. + \left(\mu_{\max+2\beta_{\max}}\right)\left(\sigma_{\max} + \mu_{\max}\right)\Delta t_{p+1}\right)\right\}.
\end{aligned}
\tag{81}
$$

We can also easily arrive

$$
VI_{E,2} \leq \frac{N\Delta t_{p+1}}{1-\beta_{\max}}\beta_{\max}\mu_{\max}, \tag{82}
$$

and

$$
\begin{aligned}
II_{E,2} &\leq \frac{N^3\Delta t_{p+1}}{1-\beta_{\max}}\left\|\beta'(t)\right\|_\infty\\
&\quad + \frac{N^3\Delta t_{p+1}}{1-\beta_{\max}}\left\{\beta_{\max}\left(\sigma_{\max} + 2\left(\mu_{\max} + \beta_{\max}\right)\right.\right.\\
&\quad \left.\left. + \left(\mu_{\max+2\beta_{\max}}\right)\left(\sigma_{\max} + \mu_{\max}\right)\Delta t_{p+1}\right)\right\}\\
&\quad + \frac{N^2\Delta t_{p+1}}{1-\beta_{\max}}\left\{\beta_{\max}\mu_{\max} + \pi_{\max}\right\}.
\end{aligned}
\tag{83}
$$

Using the same procedure given above, we get

$$
\begin{aligned}
III_{E,2} &\leq \frac{N^3\Delta t_{p+1}}{1-\beta_{\max}}\left\{\left\|\beta'(t)\right\|_\infty + \beta_{\max}\left(\mu_{\max} + 2\beta_{\max}\right)\right\}\\
&\quad + \frac{N^2\Delta t_{p+1}}{1-\beta_{\max}}\mu_{\max} + \left\{\left\|\pi'(t)\right\|_\infty + \left\|\gamma'(t)\right\|_\infty\right\}\Delta t_{p+1},
\end{aligned}
\tag{84}
$$

and

$$IV_{E,2} \leq N^3 \Delta t_{p+1}\{\beta_{\max}(\pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max})\} + N\Delta t_{p+1}(\beta_{\max} + \pi_{\max} + \mu_{\max} + \gamma_{\max})^2. \tag{85}$$

Substituting equations (83)–(85) in $I_{E,1}$, we arrive

$$\begin{aligned} I_{E,1} \leq & \frac{N^3 \Delta t_{p+1}}{1-\beta_{\max}}\{2\|\beta'(t)\|_\infty\} \\ & + \frac{N^3 \Delta t_{p+1}\beta_{\max}}{1-\beta_{\max}}\{(\sigma_{\max} + 2(\mu_{\max} + \beta_{\max})) + (\mu_{\max} + 2\beta_{\max}) + (\pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max}) + (\mu_{\max} + 2\beta_{\max})(\sigma_{\max} + \mu_{\max})\Delta t_{p+1}\} \\ & + \frac{N^2 \Delta t_{p+1}}{1-\beta_{\max}}\{\beta_{\max}\mu_{\max} + \pi_{\max} + \mu_{\max} + \frac{N\Delta t_{p+1}}{1-\beta_{\max}}(\beta_{\max} + \pi_{\max} + \mu_{\max} + \gamma_{\max})^2 + \Delta t_{p+1}\left\{\|\pi\prime(t)\|_\infty + \|\gamma\prime(t)\|_\infty\right\}. \end{aligned} \tag{86}$$

Substituting equation (86) in (73), we obtain the case.

c) For the time-discrete solution I_p at time t_p, let

$$\widetilde{I_{p+1}} = \frac{I_p + \pi_{p+1}\widetilde{E_{p+1}}\Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}}, \tag{87}$$

and for the time-continuous solutions $I(t_p)$, let

$$\widetilde{I_{p+1}} = I(t_p) - \frac{(\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1} I(t_p)}{1 + (\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}} + \frac{\pi_{p+1}\widetilde{E_{p+1}}\Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}}. \tag{88}$$

Using the triangle inequality and mean value theorem, we can easily find

$$\left|I(t_{p+1}) - \widetilde{I_{p+1}}\right| \leq \frac{(\Delta t_{p+1})^2}{2}\|I\prime\prime(t)\|_\infty + \left|\Delta t_{p+1} I\prime(t_p) + \frac{(\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1} I(t_p) - \pi_{p+1}\widetilde{E_{p+1}}\Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}}\right|, \tag{89}$$

where

$$I_{I,1} := = \left|\Delta t_{p+1} I\prime(t_p) + \frac{(\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1} I(t_p) - \pi_{p+1}\widetilde{E_{p+1}}\Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}}\right|. \tag{90}$$

Applying $I\prime(t_p)$ value in (89), we obtain

$$I_{I,1} = \Delta t_{p+1}\left|\pi_p E(t_p) - (\sigma_p + \mu_p) I(t_p) + \frac{(\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1} I(t_p) - \pi_{p+1}\widetilde{E_{p+1}}\Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1})}\right|. \tag{91}$$

Solving the above equations, we get

$$I_{I,1} \leq \Delta t_{p+1}\left\{\left|\pi_p E(t_p) - \pi_{p+1}\widetilde{E_{p+1}}\right| + \left|I(t_p)\right|\left\{\left|\sigma_{p+1} - \sigma_p\right| + \left|\mu_{p+1} - \mu_p\right|\right\} + \left|(\pi_p E(t_p) - I(t_p)(\sigma_p + \mu_p))(\sigma_{p+1} + \mu_{p+1})\Delta t_{p+1}\right|\right\}, \tag{92}$$

which yields

$$I_{I,1} \leq \frac{N(\Delta t_{p+1})^2}{1-\beta_{\max}}\left\{\|\pi\prime(t)\|_\infty + \pi_{\max}(\pi_{\max} + \mu_{\max} + \gamma_{\max} + 2\beta_{\max})\right\} + N(\Delta t_{p+1})^2\left\{\|\sigma\prime(t)\|_\infty + \|\mu\prime(t)\|_\infty + (\pi_{\max} + \sigma_{max} + \mu_{\max})(\sigma_{\max} + \mu_{\max})\right\}. \tag{93}$$

The case (c) is verified by substituting equation (93) in (89).

d) Let the time-discrete solution for Q_p at time t_p is

$$\widetilde{Q_{p+1}} = \frac{Q_p + \left(\gamma_{p+1}\widetilde{E_{p+1}} + \sigma_{p+1}\widetilde{I_{p+1}}\right)\Delta t_{p+1}}{1 + \left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}}, \tag{94}$$

and the time-continuous solutions $Q(t_p)$, we take

$$\widetilde{Q_{p+1}} = Q\left(t_p\right) - \frac{\left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1} Q\left(t_p\right)}{1 + \left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}} + \frac{\left(\gamma_{p+1}\widetilde{E_{p+1}} + \sigma_{p+1}\widetilde{I_{p+1}}\right)\Delta t_{p+1}}{1 + \left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}}, \tag{95}$$

which implies

$$\left|Q\left(t_{p+1}\right) - \widetilde{Q_{p+1}}\right| \leq \frac{\left(\Delta t_{p+1}\right)^2}{2}\|Q''(t)\|_\infty + |Q'\left(t_p\right)\Delta_{p+1} + \frac{\left(\theta_{p+1} + \mu_{p+1}\right)Q\left(t_p\right)\Delta t_{p+1} - \left(\gamma_{p+1}\widetilde{E_{p+1}} + \sigma_{p+1}\widetilde{I_{p+1}}\right)\Delta t_{p+1}}{1 + \left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}}|, \tag{96}$$

where

$$\begin{aligned} I_{Q,1} := &= \left|Q'\left(t_p\right)\Delta_{p+1} + \frac{\left(\theta_{p+1} + \mu_{p+1}\right)Q\left(t_p\right)\Delta t_{p+1} - \left(\gamma_{p+1}\widetilde{E_{p+1}} + \sigma_{p+1}\widetilde{I_{p+1}}\right)\Delta t_{p+1}}{1 + \left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}}\right| \\ &\leq \Delta t_{p+1}\left\{\left|\gamma_p E\left(t_p\right) - \gamma_{p+1}\widetilde{E_{p+1}}\right| + \left|\sigma_p I\left(t_p\right) - \sigma_{p+1}\widetilde{I_{p+1}}\right| + \left|Q\left(t_p\right)\right|\left|\theta_{p+1} - \theta_p\right| + \left|Q\left(t_p\right)\right|\left|\mu_{p+1} - \mu_p\right|\right. \\ &\quad \left. + \left|\left(\gamma_p E\left(t_p\right) + \sigma_p I\left(t_p\right) - Q\left(t_p\right)\left(\theta_p + \mu_p\right)\right)\left(\theta_{p+1} + \mu_{p+1}\right)\Delta t_{p+1}\right|\right\}. \end{aligned} \tag{97}$$

Simplifying the above equation, we arrive

$$\begin{aligned} I_{I,1} \leq &\frac{N\left(\Delta t_{p+1}\right)^2}{1 - \beta_{\max}}\left\{\|\gamma'(t)\|_\infty + \|\sigma'(t)\|_\infty\right\} + N\left(\Delta t_{p+1}\right)^2\left\{\|Q'(t)\|_\infty + \|\mu'(t)\|_\infty\right\} \\ &+ \frac{N\left(\Delta t_{p+1}\right)^2}{1 - \beta_{\max}}\left\{\gamma_{\max}\left(\pi_{\max} + \mu_{\max} + \gamma_{\max} + 2\beta_{\max}\right) + \sigma_{\max}\left(\sigma_{\max} + \mu_{\max}\right)\right\} \\ &+ N\left(\Delta t_{p+1}\right)^2\left\{\gamma_{\max}\left(\theta_{\max} + \mu_{\max}\right) + \sigma_{\max}\left(\theta_{\max} + \mu_{\max}\right) + \left(\theta_{\max} + \mu_{\max}\right)^2\right\} + \frac{\left(\Delta t_{p+1}\right)^2}{1 - \beta_{\max}}\sigma_{\max}\pi_{\max}. \end{aligned} \tag{98}$$

Substituting equation (98) in (96), we get the quarantine case.

e) Let the time-discrete solution for R_p at time t_p is

$$\widetilde{R_{p+1}} = \frac{R_p + \theta_{p+1}\widetilde{Q_{p+1}}\Delta t_{p+1}}{1 + \mu_{p+1}\Delta t_{p+1}}, \tag{99}$$

and the time-continuous solutions $R(t_p)$, let

$$\widetilde{R_{p+1}} = R\left(t_p\right) - \frac{\mu_{p+1}\Delta t_{p+1} R\left(t_p\right)}{1 + \mu_{p+1}\Delta t_{p+1}} + \frac{\theta_{p+1}\widetilde{Q_{p+1}}\Delta t_{p+1}}{1 + \mu_{p+1}\Delta t_{p+1}}, \tag{100}$$

which gives

$$\left|R\left(t_{p+1} - \widetilde{R_{p+1}}\right)\right| \leq \frac{\left(\Delta t_{p+1}\right)^2}{2}\|R''(t)\|_\infty + I_{R,1}, \tag{101}$$

where

$$I_{R,1} := = |\Delta t_{p+1} R'\left(t_p\right) + \frac{\mu_{p+1}\Delta t_{p+1} R\left(t_p\right) - \theta_{p+1}\widetilde{Q_{p+1}}\Delta t_{p+1}}{1 + \mu_{p+1}\Delta t_{p+1}}|. \tag{102}$$

The $I_{R,1}$ expression can be simplified as follows:

$$\begin{aligned} I_{R,1} \leq &\frac{N\left(\Delta t_{p+1}\right)^2}{1 - \beta_{\max}}\left\{\|\theta'(t)\|_\infty + \theta_{\max} + \mu_{\max}\right\} \\ &+ \frac{\left(\Delta t_{p+1}\right)^2}{1 - \beta_{\max}}\left\{\theta_{\max}\left(\gamma_{\max} + \sigma_{\max}\right)\right\} \\ &+ N\left\{\|\mu'(t)\|_\infty + \theta_{\max}\mu_{\max} + \mu_{\max^2}\right\}, \end{aligned} \tag{103}$$

which yields recovered instances by substituting equation (103) into (101).

(69) can be rewritten in the following form:

$$\left\|S\left(t_{p+1}\right)-\widetilde{S_{p+1}}\right\| \leq \Delta^2 t_{p+1} C_{loc,S}. \tag{104}$$

Similarly, we can get the other cases as

$$\begin{aligned}
\left\|E\left(t_{p+1}\right)-\widetilde{E_{p+1}}\right\| &\leq \Delta^2 t_{p+1} C_{loc,E},\\
\left\|I\left(t_{p+1}\right)-\widetilde{I_{p+1}}\right\| &\leq \Delta^2 t_{p+1} C_{loc,I},\\
\left\|Q\left(t_{p+1}\right)-\widetilde{Q_{p+1}}\right\| &\leq \Delta^2 t_{p+1} C_{loc,Q},\\
\left\|R\left(t_{p+1}\right)-\widetilde{R_{p+1}}\right\| &\leq \Delta^2 t_{p+1} C_{loc,R}.
\end{aligned} \tag{105}$$

□

Definition 2. Define $C_{loc}: = \max\{C_{loc,S}; C_{loc,E}; C_{loc,I}; C_{loc,Q}; C_{loc,R}\}$ which holds

$$\left\|z_{\left(t_{p+1}\right)}-\widetilde{z_{p+1}}\right\| \leq \Delta_{p+1}^2 C_{loc}, \tag{106}$$

for local errors on time interval $[t_p, t_{p+1}]$.

2) In reality, $(t_p, S_p)^T$, $(t_p, E_p)^T$, $(t_p, I_p)^T$, $(t_p, Q_p)^T$, and $(t_p, R_p)^T$ do not exactly lie on the graph of the time-continuous solution. Thus, we should examine how procedural errors such as $S_p - S(t_p)$, $E_p - E(t_p)$, $I_p - I(t_p)$, $Q_p - Q(t_p)$, and $R_p - R(t_p)$ propagate to the $(p+1)$th time step. That is,

$$\begin{aligned}
z_{p+1}-z\left(t_{p+1}\right) = &\left(z_p - z\left(t_p\right)\right)\\
&+\Delta t_{p+1}\left\{G\left(t_{p+1}, z_{p+1}\right)-G\left(t_{p+1}, z\left(t_{p+1}\right)\right)\right\},
\end{aligned} \tag{107}$$

which implies

$$\left\|z_{p+1}-z\left(t_{p+1}\right)\right\|_\infty \leq \left\|z_p - z\left(t_p\right)\right\|_\infty + \Delta t_{p+1}\left\|G\left(t_{p+1}, z_{p+1}\right)-G\left(t_{p+1}, z\left(t_{p+1}\right)\right)\right\|_\infty. \tag{108}$$

Here,

$$\begin{aligned}
&\left\|G\left(t_{p+1}, z_{p+1}\right)-G\left(t_{p+1}, z\left(t_{p+1}\right)\right)\right\|_\infty\\
&=\left\|\begin{pmatrix}
A-\mu_{p+1}\left(S\left(t_{p+1}\right)-S_{p+1}\right)-\beta_{p+1}N\left[S\left(t_{p+1}\right)\left(E\left(t_{p+1}\right)+\left(t_{p+1}\right)\right)-S_{p+1}\left(E_{p+1}+I_{p+1}\right)\right]\\
\beta_{p+1}N\left[S\left(t_{p+1}\right)\left(E_{p+1}+I_{p+1}\right)-S_{p+1}\left(E_{p+1}+I_{p+1}\right)-\left(\pi_{p+1}+\mu_{p+1}+\gamma_{p+1}\right)\left(E\left(t_{p+1}-I_{p+1}\right)\right)\right]\\
\pi_{p+1}\left(E\left(t_{p+1}\right)-E_{p+1}\right)-\left(\sigma_{p+1}+\mu_{p+1}\right)\left(I\left(t_{p+1}\right)-I_{p+1}\right)\\
\gamma_{p+1}\left(E\left(t_{p+1}\right)-E_{p+1}\right)+\sigma_{p+1}\left(I\left(t_{p+1}\right)-I_{p+1}\right)\\
-\left(\theta_{p+1}+\mu_{p+1}\right)\left(Q\left(t_{p+1}\right)-Q_{p+1}\right)\\
\theta_{p+1}\left(Q\left(t_{p+1}\right)-Q_{p+1}\right)-\mu_{p+1}\left(R\left(t_{p+1}\right)-R_{p+1}\right)
\end{pmatrix}\right\|_\infty,
\end{aligned} \tag{109}$$

which is less than equal to

$$\begin{aligned}
&\left\|\begin{pmatrix}
A+\mu_{p+1}\left\|S_{p+1}-S\left(t_{p+1}\right)\right\|_\infty+\beta_{p+1}N\left\{2\|S_{p+1}-S\left(t_{p+1}\right)\|_\infty+\left\|E_{p+1}-E\left(t_{p+1}\right)\right\|_\infty+\left\|I_{p+1}-I\left(t_{p+1}\right)\right\|_\infty\right\}\\
\beta_{p+1}N\left\{2\left\|S_{p+1}-S\left(t_{p+1}\right)\right\|_\infty+\left\|E_{p+1}-E\left(t_{p+1}\right)\right\|_\infty+\|I_{p+1}-I\left(t_{p+1}\right)\|_\infty\right\}+\left(\pi_{p+1}+\mu_{p+1}+\gamma_{p+1}\right)\left\|E_{p+1}-E\left(t_{p+1}\right)\right\|_\infty\\
\pi_{p+1}\left\|E_{p+1}-E\left(t_{p+1}\right)\right\|_\infty+\left(\sigma_{p+1}+\mu_{p+1}\right)\left\|I_{p+1}-I\left(t_{p+1}\right)\right\|_\infty\\
\gamma_{p+1}\left\|E_{p+1}-E\left(t_{p+1}\right)\right\|_\infty+\sigma_{p+1}\left\|I_{p+1}-I\left(t_{p+1}\right)\right\|_\infty-\left(\theta_{p+1}+\mu_{p+1}\right)\left\|Q_{p+1}-Q\left(t_{p+1}\right)\right\|_\infty\\
\theta_{p+1}\left\|Q_{p+1}-Q\left(t_{p+1}\right)\right\|_\infty-\mu_{p+1}\left\|R_{p+1}-R\left(t_{p+1}\right)\right\|_\infty
\end{pmatrix}\right\|_\infty\\
&\leq 6\left(\mu_{\max}+\beta_{\max}\right)+2\left(\pi_{\max}+\gamma_{\max}+\theta_{\max}\right)\cdot\left\|z_{p+1}-z\left(t_{p+1}\right)\right\|_\infty.
\end{aligned} \tag{110}$$

Thus,

$$\left\|z_{p+1}-z\left(t_{p+1}\right)\right\|_\infty \leq \left\|z_p-z\left(t_p\right)\right\|_\infty + 6\left(\mu_{\max}+\beta_{\max}\right)+2\left(\pi_{\max}+\gamma_{\max}+\theta_{\max}\right)\cdot\left\|z_{p+1}-z\left(t_{p+1}\right)\right\|_\infty. \tag{111}$$

Hence, we conclude $\|z_{p+1}-z(t_{p+1})\|_\infty$

$$\leq \frac{\left\|z_p - z\left(t_p\right)\right\|_\infty}{1-\left[6\left(\mu_{\max}+\beta_{\max}\right)+2\left(\pi_{\max}+\gamma_{\max}+\theta_{\max}\right)\right]\Delta}, \tag{112}$$

with

$$\begin{aligned}
\Delta =: &= \max_{p\in\{1,2,3,\ldots,M-1\}} \Delta t_{p+1}\\
&< \frac{1}{12\left(\mu_{\max}+\beta_{\max}\right)+4\left(\pi_{\max}+\gamma_{\max}+\theta_{\max}\right)}.
\end{aligned} \tag{113}$$

3) The upper bound between the time-discrete solution and the time-continuous solution is verified below.

For $p = 0$ in (112), it becomes $\|z_1 - z(t_1)\|_\infty = 0$.

For $p = 1$, we get $\|z_2 - z(t_2)\|_\infty \leq \|z_2 - \tilde{z}_2\|_\infty + \|\tilde{z}_2 - z(t_2)\|_\infty$. By applying equations (106) and (112), we get $\|z_2 - z(t_2)\|_\infty$

$$\leq \frac{\|z_1 - z(t_1)\|_\infty}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta} + C_{loc} \cdot \Delta^2, \tag{114}$$

which becomes

$$\|z_2 - z(t_2)\|_\infty \leq C_{loc} \cdot \Delta^2. \tag{115}$$

For $p = 2$, we get

$$\begin{aligned}\|z_3 - z(t_3)\|_\infty &\leq \|z_3 - \tilde{z}_3\|_\infty + \|\tilde{z}_3 - z(t_3)\|_\infty \\ &\leq \frac{\|z_2 - z(t_2)\|_\infty}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta} + C_{loc} \cdot \Delta^2 \\ &\leq \frac{\|z_2 - z(t_2)\|_\infty}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta} + C_{loc} \cdot \Delta^2 \\ &\leq \sum_{j=0}^{1} \frac{C_{loc} \cdot \Delta^2}{(1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta)^j}.\end{aligned} \tag{116}$$

For $p = 3$, we obtain

$$\|z_4 - z(t_4)\|_\infty \leq \|z_4 - \tilde{z}_4\|_\infty + \|\tilde{z}_4 - z(t_4)\|_\infty \leq \frac{\|z_3 - z(t_3)\|_\infty}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta} + C_{loc} \cdot \Delta^2, \tag{117}$$

which yields $\|z_4 - z(t_4)\|_\infty$

$$\leq \sum_{j=0}^{2} \frac{C_{loc} \cdot \Delta^2}{(1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta)^j}. \tag{118}$$

Proceeding like this, we can write $\|z_{p+1} - z(t_{p+1})\|_\infty$

$$\leq \sum_{j=0}^{p-1} \frac{C_{loc} \cdot \Delta^2}{(1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta)^j}, \tag{119}$$

for $p \in \{1, 2, 3, \ldots, M - 2\}$. From (119), we get $\|z_{p+2} - z(t_{p+2})\|_\infty$

$$\leq \sum_{j=0}^{p} \frac{C_{loc} \cdot \Delta^2}{(1 - [6(\mu_{\max} + \beta_{\max}) + (2\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta)^j}, \tag{120}$$

by induction method.

Now, applying the geometric series in (119), we obtain $\|z_{p+1} - z(t_{p+1})\|_\infty$.

$$\begin{aligned}&\leq C_{loc} \cdot \Delta^2 \sum_{j=0}^{p-1} \left(\frac{1}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta}\right)^j, \\ &= C_{loc} \cdot \Delta^2 \frac{(1/1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta)^p - 1}{(1/1 - [6(\mu_{\max} + \beta_{\max}) + 2Z(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta) - 1}.\end{aligned} \tag{121}$$

Assuming $\Delta < 1/(12(\mu_{\max} + \beta_{\max}) + 4(\pi_{\max} + \gamma_{\max} + \theta_{\max}))$, we conclude $(\Delta/(1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})]\Delta) - 1) \leq 1$.

Hence, we get (39).

5. Conclusion

The present paper is devoted to the analysis and optimization of the SEIQR epidemic model containing an isolation class. We derived both continuous and discrete schemes of the SEIQR model as well as the global existence of solutions and nonnegativity bounded properties for both schemes. Along with this, we have illustrated the solving technique for the discrete scheme and proposed a new optimization technique with the help of a dynamic programming problem. In addition, we have analyzed the error between continuous and discrete schemes in the last section. Finally, we conclude that the isolation class plays a vital role in controlling the COVID-19 pandemic situation. More interestingly, the results also reveal that COVID-19 can exhibit oscillatory behaviour in the future. On the other hand, social distancing methods, quarantine efficiency, and isolation can be used to keep it under control. Future research could look into the effects of current coronavirus mutations like Delta and Omicron on the COVID-19 pandemic's dynamics. We also suggested an alternative dynamic model of the SEIQR class, which can theoretically be generalized to generate continuous and discrete-time models such as SEIR, SEIRS,

SIRS, SEI, SEIS, SI, SIS, SEIQRS, SIDARTHE, and others in future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] B. Wacker and J. Schlüter, "Time-continuous and time-discrete SIR models revisited: theory and applications," *Advances in Difference Equations*, vol. 1, pp. 1–44, 2020.

[2] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society of London—Series A: Containing Papers of a Mathematical and Physical Character*, vol. 115, no. 772, pp. 700–721, 1927.

[3] J. Cao, X. Hu, W. Cheng, L. Yu, W. J. Tu, and Q. Liu, "Clinical features and short-term outcomes of 18 patients with corona virus disease 2019 in intensive care unit," *Intensive Care Medicine*, vol. 46, no. 5, pp. 851–853, 2020.

[4] I. Nesteruk, "Statistics-based predictions of coronavirus epidemic spreading in Mainland China," 2020, https://www.medrxiv.org/content/10.1101/2020.02.12.20021931v1.

[5] D. J. Gerberry and F. A. Milner, "An SEIQR model for childhood diseases," *Journal of Mathematical Biology*, vol. 59, no. 4, pp. 535–561, 2009.

[6] A. Zeb, E. Alzahrani, V. S. Erturk, and G. Zaman, "Mathematical model for coronavirus disease 2019 (covid-19) containing isolation class," *Biomed Research international*, vol. 2020, Article ID 3452402, 7 pages, 2020.

[7] M. Zima, "A certain fixed point theorem and its applications to integral-functional equations," *Bulletin of the Australian Mathematical Society*, vol. 46, no. 2, pp. 179–186, 1992.

[8] P. Zhou, X. L. Yang, X. G. Wang et al., "A pneumonia outbreak associated with a new coronavirus of probable bat origin," *Nature*, vol. 579, no. 7798, pp. 270–273, 2020.

[9] W. O. Kermack and A. G. McKendrick, "Contributions to the mathematical theory of epidemics-I 1927," *Bulletin of Mathematical Biology*, vol. 53, pp. 33–55, 1991.

[10] W. O. Kermack and A. G. McKendrick, "Contributions to the mathematical theory of epidemics-II, the problem of endemicity," *Proceedings of the Royal Society of London—Series A: Containing Papers of a Mathematical and Physical Character*, vol. 138, no. 834, pp. 55–83, 1932.

[11] S. He, Y. Peng, and K. Sun, "SEIR modeling of the COVID-19 and its dynamics," *Nonlinear Dynamics*, vol. 101, no. 3, pp. 1667–1680, 2020.

[12] I. Rahimi, A. H. Gandomi, P. G. Asteris, and F. Chen, "Analysis and prediction of covid-19 using SIR, SEIQR and machine learning models: Australia, Italy and UK cases," *Information*, vol. 12, no. 3, p. 109, 2021.

[13] T. Hussain, M. Ozair, F. Ali, S. U. Rehman, T. A. Assiri, and E. E. Mahmoud, "Sensitivity analysis and optimal control of COVID-19 dynamics based on SEIQR model," *Results in Physics*, vol. 22, Article ID 103956, 2021.

[14] H. Youssef, N. Alghamdi, M. A. Ezzat, A. A. El-Bary, and A. M. Shawky, "Study on the SEIQR model and applying the epidemiological rates of COVID-19 epidemic spread in Saudi Arabia," *Infectious Disease Modelling*, vol. 6, pp. 678–692, 2021.

[15] R. Prabakaran, S. Jemimah, P. Rawat, D. Sharma, and M. M. Gromiha, "A novel hybrid SEIQR model incorporating the effect of quarantine and lockdown regulations for COVID-19," *Scientific Reports*, vol. 11, no. 1, pp. 24073–24079, 2021.

[16] H. M. Youssef, N. Alghamdi, M. A. Ezzat, A. A. El-Bary, and A. M. Shawky, "A proposed modified SEIQR epidemic model to analyze the COVID-19 spreading in Saudi Arabia," *Alexandria Engineering Journal*, vol. 61, no. 3, pp. 2456–2470, 2022.

[17] L. J. Allen, "Some discrete-time SI, SIR, and SIS epidemic models," *Mathematical Biosciences*, vol. 124, no. 1, pp. 83–105, 1994.

[18] D. Ghosh, P. K. Santra, G. S. Mahapatra, A. Elsonbaty, and A. A. Elsadany, "A Discrete-Time epidemic model for the analysis of transmission of COVID19 based upon data of epidemiological parameters," *The European Physical Journal-Special Topics*, pp. 1–10, 2022.

[19] M. Alqhtani, K. M. Owolabi, and K. M. Saad, "Spatiotemporal (target) patterns in sub-diffusive predator-prey system with the Caputo operator," *Chaos, Solitons & Fractals*, vol. 160, Article ID 112267, 2022.

[20] N. Valliammal and C. Ravichandran, "Results on fractional neutral integro-differential systems with state-dependent delay in Banach spaces," *Nonlinear Studies*, vol. 25, no. 1, 2018.

[21] K. S. Nisar, K. Jothimani, K. Kaliraj, and C. Ravichandran, "An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain," *Chaos, Solitons & Fractals*, vol. 146, Article ID 110915, 2021.

[22] V. Vijayakumar, C. Ravichandran, K. S. Nisar, and K. D. Kucche, "New discussion on approximate controllability results for fractional sobolev type volterra-fredholm integro-differential systems of order 1<r<2," *Numerical Methods for Partial Differential Equations*, 2021.

[23] A. A. Alderremy, J. F. Gómez-Aguilar, S. Aly, and K. M. Saad, "A fuzzy fractional model of coronavirus (COVID-19) and its study with Legendre spectral method," *Results in Physics*, vol. 21, Article ID 103773, 2021.

[24] W. K. Ming, J. Huang, and C. J. Zhang, "Breaking down of healthcare system: mathematical modelling for controlling the novel coronavirus (2019-ncov) outbreak in Wuhan, China," 2020, https://www.biorxiv.org/content/10.1101/2020.01.27.922443v1.

[25] V. A. Okhuese, "Mathematical predictions for coronavirus as a global pandemic," 2020, https://www.medrxiv.org/content/10.1101/2020.03.19.20038794v1.

[26] P. Van den Driessche and J. Watmough, "Further notes on the basic reproduction number," *Mathematical Epidemiology*, pp. 159–178, Springer, Berlin, Germany, 2008.

[27] M. Batista, "Estimation of the final size of the COVID-19 epidemic," 2020, https://www.medrxiv.org/content/10.1101/2020.02.16.20023606v5.

[28] R. E. Mickens, "Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition," *Numerical Methods for Partial Differential Equations*, vol. 23, no. 3, pp. 672–691, 2007.

[29] A. K. Bairagi, M. Masud, D. H. Kim et al., "Controlling the outbreak of COVID-19: a non cooperative game perspective," *IEEE Access*, vol. 8, pp. 215570–215581, 2020.

[30] M. A. Awal, M. Masud, M. S. Hossain, A. A. M. Bulbul, S. M. H. Mahmud, and A. K. Bairagi, "A novel bayesian

optimization-based machine learning framework for COVID-19 detection from inpatient facility data," *IEEE Access*, vol. 9, pp. 10263–10281, 2021.

[31] B. Tang, N. L. Bragazzi, Q. Li, S. Tang, Y. Xiao, and J. Wu, "An updated estimation of the risk of transmission of the novel coronavirus (2019-nCov)," *Infectious Disease Modelling*, vol. 5, pp. 248–255, 2020.

[32] B. Tang, X. Wang, Q. Li et al., "Estimation of the transmission risk of the 2019-nCoV and its implication for public health interventions," *Journal of Clinical Medicine*, vol. 9, no. 2, p. 462, 2020.

[33] C. Wang, L. Liu, X. Hao et al., "Evolving epidemiology and impact of Non-pharmaceutical interventions on the outbreak of coronavirus disease 2019 in Wuhan, China," 2020, https://www.medrxiv.org/content/10.1101/2020.03.03.20030593v1.

[34] J. T. Wu, K. Leung, and G. M. Leung, "Nowcasting and forecasting the potential domestic and international spread of the 2019-nCoV outbreak originating in Wuhan, China: a modelling study," *The Lancet*, vol. 395, no. 10225, pp. 689–697, 2020.

[35] Z. Yang, Z. Zeng, K. Wang et al., "Modified SEIR and AI prediction of the epidemics trend of COVID-19 in China under public health interventions," *Journal of Thoracic Disease*, vol. 12, no. 3, pp. 165–174, 2020.

[36] S. Zhao, Q. Lin, J. Ran et al., "Preliminary estimation of the basic reproduction number of novel coronavirus (2019-nCoV) in China, from 2019 to 2020: a data-driven analysis in the early phase of the outbreak," *International Journal of Infectious Diseases*, vol. 92, pp. 214–217, 2020.

[37] S. Mandal, T. Bhatnagar, and N. Arinaminpathy, "Prudent public health intervention strategies to control the coronavirus disease 2019 transmission in India: A mathematical model-based approach," *Indian Council of Medical Research*, vol. 151, 2020.

[38] B. F. Maier and D. Brockmann, "Effective containment explains subexponential growth in recent confirmed COVID-19 cases in China," *Science*, vol. 368, no. 6492, pp. 742–746, 2020.

[39] C. Anastassopoulou, L. Russo, A. Tsakris, and C. Siettos, "Data-based analysis, modelling and forecasting of the COVID-19 outbreak," *PLoS One*, vol. 15, no. 3, Article ID 0230405, 2020.

[40] S. E. Eikenberry, M. Mancuso, E. Iboi et al., "To mask or not to mask: modeling the potential for face mask use by the general public to curtail the COVID-19 pandemic," *Infectious Disease Modelling*, vol. 5, pp. 293–308, 2020.

[41] Z. Liu, P. Magal, O. Seydi, and G. Webb, "Predicting the cumulative number of cases for the COVID-19 epidemic in China from early data," *Mathematical Bioscience and Engineering*, vol. 17, 2020.

[42] F. Ndaïrou, I. Area, J. J. Nieto, and D. F. M. Torres, "Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan," *Chaos, Solitons & Fractals*, vol. 135, Article ID 109846, 2020.

[43] Z. Liu, P. Magal, O. Seydi, and G. Webb, "A COVID-19 epidemic model with latency period," *Infectious Disease Modelling*, vol. 5, pp. 323–337, 2020.

[44] S. Choi and M. Ki, "Estimating the reproductive number and the outbreak size of COVID-19 in Korea," *Epidemiology and Health*, vol. 42, Article ID e2020011, 2020.

[45] S. O. Akindeinde, E. Okyere, A. O. Adewumi, R. S. Lebelo, O. O. Fabelurin, and S. E. Moore, "Caputo fractional-order SEIRP model for COVID-19 Pandemic," *Alexandria Engineering Journal*, vol. 61, no. 1, pp. 829–845, 2022.

[46] G. David, C. Schaeffer, and W. John, *Ordinary Differential Equations: Basics and beyond*, Springer, Berlin, Germany, 2018.

[47] B. Wacker, T. Kneib, and J. Schlüter, *Revisiting Maximum Log-Likelihood Parameter Estimation for Two-Parameter Weibull Distributions: Theory and Applications*, 2020.

[48] M. Reed and B. Simon, "Functional analysis," *Gulf Professional Publishing*, vol. 1, pp. 1–413, 1980.

[49] B. G. Pachpatte, *Inequalities for Differential and Integral Equation*, Annals of the Institute of Statistical Mathematics, Elsevier, Amsterdam, Netherlands, 1997.

[50] M. Martcheva, *An Introduction to Mathematical Epidemiology*, Springer, Berlin, Germany, 2015.

[51] F. Brauer, C. Castillo-Chavez, and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, Berlin, Germany, 2012.

[52] T. Harko, F. S. Lobo, and M. Mak, "Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates," *Applied Mathematics and Computation*, vol. 236, pp. 184–194, 2014.