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Research Article

Finding the Number of Weak Homomorphisms of Paths

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Let G and H be graphs. A mapping f from $V(G)$ to $V(H)$ is called a weak homomorphism from G to H if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. In this paper, we provide an algorithm to determine the number of weak homomorphisms of paths.

1. Introduction

Let G and H be graphs. A mapping $f: V(G) \longrightarrow V(H)$ is a homomorphism from G to H if f preserves the edges, i.e., if $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A homomorphism from G to G itself is called an endomorphism on G. Denote the set of homomorphisms from G to H by $\mathrm{Hom}(G, H)$ and the set of endomorphisms on G by $\mathrm{End}(G)$. Clearly, $\mathrm{End}(G)$ forms a monoid under composition of mappings. Let P_n denote a path of order n such that $V(P_n) = \{0, 1, \cdots, n-1\}$ and $E(P_n) = \{\{i, i+1\} \mid i = 0, 1, \cdots, n-2\}$. Let C_n denote a cycle of order $n (n \geq 3)$ such that $V(C_n) = \{0, 1, \cdots, n-1\}$ and $E(C_n) = \{\{i, i+1\} \mid i = 0, 1, \cdots, n-1\}$, where $+$ is the addition modulo n. Furthermore, we will refer to [1, 2] for more information about graphs and algebraic graphs.

There are many interesting results concerning graphs and their homomorphisms (or endomorphism monoids). In 1992, Böttcher and Knauer [3] gave an account of the different ways to define homomorphisms of graphs, which leads to six classes of endomorphisms for each graph, i.e., endomorphisms, half-strong endomorphisms, locally-strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms, and automorpisms. The formulas for the number of graph homomorphisms and the number of graph endomorphisms are the important tools for studying the structures of $\mathrm{Hom}(G, H)$ and $\mathrm{End}(G)$, respectively. The formula for the number of endomorphisms on paths $\mathrm{End}(P_n)$ was introduced by Arworn [4] in 2009. She gave the formula in terms of the numbers of shortest paths from the point $(0, 0)$ to any point (i, j) in an r-ladder square lattice. Furthermore, in the same year, Arworn and Wojtylak [5] gave the formula for the number of homomorphisms of paths $\mathrm{Hom}(P_m, P_n)$ in terms of the order of $\mathrm{Hom}_j^i(P_m, P_n)$ where $\mathrm{Hom}_j^i(P_m, P_n) = \{f \in \mathrm{Hom}(P_m, P_n) \mid f(0) = i, f(m-1) = j\}$ for all $i, j \in \{0, 1, \cdots, n-1\}$.

For a mapping $f: V(G) \longrightarrow V(H)$, we say that f contracts an edge $\{x, y\}$ if $f(x) = f(y)$. The point is that homomorphisms have to preserve edges. If we have the possibility of contracting edges as well, then this could also be achieved with usual homomorphisms when our graphs contain a loop at every vertex.

A mapping $f: V(G) \longrightarrow V(H)$ is called a weak homomorphism from a graph G to a graph H (also called an egamorphism) if f contracts or preserves the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A weak homomorphism from G to G itself is called a weak endomorphism on G. Denote the set of weak homomorphisms from G to H by $\mathrm{WHom}(G, H)$, and the set of weak endomorphisms on G by $\mathrm{WEnd}(G)$. Clearly, $\mathrm{WEnd}(G)$ forms a monoid under composition of mappings.

In 2010, Sirisathianwatthana and Pipattanajinda [6] gave the number of weak homomorphisms of cycles $\mathrm{WHom}(C_m, C_n)$ in terms of the order of $\mathrm{WHom}^i_j(P_{m-1}, C_n)$ where $\mathrm{WHom}^i_j(P_{m-1}, C_n) = \{f \in \mathrm{WHom}(P_{m-1}, C_n) \mid f(0) = i, f(m-1) = j\}$ for all $i, j \in \{0, 1, \cdots, n-1\}$. Recently, in 2018, Knauer and Pipattanajinda [7] introduced the number of weak endomorphisms on paths $\mathrm{WEnd}(P_n)$ in terms of the numbers of shortest paths from the point (0,0,0) to any point (i, j, k) in the three-dimensional square lattice and in the r-ladder three-dimensional square lattice. In this paper, we are interested in finding the number of weak homomorphisms of paths $\mathrm{WHom}(P_m, P_n)$ in terms of $|A^i_{m-1,n}|$, $|B^i_{m-1,n}|$, and $|C^i_{m-1,n}|$ where $A^i_{m-1,n} = \{f \in \mathrm{WHom}(P_{m-1}, P_n) \mid f(0) = i\}$, $B^i_{m-1,n} = \{f \in \mathrm{WHom}(P_{m-1}, P_n) \mid f(0) = i \text{ and } f(m-2) = 0\}$, $C^i_{m-1,n} = \{f \in \mathrm{WHom}(P_{m-1}, P_n) \mid f(0) = i \text{ and } f(m-2) = n-1\}$ for all $i \in \{0, 1, \cdots, n-1\}$.

2. The Number of Weak Homomorphisms of Paths

In this section, we give the number of weak homomorphisms from P_m to P_n,i.e.,$|\mathrm{WHom}(P_m, P_n)|$. We see at once that $|\mathrm{WHom}(P_1, P_n)| = n$ and $|\mathrm{WHom}(P_m, P_1)| = 1$. The task is now to find $|\mathrm{WHom}(P_m, P_n)|$ for all $m, n \in \mathbb{N} - \{1\}$. For $i \in \{0, 1, \cdots, n-1\}$, let $A^i_{m,n} = \{f \in \mathrm{WHom}(P_m, P_n) \mid f(0) = i\}$, $B^i_{m,n} = \{f \in \mathrm{WHom}(P_m, P_n) \mid f(0) = i \text{ and } f(m-1) = 0\}$, $C^i_{m,n} = \{f \in \mathrm{WHom}(P_m, P_n) \mid f(0) = i \text{ and } f(m-1) = n-1\}$, and $D^i_{m,n} = \{f \in \mathrm{WHom}(P_m, P_n) \mid f(0) = i \text{ and } f(m-1) \neq 0 \text{ and } f(m-1) \neq n-1\}$. It is clear that $\{A^i_{m,n} \mid i = 0, 1, \cdots, n-1\}$ is a partition of $\mathrm{WHom}(P_m, P_n)$, and $\{B^i_{m,n}, C^i_{m,n}, D^i_{m,n}\}$ is a partition of $A^i_{m,n}$. By the definition of a weak homomorphism, if $f(0) = i$ then $i - j \leq f(j) \leq i + j$ for all $i \in \{0, 1, \cdots, n-1\}$ and $j \in \{0, 1, \cdots, m-1\}$. It follows that if $f(0) \geq m$, then $1 \leq f(m-1)$. This gives $B^i_{m,n} = \varnothing$ for all $i \geq m$. Similary, if $f(0) \leq m-2$, then $f(m-1) \leq 2m-3 < n-1$. We thus get $C^i_{m,n} = \varnothing$ for all $i \leq m-2$. For simplicity of notation, we some time write $f = f(0)f(1)\cdots f(m-1)$ instead of $f = \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ f(0) & f(1) & \cdots & f(m-1) \end{pmatrix}$.

Example 1. Consider the weak homomorphisms from P_3 to P_6. We see that $\mathrm{WHom}(P_3, P_6) = A^0_{3,6} \cup A^1_{3,6} \cup A^2_{3,6} \cup A^3_{3,6} \cup A^4_{3,6} \cup A^5_{3,6}$ where $A^0_{3,6} = \{000,001,010,011,012\}$, $A^1_{3,6} = \{100,101,110,111,112,121,122,123\}$, $A^2_{3,6} = \{210,211,212,221,222,223,232,233,234\}$, $A^3_{3,6} = \{321,322,323,332,333,334,343,344,345\}$, $A^4_{3,6} = \{432,433,434,443,444,445,454,455\}$, and $A^5_{3,6} = \{543,544,545,554,555\}$. Moreover, we get $A^i_{3,6} = B^i_{3,6} \cup C^i_{3,6} \cup D^i_{3,6}$ for all $i = 0, 1, \cdots, 5$ where

$$\begin{array}{lll} B^0_{3,6} = \{000,010\}, & C^0_{3,6} = \varnothing, & D^0_{3,6} = \{001,011,012\}, \\ B^1_{3,6} = \{100,110\}, & C^1_{3,6} = \varnothing, & D^1_{3,6} = \{101,111,112,121,122,123\}, \\ B^2_{3,6} = \{210\}, & C^2_{3,6} = \varnothing, & D^2_{3,6} = \{211,212,221,222,223,232,233,234\}, \\ B^3_{3,6} = \varnothing, & C^3_{3,6} = \{345\}, & D^3_{3,6} = \{321,322,323,332,333,334,343,344\}, \\ B^4_{3,6} = \varnothing, & C^4_{3,6} = \{445,455\}, & D^4_{3,6} = \{432,433,434,443,444,454\}, \\ B^5_{3,6} = \varnothing, & C^5_{3,6} = \{545,555\}, & D^5_{3,6} = \{543,544,554\}. \end{array} \tag{1}$$

Let us look at the order of the sets $A^i_{3,6}$, $B^i_{3,6}$, and $C^i_{3,6}$ in Example 1, we see that $|A^0_{3,6}| = |A^5_{3,6}|$, $|A^1_{3,6}| = |A^4_{3,6}|$, $|A^2_{3,6}| = |A^3_{3,6}|$, and $|B^i_{3,6}| = |C^{6-(i+1)}_{3,6}|$ for all $i = 0, 1, 2$. In the first two lemmas, we consider about some relations of the order of $A^i_{m,n}$, $B^i_{m,n}$, and $C^i_{m,n}$ for all $i = 0, 1, \cdots, n-1$.

Lemma 2. *Let $m, n \in \mathbb{N} - \{1\}$. Then $|A^i_{m,n}| = |A^{n-(i+1)}_{m,n}|$ for all $i = 0, 1, \cdots, n-1$.*

Proof. Let $i \in \{0, 1, \cdots, n-1\}$ and $f \in A^i_{m,n}$. Define $g : V(P_m) \longrightarrow V(P_n)$ by $g(j) = f(j) + n - (2f(j) + 1)$ for all $j \in \{0, 1, \cdots, m-1\}$. We first prove that g is a weak homomorphism. Let $\{x, y\} \in E(P_m)$. To show that $g(x) = g(y)$ or $\{g(x), g(y)\} \in E(P_n)$. Since $\{x, y\} \in E(P_m)$ and f is a weak homomorphism, we get $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(P_n)$.

Case 1. If $f(x) = f(y)$, then

$$g(x) = f(x) + n - (2f(x) + 1) = f(y) + n - (2f(y) + 1) = g(y). \tag{2}$$

Case 2. If $\{f(x), f(y)\} \in E(P_n)$, then $f(x) = f(y) + 1$ or $f(x) = f(y) - 1$.

Case 2.1. If $f(x) = f(y) + 1$, then

$$\begin{aligned} g(x) &= f(x) + n - (2f(x) + 1) \\ &= f(y) + 1 + n - (2(f(y) + 1) + 1) \\ &= f(y) + n - (2f(y) + 1) - 1 = g(y) - 1. \end{aligned} \tag{3}$$

It follows that $\{g(x), g(y)\} \in E(P_n)$.

Case 2.2. If $f(x) = f(y) - 1$, then

$$\begin{aligned} g(x) &= f(x) + n - (2f(x) + 1) \\ &= f(y) - 1 + n - (2(f(y) - 1) + 1) \\ &= f(y) + n - (2f(y) + 1) + 1 = g(y) + 1. \end{aligned} \tag{4}$$

This gives $\{g(x), g(y)\} \in E(P_n)$.

From Case 1 and Case 2, we conclude that $g \in \mathrm{WHom}(P_m, P_n)$. Since $g(0) = f(0) + n - (2f(0) + 1) = i + n - (2i + 1) = n - (i + 1)$, we get $g \in A^{n-(i+1)}_{m,n}$. Define $\phi : A^i_{m,n} \longrightarrow A^{n-(i+1)}_{m,n}$ by $\phi(f) = f + n - (2f + 1)$ for all $f \in A^i_{m,n}$. We will show that ϕ is bijective. Let $f_1, f_2 \in A^i_{m,n}$ such that $\phi(f_1$

$) = \phi(f_2)$. Therefore, for all $x \in V(P_m)$,

$$\begin{aligned}(\phi(f_1))(x) &= (\phi(f_2))(x),\\ f_1(x) + n - (2f_1(x)+1) &= f_2(x) + n - (2f_2(x)+1),\\ f_1(x) &= f_2(x).\end{aligned} \tag{5}$$

The result is $f_1 = f_2$. So ϕ is injective. Let $g \in A^{n-(i+1)}_{m,n}$. Define $f : V(P_m) \longrightarrow V(P_n)$ by $f(j) = -g(j) + n - 1$. In the same manner as proved above, we get $f \in \mathrm{WHom}(P_m, P_n)$. Since $f(0) = i$, we have $f \in A^i_{m,n}$ for all $j \in \{0, 1, \cdots, m-1\}$. As $\phi(f) = g$, we get ϕ is surjective. Hence ϕ is bijective and so $|A^i_{m,n}| = |A^{n-(i+1)}_{m,n}|$ for all $i = 0, 1, \cdots, n-1$.

Lemma 3. *Let $m, n \in \mathbb{N} - \{1\}$.*

(1) $|B^i_{m,n}| = |C^{n-(i+1)}_{m,n}|$ for all $i = 0, 1, \cdots, n-1$.

(2) If $n > 2m-2$, then $|A^j_{m,n}| = |A^k_{m,n}|$ for all $j, k \in \{m-1, m, \cdots, n-m\}$.

Proof. 1. Consider the function ϕ in the proof of Lemma 2. It is easy to check that $\phi(B^i_{m,n}) = C^{n-(i+1)}_{m,n}$. This gives $|B^i_{m,n}| = |C^{n-(i+1)}_{m,n}|$ for all $i = 0, 1, \cdots, n-1$.

2. Let $n > 2m-2$. Then $|\{m-1, m, \cdots, n-m\}| \geq 1$. Define ϕ: $A^j_{m,n} \longrightarrow A^k_{m,n}$ by $\phi(f) = f + (k-j)$ for all $f \in A^j_{m,n}$. By similar arguments as in Lemma 2, we also conclude that ϕ is bijective.

The following lemma gives the order of $A^i_{m,n}$ in terms of $|A^i_{m-1,n}|$, $|B^i_{m-1,n}|$ and $|C^i_{m-1,n}|$.

Lemma 4. *Let $m, n \in \mathbb{N} - \{1\}$. Then $|A^i_{m,n}| = 3|A^i_{m-1,n}| - |B^i_{m-1,n}| - |C^i_{m-1,n}|$ for all $i \in \{0, 1, \cdots, n-1\}$.*

Proof. Let $f \in A^i_{m-1,n}$ and $f' : V(P_m) \longrightarrow V(P_n)$ such that $f'|_{\{0,1,\cdots,m-2\}} = f$.

Since $f \in B^i_{m-1,n} \longleftrightarrow f' \in \{g_1, g_2\} \subseteq A^i_{m,n} \longleftrightarrow f' \in A^i_{m,n}$ and $f'(m-2) = 0$ where

$$g_1(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ 0 & \text{if } x = m-1, \end{cases}$$
$$g_2(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ 1 & \text{if } x = m-1, \end{cases} \tag{6}$$

then in this case we obtain $2|B^i_{m-1,n}|$ weak homomorphisms in $A^i_{m,n}$.

Since $f \in C^i_{m-1,n} \longleftrightarrow f' \in \{t_1, t_2\} \subseteq A^i_{m,n} \longleftrightarrow f' \in A^i_{m,n}$ and $f'(m-2) = n-1$, where

$$t_1(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ n-1 & \text{if } x = m-1, \end{cases}$$
$$t_2(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ n-2 & \text{if } x = m-1, \end{cases} \tag{7}$$

then in this case we get $2|C^i_{m-1,n}|$ weak homomorphisms in $A^i_{m,n}$.

Since $f \in D^i_{m-1,n} \longleftrightarrow f' \in \{h_1, h_2, h_3\} \subseteq A^i_{m,n} \longleftrightarrow f' \in A^i_{m,n}$ and $f'(m-2) \notin \{0, n-1\}$, where

$$h_1(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ f(m-2) & \text{if } x = m-1, \end{cases}$$
$$h_2(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ f(m-2)+1 & \text{if } x = m-1, \end{cases} \tag{8}$$
$$h_3(x) = \begin{cases} f(x) & \text{if } x = 0, 1, \cdots, m-2 \\ f(m-2)-1 & \text{if } x = m-1, \end{cases}$$

then in this case we have $3|D^i_{m-1,n}|$ weak homomorphisms in $A^i_{m,n}$.

Thus, we conclude that $|A^i_{m,n}| = 3|D^i_{m-1,n}| + 2|B^i_{m-1,n}| + 2|C^i_{m-1,n}| = 3|D^i_{m-1,n}| + (3|B^i_{m-1,n}| - |B^i_{m-1,n}|) + (3|C^i_{m-1,n}| - |C^i_{m-1,n}|) = 3(|D^i_{m-1,n}| + |B^i_{m-1,n}| + |C^i_{m-1,n}|) - |B^i_{m-1,n}| - |C^i_{m-1,n}|$. Since $|A^i_{m-1,n}| = |D^i_{m-1,n}| + |B^i_{m-1,n}| + |C^i_{m-1,n}|$, it follows that $|A^i_{m,n}| = 3|A^i_{m-1,n}| - |B^i_{m-1,n}| - |C^i_{m-1,n}|$.

Example 5. Consider the weak homomorphisms from P_2 to P_6. We have $\mathrm{WHom}(P_2, P_6) = A^0_{2,6} \cup A^1_{2,6} \cup A^2_{2,6} \cup A^3_{2,6} \cup A^4_{2,6} \cup A^5_{2,6}$ where

$$\begin{array}{llll} A^0_{2,6} = \{00, 01\}, & B^0_{2,6} = \{00\}, & C^0_{2,6} = \varnothing, & D^0_{2,6} = \{01\}, \\ A^1_{2,6} = \{10,11,12\}, & B^1_{2,6} = \{10\}, & C^1_{2,6} = \varnothing, & D^1_{2,6} = \{11, 12\}, \\ A^2_{2,6} = \{21,22,23\}, & B^2_{2,6} = \varnothing, & C^2_{2,6} = \varnothing, & D^2_{2,6} = \{21,22,23\}, \\ A^3_{2,6} = \{32,33,34\}, & B^3_{2,6} = \varnothing, & C^3_{2,6} = \varnothing, & D^3_{2,6} = \{32,33,34\}, \\ A^4_{2,6} = \{43,44,45\}, & B^4_{2,6} = \varnothing, & C^4_{2,6} = \{45\}, & D^4_{2,6} = \{43, 44\}, \\ A^5_{2,6} = \{54, 55\}, & B^5_{2,6} = \varnothing, & C^5_{2,6} = \{55\}, & D^5_{2,6} = \{54\}. \end{array} \tag{9}$$

Consider the sets $A^i_{3,6}$ in Example 1, we see that $|A^i_{3,6}| = 3|A^i_{2,6}| - |B^i_{2,6}| - |C^i_{2,6}|$.

The next theorem gives the order of $\mathrm{WHom}(P_m, P_n)$ in terms of $|A^i_{m-1,n}|$, $|B^i_{m-1,n}|$ and $|C^i_{m-1,n}|$ for all $i = 0, 1, \cdots, m-1$.

Theorem 6. *Let $m, n \in \mathbb{N} - \{1\}$.*

(1) If n is even, then $|WHom(P_m, P_n)| = 2(3|A^0_{m-1,n}| - |B^0_{m-1,n}| - |C^0_{m-1,n}|) + 2(3|A^1_{m-1,n}| - |B^1_{m-1,n}| - |C^1_{m-1,n}|) + \cdots + 2(3|A^{n/2-1}_{m-1,n}| - |B^{n/2-1}_{m-1,n}| - |C^{n/2-1}_{m-1,n}|)$.

(2) If n is odd, then $|WHom(P_m, P_n)| = 2(3|A^0_{m-1,n}| - |B^0_{m-1,n}| - |C^0_{m-1,n}|) + 2(3|A^1_{m-1,n}| - |B^1_{m-1,n}| - |C^1_{m-1,n}|) + \cdots + 2(3|A^{\lfloor n/2\rfloor-1}_{m-1,n}| - |B^{\lfloor n/2\rfloor-1}_{m-1,n}| - |C^{\lfloor n/2\rfloor-1}_{m-1,n}|) + (3|A^{\lfloor n/2\rfloor}_{m-1,n}| - |B^{\lfloor n/2\rfloor}_{m-1,n}| - |C^{\lfloor n/2\rfloor}_{m-1,n}|)$.

Proof. Since $|WHom(P_m, P_n)| = |A^0_{m,n}| + |A^1_{m,n}| + \cdots + |A^{n-1}_{m,n}|$, by Lemma 2, we have

$$|WHom(P_m, P_n)| = \begin{cases} 2|A^0_{m,n}| + 2|A^1_{m,n}| + \cdots + 2|A^{n/2-1}_{m,n}| & \text{if } n \text{ is even}, \\ 2|A^0_{m,n}| + 2|A^1_{m,n}| + \cdots + 2\left|A^{\lfloor \frac{n}{2} \rfloor - 1}_{m,n}\right| + \left|A^{\lfloor n/2 \rfloor}_{m,n}\right| & \text{if } n \text{ is odd}. \end{cases} \quad (10)$$

By Lemma 4, we get the statements 1 and 2.

Theorem 7. *Let $m, n \in \mathbb{N} - \{1\}$. If $n > 2m - 2$, then $|WHom(P_m, P_n)| = 2(3|A^0_{m-1,n}| - |B^0_{m-1,n}|) + 2(3|A^1_{m-1,n}| - |B^1_{m-1,n}|) + \cdots + 2(3|A^{m-2}_{m-1,n}| - |B^{m-2}_{m-1,n}|) + (n - 2m + 2)(3|A^{m-1}_{m-1,n}| - |B^{m-1}_{m-1,n}|)$.*

Proof. Suppose that $n > 2m - 2$. Then $|\{m - 1, m, \cdots, n - m\}| = n - 2m + 2$. Since $|WHom(P_m, P_n)| = |A^0_{m,n}| + |A^1_{m,n}| + \cdots + |A^{n-1}_{m,n}|$, by Lemma 2 and Lemma 3 (2), we have $|WHom(P_m, P_n)| = 2|A^0_{m,n}| + 2|A^1_{m,n}| + \cdots + 2|A^{m-2}_{m,n}| + (n - 2m + 2)|A^{m-1}_{m,n}|$. Consider $C^i_{m-1,n} = \{f \in WHom(P_{m-1}, P_n) \mid f(0) = i \text{ and } f(m-2) = n-1\}$. We see that if $f(0) = i$ then $f(m-2) \le i + (m-2)$. It follows that if $f(0) \le m - 1$ then $f(m-2) \le 2m - 3 < n - 1$. Thus $C^i_{m-1,n} = \varnothing$ for all $i = 0, 1, \cdots, m - 1$. So, by Lemma 4, $|A^i_{m,n}| = 3|A^i_{m-1,n}| - |B^i_{m-1,n}|$ for all $i = 0, 1, \cdots, m - 1$. It follows that $|WHom(P_m, P_n)| = 2(3|A^0_{m-1,n}| - |B^0_{m-1,n}|) + 2(3|A^1_{m-1,n}| - |B^1_{m-1,n}|) + \cdots + 2(3|A^{m-2}_{m-1,n}| - |B^{m-2}_{m-1,n}|) + (n - 2m + 2)(3|A^{m-1}_{m-1,n}| - |B^{m-1}_{m-1,n}|)$.

Example 8. From Example 1, we have $|\mathrm{WHom}(P_3, P_6)| = 44$. Consider $\mathrm{WHom}(P_2, P_6)$ in Example 5 we see that

$$\begin{aligned} |\mathrm{WHom}(P_3, P_6)| &= 2\left(3|A^0_{2,6}| - |B^0_{2,6}|\right) + 2\left(3|A^1_{2,6}| - |B^1_{2,6}|\right) \\ &\quad + 2\left(3|A^2_{2,6}| - |B^2_{2,6}|\right) \\ &= 2(3(2) - 1) + 2\left(3^2 - 1\right) + 2\left(3^2 - 0\right) \\ &= 10 + 16 + 18 = 44. \end{aligned} \quad (11)$$

In the following lemmas, we give the order of $A^{m-1}_{m,n}$ and $A^{m-2}_{m,n}$.

Lemma 9. *Let $m, n \in \mathbb{N} - \{1\}$. If $n > 2m - 2$, then $|A^{m-1}_{m,n}| = 3|A^{m-1}_{m-1,n}| - |B^{m-1}_{m-1,n}| = 3^{m-1}$ for all $m \in \mathbb{N}$.*

Proof. Let $f \in A^{m-1}_{m-1,n}$. Then $f(0) = m - 1$, and so $1 = (m-1) - (m-2) \le f(m-2) \le (m-1) + (m-2) = 2m - 3$. So, for each $i \in \{0, 1, \cdots, m - 3\}$, if $f(i) = k$ then there are 3 possible choises for $f(i+1)$ which are $k - 1, k$, and $k + 1$. By the multiplication principle, $|A^{m-1}_{m-1,n}| = 3^{m-2}$. Since $B^{m-1}_{m-1,n} = \varnothing$ and $C^{m-1}_{m-1,n} = \varnothing$, we have $|A^{m-1}_{m,n}| = 3(3^{m-2}) = 3^{m-1}$ by Lemma 4.

Lemma 10. *Let $m, n \in \mathbb{N} - \{1\}$. If $n > 2m - 2$, then $|A^{m-2}_{m,n}| = 3|A^{m-2}_{m-1,n}| - |B^{m-2}_{m-1,n}| = 3^{m-1} - 1$ for all $m \in \mathbb{N}$.*

Proof. Let $f \in A^{m-2}_{m-1,n}$. Then $f(0) = m - 2$, and $1 = (m-2) - (m-3) \le f(m-3) \le (m-2) + (m-3) = 2m - 5$. So, for each $i \in \{0, 1, \cdots, m - 3\}$, if $f(i) = k$ then there are 3 possible choices for $f(i+1)$ which are $k - 1, k$, and $k + 1$. By the multiplication principle, $|A^{m-2}_{m-1,n}| = 3^{m-2}$. It easy to see that $f \in B^{m-2}_{m-1,n} \leftrightarrow f(0) = m - 2$ and $f(m-2) = 0 \leftrightarrow f = (m-2)(m-3)\cdots(1)(0)$. Thus $|B^{m-2}_{m-1,n}| = 1$. Since $C^{m-2}_{m-1,n} = \varnothing$, we get $|A^{m-2}_{m,n}| = 3(3^{m-2}) - 1 = 3^{m-1} - 1$ by Lemma 4.

In light of Theorem 7, Lemma 9, and Lemma 10, we have the following.

Corollary 11. *Let $m, n \in \mathbb{N} - \{1\}$. If $n > 2m - 2$, then $|WHom(P_m, P_n)| = 2(3|A^0_{m-1,n}| - |B^0_{m-1,n}|) + 2(3|A^1_{m-1,n}| - |B^1_{m-1,n}|) + \cdots + 2(3|A^{m-3}_{m-1,n}| - |B^{m-3}_{m-1,n}|) + 2(3^{m-1} - 1) + (n - 2m + 2)(3^{m-1})$.*

Data Availability

All data are available in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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